Put away and turn off all cell phones and electronic devices. You do not need a calculator. Put your answers on the sheets of paper handed out.

1. For a regression model with pairs \((Y_1, x_1), \ldots, (Y_n, x_n)\), it is assumed that

\[
\frac{1}{Y_i} = x_i^T \beta + e_i,
\]

where \(e_1, \ldots, e_n\) are iid \(N(0, \sigma^2)\), the \(Y_i\) are strictly positive, \(0 < \gamma < 1\), and the \(x_i\) are fixed constants. (We can make the normal assumption for the purposes of finding a likelihood even though it is not possible to add normal errors and guarantee \(Y_i > 0\).)

a. Write down the likelihood assuming \(\gamma\) is unknown.

b. After estimating \(\gamma\), the estimates for \(\beta\) and \(\sigma^2\) are found by the usual least squares program with dependent variable \(1/Y_i^\gamma\). Two students are asked to give a confidence interval for \(\beta_1\). Ralph ignores that \(\gamma\) has been estimated and gives a confidence interval for \(\beta_1\) based on the least squares output. Mary estimates the information matrix taking into account that \(\gamma\) has been estimated, inverts it, and constructs a confidence interval for \(\beta_1\). How are these two confidence intervals likely to be different? (Ignore the difference between normal and \(t\) percentiles.)

2. Suppose that we have a one-way random effects model from a meta-analysis:

\[
Y_i = \mu + a_i + e_i, \quad i = 1, \ldots, k,
\]

where \(a_1, \ldots, a_k\) are iid \(N(0, \sigma^2_a)\) and independent of the errors \(e_i\) that are all independent and distributed as \(N(0, V_i)\), where \(V_i\) is known, \(i = 1, \ldots, k\). Recall from last year’s mid-term solution that the log likelihood is

\[
C - \frac{1}{2} \sum_{i=1}^{k} \log(\sigma^2_a + V_i) - \frac{1}{2} \sum_{i=1}^{k} w_i (Y_i - \mu)^2,
\]

where \(w_i = (\sigma^2_a + V_i)^{-1}\).

a. Find \(I_k(Y, \theta)\).

b. Find the maximum likelihood estimator (MLE) of \(\mu\) for a given fixed value of \(\sigma^2_a\), say \(\hat{\mu}(\sigma^2_a)\). Use one of the theorems on p. 3 to justify that it is the unique MLE.

c. Using b., give the profile likelihood of \(\sigma^2_a\).
3. Ralph goes to the NC State Fair and wants to decide which of two games he is most likely to win at (to later impress his girlfriend on a return trip to the Fair). So he plays Game 1 until he wins; it takes $Y_1$ attempts. Then he plays Game 2 until he wins; it takes $Y_2$ attempts. If the probabilities that he wins on any one attempt is $p_1$ for the first game and $p_2$ for the second, he wants to test $H_0 : p_1/p_2 = 1$. (From his biostatistics background, he prefers to think in terms of relative risk $p_1/p_2$ when probabilities are small.) Assuming independence among attempts and no learning over time, the probability mass functions are geometric,

$$f(y; p_i) = p_i(1 - p_i)^{y-1} \quad y = 0, 1, \ldots \quad 0 \leq p_i \leq 1, \ i = 1, 2.$$ 

The mean and variance of a geometric random variable with parameter $p$ are $1/p$ and $(1 - p)/p^2$, respectively.

a. Find the maximum likelihood estimates.

b. Give the total Fisher information matrix $I_T(p_1, p_2)$ for the data.

c. Derive the Wald statistic $T_W$ based on $h(p_1, p_2) = p_1/p_2 - 1$. Express it in terms of the maximum likelihood estimates and simplify it as far as you can (without wasting too much time).

4. Ralph attended the International Dinner last week and heard so much about being a Bayesian that he decided to try it out on his fair data from the previous problem.

a. What is Jeffreys’s prior for $(p_1, p_2)$ from 3.?

b. Give the posterior for $(p_1, p_2)$ based on your answer in a.

c. Explain how to get a credible interval for $p_1/p_2$ based on the joint posterior in b., but do not carry out the calculus.
Theorem 1 (Makelainen, et al., 1981, Corollary 2.5) Let \( \Theta \) be a connected open subset of \( \mathbb{R}^b \), \( b \geq 1 \), and let \( \ell(\theta) \) be a twice continuously differentiable real-valued function on \( \Theta \) with \( \lim_{\theta \to \partial \Theta} \ell(\theta) = c \), where \( c \) is either a real number or \( -\infty \). Suppose that \( I_T(Y, \theta) = -\partial^2 \ell(\theta)/\partial \theta \partial \theta^T \) is positive definite at every point \( \theta \in \Theta \) for which \( \partial \ell(\theta)/\partial \theta^T = 0 \). Then \( \ell(\theta) \) has a unique global maximum and no other critical points. Furthermore, \( \ell(\theta) > c \) for every \( \theta \in \Theta \).

Theorem 2 (Makelainen, et al., 1981, Theorem 2.6) Let \( \ell(\theta) \) be twice continuously differentiable with \( \theta \) varying in a connected open subset \( \Theta \subset \mathbb{R}^b \). Suppose that

(i) the likelihood equations \( \partial \ell(\theta)/\partial \theta^T = 0 \) have at least one solution \( \theta \in \Theta \)

and that

(ii) \( I_T(Y, \theta) = -\partial^2 \ell(\theta)/\partial \theta \partial \theta^T \) is positive definite at every point \( \theta \in \Theta \).

Then

(a) \( \ell(\theta) \) is a strictly concave function of \( \theta \);

(b) there is a unique maximum likelihood estimate \( \hat{\theta}_{MLE} \in \Theta \);

(c) \( \ell(\theta) \) has no other maxima or minima or other stationary points in \( \Theta \).