Satterthwaite’s formula

When you have random effects, there are occasions in which you have to combine mean squares to get a proper denominator. Consider a split plot in blocks with s levels of the split plot factor, t levels of the whole plot factor, and b blocks. The model is

\[ Y_{ijk} = \mu + B_i + \tau_j + E(1)_{ij} + \gamma_k + (\tau \gamma)_{jk} + E(2)_{ijk} \]

Comparing 2 whole plot means within the same split plot we get a variance \( 2V/r \) where \( r \) is the number of things averaged in each mean and \( V = \sigma_1^2 + \sigma_2^2 \) is the whole plot error variance plus the spit plot error variance (see note 1 below).

The error 1 (or error A) mean square estimates \( s\sigma_1^2 + \sigma_2^2 \) and the error 2 (or error B) mean square estimates \( \sigma_2^2 \) so that \( [\text{MS}(1) + (s-1)\text{MS}(2)]/s \) estimates \( V \) and we can use the square root of that estimate as our standard error in getting a confidence interval, but what would be our degrees of freedom? Is it \( df_1 \) from error 1 or \( df_2 \) from error 2 or what? Satterthwaite suggested the following approximation (stat majors: we are dealing with a weighted average of two Chi-square variables here)

Write the linear combination of \( m \) mean squares as \( \sum_{i=1}^{m} a_i \text{MSE}(i) \)

The approximate degrees of freedom for that linear combination is

\[ df = \left( \frac{\sum_{i=1}^{m} a_i \text{MSE}(i)}{\sum_{i=1}^{m} \frac{(a_i \text{MSE}(i))^2}{df_i}} \right)^2 \]

Note 1: \( \bar{Y}_{21} - \bar{Y}_{*11} = \tau_2 - \tau_1 + (\tau \gamma)_{21} - (\tau \gamma)_{11} + \bar{e}(1)_{*2} - \bar{e}(1)_{*1} + \bar{e}(2)_{*2} - \bar{e}(2)_{*1} \)

Both the whole and split plot error terms are averaged over the \( b \) blocks (or in general over the number of things you averaged in each sample mean) so the variance would be \( 2(\sigma_1^2 + \sigma_2^2)/r \) where \( r \) is the number of observations averaged in each mean. In a split plot with just one whole plot factor as above, \( r \) is \( b \).

Note 2: In PROC GLM, I cannot specify an error term that is a mixture and so cannot get this computed within PROC GLM. In MIXED, I have to specify that the default (“containment”) degrees of freedom method be overridden with DDFM=SATTERTHWAITE in my model statement. Notice that in MIXED, there are no mean squares so that research on how to mimic Satterthwaite’s method in REML applications had to be done.