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RESOLVABLE INCOMPLETE BLOCK DESIGNS WITH TWO REPLICATIONS

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I. Introduction

Incomplete block designs become most useful when the number of treatments or varieties is very large. In such cases it is seldom that the research worker will have experimental material for more than a small number of replications. It is therefore of practical importance to make available to him a repertoire of designs involving as few replications as possible.

For a systematic search for such designs, their construction and analysis, it is advisable to go step by step from designs requiring two replications to those requiring three replications, four replications, and so on.

In this paper, we shall consider two-replicate designs only. There are two main groups of incomplete block designs already investigated, namely, the balanced incomplete block designs due to Yates (1936) and the partially balanced incomplete block (p.b.i.b) designs developed by the authors (1939) and whose definition was slightly modified by Nair and Rao (1942).

In virtue of the fact that the number of replications of each treatment in a balanced incomplete block design cannot be less than the number of plots per block and since the latter cannot be less than two, the search for two-replicate balanced incomplete block designs has to be confined to the case where there are two plots per block. There is only one set of such balanced designs, namely, the one where every pair of treatments is made to constitute one block. In the familiar notation:

$$\lambda(v-1) = r(k-1)$$

the only two-replicate balanced design we can construct is for the trivial case,

$$v = 3. \; k = 2, \; r = 2, \; b = 3, \; \lambda = 1 \; .$$

Since partially balanced incomplete block designs are free from the restriction $r \geq k$ a number of p.b.i.b. designs can be constructed for which $r = 2$. Bose (1951) made an exhaustive study of two-replicate p.b.i.b. designs having two associate classes. Nair (1950, 1951) gave some examples of two-replicate p.b.i.b. designs involving three or four associate classes. Among the designs considered by them there is a sub-class where the blocks can be arranged in two separate replication groups. This is a feature which, if taken advantage of while laying out the experiment, can help in recovering inter-block information, and in thereby improving the efficiency of the design. Such designs will be called resolvable designs.

The only resolvable two-replicate p.b.i.b. designs found so far are basically the simple square and rectangular lattices involving p^2 and $p(p-1)$ treatments respectively in blocks of p and $(p-1)$ plots. The former involves two and the latter four associate classes of treatment comparisons. By replacing each treatment by a set of q treatments in each of these lattices we can evolve two-replicate p.b.i.b. designs with three and five associate classes for p^2q and $p(p-1)q$ treatments in blocks of pq and $(p-1)q$ treatments respectively.

In this paper we shall evolve a new class of resolvable incomplete block designs having only two replications. In general, these designs have seven associate classes of treatment comparisons, but in particular cases they may be less than seven. This new class of designs does not in general belong to the category of p.b.i.b. designs defined by Bose, Nair and Rao, but the dual designs of this class i.e., the designs obtained by interchanging treatments and blocks, are p.b.i.b. designs with three associate classes. The four p.b.i.b. designs mentioned in the last paragraph are special cases of this new class of designs.

The method of analysis of the new class of designs with recovery of inter-block information is simple and straightforward. The necessary formulae have been derived and illustrated by working out a numerical example.

A list of all the designs with $k \leq 10$ has been prepared and their methods of construction explained.

II. The Design

1. Consider the uxu incidence matrix of a symmetrical balanced incomplete block design where u treatments are replicated r times so that there are u blocks of r plots each and

(2.10) $$\lambda(u-1) = r(r-1) .$$

Let the matrix be denoted by

(2.11) $$N = (n_{ij})$$

where n_{ij}, the element at the junction of i-th row and j-th column, is either 0 or 1.

For example, take the symmetrical balanced incomplete block design with $u = 7$, $r = 3$, $\lambda = 1$. The incidence matrix of this design is

$$N = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

If rows stand for blocks and columns for treatments the incidence matrix tells us that treatment number 4, say, occurs in blocks no. 1, 3 and 4. Similarly, block number 4, say, contains treatments no. 4, 5 and 7.

2. The following conditions are satisfied by the elements of matrix (2.11)

(2.200) $$\sum_{i=1}^{u} n_{ij} = \sum_{i=1}^{u} n_{ij}^2 = r \qquad (j = 1, 2, \ldots u)$$

(2.205) $$\sum_{j=1}^{u} n_{ij} = \sum_{j=1}^{u} n_{ij}^2 = r \qquad (i = 1, 2, \ldots u)$$

(2.210) $$\sum_{j=1}^{u} n_{hj}\, n_{ij} = \lambda \qquad (h \neq i = 1, 2, \ldots u)$$

(2.215) $$\sum_{i=1}^{u} n_{ij}\, n_{ih} = \lambda \qquad (j, h = 1, 2, \ldots u) \quad .$$

3. In the matrix (2.11) we shall replace every n_{ij} having value 1 by a set of p treatments and every n_{ij} having value 0 by a set of q treatments. No treatment is repeated inside the matrix. The total number of treatments in

any row or column of the matrix will be

(2.30) $$k = rp + (u-r)q$$

and in the whole matrix:

(2.31) $$v = ku \quad .$$

4. To construct the design we take all the k treatments in the same row of the matrix to form a single block and thus build up u blocks from the u rows to form one replication. We do likewise with the k treatments in the same column of the matrix and form u blocks of a second replication from the u columns. The design will therefore have two resolvable replications with 2u blocks of k plots each.

5. We will consider a few examples at this stage.

Example 1. Let $u = 3$, $r = 2$. Then $\lambda = 1$ and

$$N = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix} \quad .$$

Let $p = 2$, $q = 1$. Representing the ℓ-th treatment in the cell (ij) by t_{ij}^{ℓ}, the scheme of the design is:

t_{11}^{1} t_{11}^{2}	t_{12}^{1} t_{12}^{2}	t_{13}
t_{21}	t_{22}^{1} t_{22}^{2}	t_{23}^{1} t_{23}^{2}
t_{31}^{1} t_{31}^{2}	t_{32}	t_{33}^{1} t_{33}^{2}

Re-identifying the treatments and calling them 1, 2, ... 15, we could write the scheme in a simpler way as

1,2	3,4	5
6	7,8	9,10
11,12	13	14,15

The design is then:

Replication I						Replication II					
Block Number	Treatment Number					Block Number	Treatment Number				
(1)	1	2	3	4	5	(4)	1	2	6	11	12
(2)	6	7	8	9	10	(5)	3	4	7	8	13
(3)	11	12	13	14	15	(6)	5	9	10	14	15

We thus have a resolvable two-replicate design for 15 treatments in six blocks of 5 plots each.

Example 2. Let $u = r$. In other words we are dealing with symmetrical complete blocks, so that the n_{ij} in the incidence matrix is 1 in every cell and $\lambda = r$. Putting $p = 1$, we get the simple square lattice for $v = u^2$, $k = u$. When $p > 1$, we get the extended square lattice design: $v = u^2p$, $k = up$.

Example 3. Let $u = r+1$. Then $\lambda = r-1$. By putting $p = 1$, $q = 0$, we get the simple rectangular lattice for $v = u(u-1)$, $k = (u-1)$. If $p > 1$, $q = 0$, we get the extended rectangular lattice design: $v = u(u-1)p$, $k = (u-1)p$.

III. Dual of the Design

1. Let us obtain the dual of the design given in Example 1 above. Using treatment numbers of the original design to represent block numbers of the dual design and *vice versa*, we get

Block No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Treatment	(1)	(1)	(1)	(1)	(1)	(2)	(2)	(2)	(2)	(2)	(3)	(3)	(3)	(3)	(3)
Number	(4)	(4)	(5)	(5)	(6)	(4)	(5)	(5)	(6)	(6)	(4)	(4)	(5)	(6)	(6)

This is a p.b.i.b. design with the following parameters:

$$v = 6 \qquad k = 2 \qquad r = 5 \qquad b = 15$$

$$\lambda_1 = 2 \qquad \lambda_2 = 1 \qquad \lambda_3 = 0$$

$$n_1 = 2 \qquad n_2 = 1 \qquad n_3 = 2$$

$$p^1_{jk} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad p^2_{jk} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \qquad p^3_{jk} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. It can be seen that, in general, the dual of the two-replicate design considered in the previous section is a p.b.i.b. design having the following parameters:

$$v^* = 2u \qquad k^* = 2 \qquad r^* = rp + (u-r)q \qquad b^* = ur^*$$

$$\lambda_1 = p \qquad \lambda_2 = q \qquad \lambda_3 = 0$$

$$n_1 = r \qquad n_2 = (u-r) \qquad n_3 = (u-1)$$

$$p^1_{jk} = \begin{pmatrix} 0 & 0 & (r-1) \\ 0 & 0 & (u-r) \\ (r-1) & (u-r) & 0 \end{pmatrix} \qquad p^2_{jk} = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & (u-r-1) \\ r & (u-r-1) & 0 \end{pmatrix}$$

$$p^3_{jk} = \begin{pmatrix} \lambda & (r-\lambda) & 0 \\ (r-\lambda) & (u-2r+\lambda) & 0 \\ 0 & 0 & (u-2) \end{pmatrix} .$$

When $u = r$, n_2 vanishes and hence the dual of the extended square lattice ($v = u^2p$, $k = up$) is a p.b. .b. design with two associate classes,

$$v^* = 2u \qquad k^* = 2 \qquad r^* = up \qquad b^* = u^2p$$
$$\lambda_1 = p \qquad \lambda_2 = 0$$
$$n_1 = u \qquad n_2 = u-1$$

$$p^1_{jk} = \begin{pmatrix} 0 & (u-1) \\ (u-1) & 0 \end{pmatrix} \qquad p^2_{jk} = \begin{pmatrix} u & 0 \\ 0 & (u-2) \end{pmatrix}$$

When $p = 1$, we have the dual of the simple square lattice ($v = u^2$, $k = u$)

$$v^* = 2u \qquad k^* = 2 \qquad r^* = u \qquad b^* = u^2$$
$$\lambda_1 = 1 \qquad \lambda_2 = 0$$
$$n_1 = u \qquad n_2 = u-1$$

and same values of p^1_{jk} and p^2_{jk} as for the extended square lattice.

IV. Estimation of Treatment Effects

1. Let us denote the n_{ij} treatments of cell (ij) by $ij(1)$, $ij(2)$, ..., $ij(n_{ij})$ and their effects by

$$t_{ij}^{(1)}, t_{ij}^{(2)}, \ldots, t_{ij}^{(n_{ij})} \quad .$$

Let

$$(4.105) \qquad R_i(t) = \sum_{j=1}^{u} \sum_{\ell=1}^{n_{ij}} t_{ij}^{(\ell)}$$

= Sum of the treatment effects for row i

$$(4.110) \qquad C_j(t) = \sum_{i=1}^{u} \sum_{\ell=1}^{n_{ij}} t_{ij}^{(\ell)}$$

= Sum of the treatment effects for column j.

For the sake of unique estimation of the effects, we shall make the usual assumption that $\Sigma\ \Sigma\ \Sigma\ t_{ij}^{(\ell)} = 0$, or, that

$$(4.12) \qquad \sum_{i=1}^{u} R_i(t) = \sum_{j=1}^{u} C_j(t) = 0.$$

For treatment $ij(\ell)$, let $x_{ij}^{(\ell)}$ be the observed value of the character under study in the first replication and $y_{ij}^{(\ell)}$ that in the second replication. We shall call these replications x- and y-replications.

Let

$$(4.130) \qquad R_i(x) = \sum_{j=1}^{u} \sum_{\ell=1}^{n_{ij}} x_{ij}^{(\ell)}$$

(4.135) $$C_j(x) = \sum_{i=1}^{u} \sum_{\ell=1}^{n_{ij}} x_{ij}^{(\ell)} \quad ..$$

Let

$$T(x) = \sum_{i=1}^{u} R_i(x) = \sum_{j=1}^{u} C_j(x)$$

(4.140) $$R_i(y) = \sum_{j=1}^{u} \sum_{\ell=1}^{n_{ij}} y_{ij}^{(\ell)}$$

(4.145) $$C_j(y) = \sum_{i=1}^{u} \sum_{\ell=1}^{n_{ij}} y_{ij}^{(\ell)} \quad .$$

Let

$$T(y) = \Sigma R_i(y) = \Sigma C_j(y) \ .$$

It will be noted that $R_i(x)$ is the total value of x for the i-th block of the x-replication and that $C_j(y)$ is the total value of y for the j-th block of the y-replication.

Let $Q_{ij}^{(\ell)}$ be the total observed value of the character under study for the two replications of treatment $ij(\ell)$ minus the sum of the corresponding block means.

Hence,

(4.15) $$Q_{ij}^{(\ell)} = x_{ij}^{(\ell)} + y_{ij}^{(\ell)} - \frac{1}{k} \left\{ R_i(x) + C_j(y) \right\} \quad .$$

Let us introduce a quantity $Q_{ij}^{\prime(\ell)}$ defined by

$$(4.16) \qquad Q_{ij}^{\prime(\ell)} = \frac{1}{k}\left\{R_i(x) + C_j(y)\right\} - \frac{1}{v}\left\{T(x) + T(y)\right\} .$$

Let

$$(4.170) \qquad R_i(Q) = \sum_{j=1}^{u} \sum_{\ell=1}^{n_{ij}} Q_{ij}^{(\ell)} \; ; \quad R_i(Q') = \sum_{j=1}^{u} \sum_{j=1}^{n_{ij}} Q_{ij}^{\prime(\ell)} .$$

$$(4.175) \qquad C_j(Q) = \sum_{i=1}^{u} \sum_{\ell=1}^{n_{ij}} Q_{ij}^{(\ell)} \; ; \quad C_j(Q') = \sum_{i=1}^{u} \sum_{j=1}^{n_{ij}} Q_{ij}^{\prime(\ell)} .$$

2. The normal equation corresponding to the treatment $ij(\ell)$ may be written as

$$(4.20) \qquad k\left\{wQ_{ij}^{(\ell)} + w'Q_{ij}^{\prime(\ell)}\right\} = 2kwt_{ij}^{(\ell)} - (w-w')\left\{R_i(t) + C_j(t)\right\}$$

where w and w' are reciprocals of the intra- and inter-block variance per plot.

Summing up over j,

$$(4.21) \qquad k\left\{wR_i(Q) + w'R_i(Q')\right\} = k(w+w')R_i(t) - (w-w') \sum_{j=1}^{u} n_{ij}C_j(t)$$

Summing up over i,

$$(4.22) \qquad k\left\{wC_j(Q) + w'C_j(Q')\right\} = k(w+w')C_j(t) - (w-w') \sum_{i=1}^{u} n_{ij}R_i(t) .$$

Substituting for $C_j(t)$ from (4.22) in (4.21) and putting

$$(4.23) \qquad \frac{w-w'}{w+w'} = \gamma ,$$

we get,

$$(4.24) \qquad C_{i1}R_1(t) + C_{i2}R_2(t) + \ldots + C_{iu}R_u(t) = k \left\{ wR_i(Q) + w'R_i(Q') \right\} + \gamma \sum_{j=1}^{u} n_{ij} \left\{ wC_j(Q) + w'C_j(Q') \right\}$$

where

$$(4.250) \qquad C_{ii} = k(w+w') \left\{ 1 - \left(\frac{\gamma}{k}\right)^2 (n_{i1}^2 + n_{i2}^2 + \ldots + n_{iu}^2) \right\}$$

$$(4.255) \qquad C_{ih} = - \frac{\gamma^2}{k} (w+w') \left\{ n_{i1}n_{h1} + n_{i2}n_{h2} + \ldots + n_{iu}n_{hu} \right\} \; i \neq h \; .$$

In the case of the design considered in Section II, n_{ij} takes either of the valuesp or q according as the corresponding call of the incidence matrix of the symmetrical balanced incomplete block design has the number 1 or 0. Hence (4.250) and (4.255) reduce to

$$(4.260) \qquad C_{ii} = k(w+w') \left[1 - \left(\frac{\gamma}{k}\right)^2 \; rp^2 + (u-r)q^2 \right]$$

$$(4.265) \qquad C_{ih} = - \frac{\gamma^2}{k} (w+w') \left\{ \lambda p^2 + 2(r-\lambda)pq + (u-2r+\lambda)q^2 \right\} .$$

Using the condition (4.12), we can simplify (4.24) to the form

$$(4.27) \quad (C_{ii} - C_{ih})R_i(t) = k \left\{ wR_i(Q) + w'R_i(Q') \right\} + \gamma \sum_{j=1}^{u} n_{ij} \left\{ wC_j(Q) + w'C_j(Q') \right\} .$$

Similarly, we get

$$(4.28) \quad (C_{ii} - C_{ih})C_j(t) = k \left\{ wC_j(Q) + w'C_j(Q') \right\} + \gamma \sum_{i=1}^{u} n_{ij} \left\{ wR_i(Q) + w'R_i(Q') \right\} .$$

The value of $(C_{ii} - C_{ih})$ occurring in (4.27) and (4.28) can be simplified to the form

$$(4.29) \qquad C_{ii} - C_{ih} = \frac{1}{k}(w+w')\left\{k^2 - (r-\lambda)(p-q)^2 \gamma^2\right\} .$$

3. Substituting (4.27) and (4.28) in (4.20), we have

$$(4.30) \qquad t_{ij}^{(\ell)} = \frac{1}{2w}(wQ_{ij}^{(\ell)} + w'Q_{ij}^{'(\ell)}) + \frac{k\gamma}{2w\mu}\left[\left\{wR_i(Q) + w'R_i(Q')\right\} + \left\{wC_j(Q) + w'C_j(Q')\right\}\right] + \frac{\gamma^2}{2w\mu}\left[\sum_{i=1}^{u} n_{ij}\left\{wR_i(Q) + w'R_i(Q')\right\} + \sum_{j=1}^{u} n_{ij}\left\{wC_j(Q) + w'C_j(Q')\right\}\right]$$

where,

$$(4.31) \qquad \mu = k^2 - (r-\lambda)(p-q)^2 \gamma^2 .$$

By putting $w' = 0$ in (4.30) and (4.31) we can obtain the intra-block estimate of the treatment effect, namely,

$$(4.32) \qquad \underset{\text{(intra-block)}}{t_{ij}^{(\ell)}} = \frac{1}{2} Q_{ij}^{(\ell)} + \frac{k\left\{R_i(Q) + C_j(Q)\right\} + \sum_{i=1}^{u} n_{ij}R_i(Q) + \sum_{j=1}^{u} n_{ij}C_j(Q)}{w \quad k^2 - (r-\lambda)(p-q)^2} .$$

V. Variances of Estimated Treatment Differences

1. We are interested to compare the effects, as estimated by (4.30), of any two of the v treatments. There are seven types of treatment pairs each having a different expression for the variance of difference between the two members of the pair. They are separately considered below.

(1) For two treatments belonging to the same cell of the uxu matrix, the variance of the difference of their effects is $1/w$.

(2) For two treatments belonging to different cells of the same row or column of the uxu matrix, the variance of the difference in effects is $\frac{1}{w}\left(1 + \frac{k\gamma}{\mu}\right)$.

(3) Two treatments not belonging to the same row or column of the uxu matrix fall into five sub-classes. Let us denote the cells to which these treatments belong by (gh) and (ij) where $g \neq i$ and $h \neq j$. We know that there are two types of cells, namely, those having p treatments and the others having q treatments. If we take the quadrant formed by the four cells:

(gh)	(gj)
(ih)	(ij)

it is clear that there are 16 types of quadrants. They fall into five groups according as the value of the cross difference

(5.10) $$d = n_{gh} + n_{ij} - n_{gj} - n_{ih}$$

is either 0, (p-q), (q-p), 2(p-q) or 2(q-p).

(i) <u>d = 0</u>. There are six quadrants in this group, namely,

	h j					
g	p p	q q	p p	p q	q p	q q
i	p p	q q	q q	p q	q p	p p

Variance of the difference between the effects of a treatment in cell (gh) and in cell (ij) is $\frac{1}{w}\left(1 + \frac{2k\gamma}{\mu}\right)$.

(ii) d = (p-q). There are four quadrants in this group, namely

	h j			
g	p p	p q	p q	q q
i	q p	p p	q q	q p

Variance of the difference between the effects of a treatment in cell (gh) and in cell (ij) is $\frac{1}{w}\left[1 + \frac{2k\gamma + (p-q)\gamma^2}{\mu} \right]$.

(iii) d = (q-p). There are four quadrants in this group, namely,

q q	q p	q p	p p
p q	q q	p p	p q

Variance in this case is $\frac{1}{w}\left[1 + \frac{2k\gamma + (q-p)\gamma^2}{\mu} \right]$.

(iv) d = 2(p-q). There is only one quadrant in this group, namely,

p q
q p

Variance is $\frac{1}{w}\left[1 + \frac{2k\gamma + 2(p-q)\gamma^2}{\mu}\right]$.

(v) <u>d = 2(q-p)</u> . There is only one quadrant in this group, namely,

q	p
p	q

Variance is $\frac{1}{w}\left[1 + \frac{2k\gamma + 2(q-p)\gamma^2}{\mu}\right]$.

The variances in the cases (i) to (v) may be denoted by the common expression

$$\frac{1}{w}\left[1 + \frac{2k\gamma + d\gamma^2}{\mu}\right]$$

where d = 0, (p-q), (q-p), 2(p-q), 2(q-p) respectively.

2. Of the total number of $\frac{1}{2}$ v(v-1) differences between pairs of treatments, the number belonging to type (1) is:

$$\frac{1}{2}\, u \left\{rp(p-1) + (u-r)q(q-1)\right\} .$$

The number belonging to type (2) is:

$$u \left\{rp(k-p) + (u-r)q(k-q)\right\} .$$

The number belonging to type (3), sub-types (i) to (v) combined is:

$$\frac{1}{2}\, u \left\{rp(uk-2k+p) + (u-r)q(uk-2k+q)\right\} .$$

It is not easy in the general case to split up the last number into those belonging to each of the sub-types (i) to (v) and hence to calculate the mean variance of all comparisons. For particular designs the number of comparisons of each of the sub-types (i) to (v) can be determined and the mean variance calculated.

3. Special cases.

(a) When $u = 3$, r becomes 2 and the values of n_{ij} in the final scheme for the design: $v = 3k$, $k = 2p+q$ are distributed in the form

$$\begin{matrix} p & p & q \\ q & p & p \\ p & q & p \end{matrix}$$

In this case quadrants with $d = 0$ do not occur. Hence the number of different types of comparisons reduces to six.

Example 1 of Section II is a particular case when $p = 2$, $q = 1$.

(b) When $p > 1$, $q = 0$, comparisons of the sub-types (iii) and (v) do not occur. Hence the number of different types of comparisons reduces to five.

(c) When $p = 0$, $q > 1$, comparisons of the sub-types (ii) and (iv) do not occur. Hence, as in (b), the number of different types of comparisons reduces to five.

(d) When $p = 1$, $q = 0$ or $p = 0$, $q = 1$ there are only four types of comparisons. The corresponding variances are

$$\frac{1}{w}\left(1 + \frac{k\gamma}{\mu}\right), \qquad \frac{1}{w}\left[1 + \frac{2k\gamma + d\gamma^2}{\mu}\right] \quad d = 0, 1, 2$$

where $k = r$ or $u-r$ and $\mu = k^2 - (r-\lambda)\gamma^2$.

(e) If, in (b), $u = r+1$ so that $\lambda = r-1$, we have the extended rectangular lattice: $v = u(u-1)p$, $k = (u-1)p$ having five variances for treatment comparisons.

(f) If, in (d), $u = r+1$ so that $\lambda = r-1$, we have the simple rectangular lattice: $v = u(u-1)$, $k = (u-1)$. Here, $\mu = k^2 - \gamma^2$ in the expressions for the four variances.

(g) When $p = q$, which is the same as the case $u = r = \lambda$ the sub-types (i) to (v) merge into a single type of comparison and hence there are only three different variances instead of seven. They are:

$$\frac{1}{w},\ \frac{1}{w}\left(1 + \frac{\gamma}{k}\right),\ \ \frac{1}{w}\left(1 + \frac{2\gamma}{k}\right) \ .$$

The design is now the etended square lattice: $v = u^2p$, $k = up$.

When $p = q = 1$, there will be only two different variances, namely, $\frac{1}{w}\left(1 + \frac{\gamma}{k}\right)$ and $\frac{1}{w}\left(1 + \frac{2\gamma}{k}\right)$. The design becomes the simple square lattice: $v = u^2$, $k = u$.

VI. Estimation of w and w'.

1. The values of w and w' entering into the estimates of treatment effects given by (4.30) and in the variances of the differences of these estimates calculated in Section V are not known a priori and have to be estimated from the data. This is done with the aid of the analysis of variance given in Table 1.

2. The following sums of squares must be calculated first:

$$(6.20) \qquad \text{Total S.S.} = \sum_{i=1}^{u} \sum_{j=1}^{u} \sum_{\ell=1}^{n_{ij}} \left[\left\{x_{ij}^{(\ell)}\right\}^2 + \left\{y_{ij}^{(\ell)}\right\}^2 \right] - \frac{\left\{T(x) + T(y)\right\}^2}{2v} .$$

(6.21) Replication S.S. $= \dfrac{\{T(x) - T(y)\}^2}{2v}$.

(6.22) Blocks S.S. (unadjusted) $= \dfrac{1}{k}\left[\sum_{i=1}^{u}\{R_i(x)\}^2 + \sum_{j=1}^{u}\{C_j(y)\}^2\right] - \dfrac{1}{v}\left[\{T(x)\}^2 + \{T(y)\}^2\right]$.

(6.23) Treatment S.S. (unadjusted) $= \dfrac{1}{2} \sum_{i=1}^{u} \sum_{j=1}^{u} \sum_{\ell=1}^{n_{ij}} \{x_{ij}^{(\ell)} + y_{ij}^{(\ell)}\}^2 - \dfrac{\{T(x) + T(y)\}^2}{2v}$

3. The treatment sum of squares (adjusted) is calculated using the formula

$$(6.30) \qquad \sum_{i=1}^{u} \sum_{j=1}^{u} \sum_{\ell=1}^{n_{ij}} t_{ij}^{(\ell)} \; Q_{ij}^{(\ell)}$$

where $t_{ij}^{(\ell)}$ is the intra-block estimate given in (4.32).

Table 1. Analysis of Variance

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Squares
Replication	1	see (6.21)	
Blocks within replications (unadjusted)	2(u-1)	see (6.22)	
Treatments (adjusted)	(v-1)	see (6.30)	E_t
Intra-block error	v-2u+1	By subtraction	E_e
Total	2v-1	see (6.20)	
Blocks within replications (adjusted)	2(u-1)	(6.22) + (6.30) - (6.23)	E_b

4. The Blocks Sum of Squares within replications (adjusted) is obtained by using the relation

Block (adj.) + Treatment (unadj.) = Block (unadj.) + Treatment (adj.) .

5. A test of significance of the differences among the treatment effects (intra-block) can be performed by F-test where

$$F = E_t/E_e$$

with degrees of freedom (v-1) and (v-2u + 1).

6. Using formulae given by Nair (1944) the estimates of w and w' are as follows:

(6.60) $$w' = \frac{1}{2E_b - E_e} \quad ; \qquad w = \frac{1}{E_e}$$

Hence,

(6.61) $$\frac{w'}{w} = \frac{E_e}{2E_b - E_e}$$

and

(6.62) $$\gamma = \frac{E_b - E_e}{E_b} .$$

VII. A Numerical Example

1. We shall consider an example using the design for which u = 3, r = 2, λ = 1; p = 2, q = 1 so that the experiment will consist of two replications of 15 treatments in blocks of 5 plots each. Let us number the treatments: 1, 2, ..., 15. They fill the u x u matrix as follows:

1,2	3,4	5
6	7,8	9,10
11,12	13	14,15

The treatments were randomized within blocks and the blocks within replications. The lay-out of the experiment came out to be as follows:

Block Number	Replication I				
(1)	8	10	6	7	9
(2)	13	14	11	15	12
(3)	4	3	2	1	5

Block Number	Replication II				
(4)	4	8	7	13	3
(5)	12	6	2	1	11
(6)	10	15	5	9	14

2. These arrangements were super-imposed on a uniformity trial data on yield of peanuts published by Robinson, Rigney and Harvey (1948). Each individual yield is for a 50-feet single row plot of peanuts in units of 10 grams.

The yield of each plot of the lay-out is given below.

Block Number	Replication I					Block Total	Block Mean
	8	10	6	7	9		
(1)	370	342	319	321	339	1691	338.2
	13	14	11	15	12		
(2)	265	276	304	316	254	1415	283.0
	4	3	2	1	5		
(3)	299	314	272	222	280	1387	277.4
	Replication Total					4493	

Block Number	Replication II					Block Total	Block Mean
	4	8	7	13	3		
(4)	273	285	269	253	249	1329	265.8
	12	6	2	1	11		
(5)	293	276	313	254	283	1419	283.8
	10	15	5	9	14		
(6)	197	238	247	313	257	1252	250.4
	Replication Total					4000	

$$\text{General Mean} = \frac{4493 + 4000}{30} = 283.1$$

We may at this stage calculate the following sums of squares

$$\text{Total} = (370)^2 + \ldots + \ldots + (257)^2 - \frac{(8493)^2}{30} = 41276.7$$

$$\text{Replications} = (4493 - 4000)^2/30 = 8101.6$$

$$\begin{array}{l}\text{Blocks within} \\ \text{replications} \\ \text{(unadjusted)}\end{array} = \frac{(1691)^2 + (1415)^2 + (1387)^2}{5} - \frac{(4493)^2}{15}$$

$$+ \frac{(1329)^2 + (1419)^2 + (1252)^2}{5} - \frac{(4000)^2}{15} = 14086.[illegible]$$

3. We will next calculate the values of $Q_{ij}^{(\ell)}$ and $Q_{ij}^{\prime(\ell)}$. They are given in columns (4) and (6) of Table 2.

Table 2. Calculation of Treatment Effects

Treat. No.	Treat. Total	Total of Corr. Block Means	$Q_{ij}^{(\ell)}$	$t_{ij}^{(\ell)}$ (intra-block)	$Q_{ij}^{\prime(\ell)}$	$P = Q + \frac{w'}{w}$	$t_{ij}^{(\ell)}$ (intra + inter-block)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	476	561.2	-85.2	-54.37	- 5.0	-85.9960	-51.90
2	585	561.2	+23.8	+ 0.13	- 5.0	+23.0040	+ 2.60
3	563	543.2	+19.8	+13.71	-23.0	+16.1384	+10.15
4	572	543.2	+28.8	+18.21	-23.0	+25.1384	+14.65
5	527	527.8	- 0.8	+ 1.23	-38.4	- 6.9133	- 3.76
6	595	622.0	-27.0	-18.85	+55.8	-18.1166	-10.87
7	590	604.0	-14.0	+ 3.23	+37.8	- 7.9822	+ 5.17
8	655	604.0	+51.0	+35.73	+37.8	+57.0178	+37.67
9	652	588.6	+63.4	+39.75	+22.4	+66.9661	+40.26
10	539	588.6	-49.6	-16.75	+22.4	-46.0339	-16.24
11	587	566.8	+20.2	- 1.77	+ 0.6	+20.2955	+ 1.29
12	547	566.8	-19.8	-21.77	+ 0.6	-19.7045	-18.71
13	518	548.8	-30.8	-11.68	-17.4	-33.5701	-14.66
14	533	533.4	- 0.4	+ 1.34	-32.8	- 5.6218	- 3.07
15	554	533.4	+20.6	+11.84	-32.8	+15.3782	+ 7.43

The intra-block estimates of the treatment effects have next to be calculated using the formula (4.32). In this example, that formula may be written

$$t_{ij}^{(\ell)} = \frac{1}{2} \left[Q_{ij}^{(\ell)} + \alpha_i(Q) + \beta_j(Q) \right]$$

where,

$$\alpha_i(Q) = \frac{1}{24} \left[5R_i(Q) + \sum_{i=1}^{3} n_{ij}C_j(Q) \right]$$

$$\beta_j(Q) = \frac{1}{24} \left[5C_j(Q) + \sum_{i=1}^{3} n_{ij}R_i(Q) \right]$$

The calculation of $\alpha_i(Q)$ and $\beta_j(Q)$ become simple if the values of $Q_{ij}^{(\ell)}$ given in column (4) of Table 2 are arranged in the 3 x 3 matrix form as below.

	j = 1		j = 2		j = 3		$R_i(Q)$	$\sum_{j=1}^{3} n_{ij}C_j(Q)$	$\alpha_i(Q)$
i = 1	Q_1 -85.2	Q_2 +23.8	Q_3 +19.8	Q_4 +28.8	Q_5 -0.8		-13.6	-33.2	-4.217
i = 2	Q_6 -27.0		Q_7 -14.0	Q_8 +51.0	Q_9 +63.4	Q_{10} -49.6	+23.8	+88.0	+8.625
i = 3	Q_{11} +20.2	Q_{12} -19.8	Q_{13} -30.8		Q_{14} - 0.4	Q_{15} +20.6	-10.2	-54.8	-4.408
$C_j(Q)$	-88.0		-54.8		+33.2				
$\sum_{i=1}^{3} n_{ij}R_i(Q)$	-23.8		+10.2		+13.6				
$\beta_j(Q)$	-19.325		+11.842		+ 7.483				

The intra-block estimate of effect of treatment No. 1 is

$$\frac{1}{2} [-85.2 - 4.217 - 19.325] = -54.37 .$$

Estimates for the remaining treatments have similarly been calculated. They are given in column (5) of Table 2.

4. The sum of squares due to treatments adjusted for block effects can now be calculated by multiplying the entries of columns (4) and (5) of Table 2 and adding up. This comes out to be 12061.5.

The sum of squares due to treatments unadjusted for block effects is obtained from the entries of column (2) of Table 2 in the usual way. It comes out as 15914.2.

We can now complete the table of analysis of variance:

Table 3. Analysis of Variance

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square
Replication	1	8101.6	
Blocks within replications (unadj.)	4	14086.3	
Treatments (adj.)	14	12061.5	861.54 (E_t)
Intra-block error	10	7027.3	702.73 (E_e)
Total	29	41276.7	
Treatment (unadj.)	14	15914.2	
Blocks within replications (adj.)	4	10233.6	2558.40 (E_b)

5. To test whether the intra-block estimates of treatment effects given in column (5) of Table 2 are significant we perform the variance - ratio test

$$F = E_t/E_e = \frac{861.54}{702.73} + 1.23$$

with 14 and 10 degrees of freedom. We find that the observed value of F is not significant even at the 10 percent level of significance. As there had been no <u>real</u> treatments in the experiment this non-significance is in conformity with our expectation.

6. We will now proceed to calculate the treatment effects with recovery of inter-block information using the formula (4.30). For doing this, we estimate the following quantities from Table 3.

$$\frac{w'}{w} = \frac{E_e}{2E_b - E_e} = 0.15920$$

$$\gamma = \frac{E_b - E_e}{E_b} = 0.72532$$

$$\mu = 25 - \gamma^2 = 24.47391$$

$$\frac{\gamma}{\mu} = 0.02964 \qquad \frac{\gamma^2}{\mu} = 0.02150$$

Values of the quantity

$$P_{ij}^{(\ell)} = Q_{ij}^{(\ell)} + \frac{w'}{w} Q_{ij}^{'(\ell)} = Q_{ij}^{(\ell)} + 0.15920 \quad Q_{ij}^{'(\ell)}$$

are then calculated in column (7) of Table 2. These values are next arranged in the form of a 3 x 3 matrix as done earlier for values of $Q_{ij}^{(\ell)}$.

	$j = 1$	$j = 2$	$j = 3$	$\alpha_i = R_i(P)$	$\alpha_i' = \sum_{j=1}^{3} n_{ij}\beta_j$	$\frac{\Upsilon}{\mu}(k\alpha_i + \gamma\alpha_i')$
$i = 1$	P_1 -85.9960 P_2 +23.0040	P_3 +16.1384 P_4 +25.1384	P_5 - 6.9133	-28.6285	-23.7753	-4.7539
$i = 2$	P_6 -18.1166	P_7 - 7.9822 P_8 +57.0178	P_9 +66.9661 P_{10} -46.0339	+51.8512	+80.5176	+9.4153
$i = 3$	P_{11} +20.2955 P_{12} -19.7045	P_{13} -33.5701	P_{14} - 5.6218 P_{15} +15.3782	-23.2227	-56.7423	-4.6614
$\beta_j = C_j(P)$	-80.5176	+56.7423	+23.7753			
$\beta_j' = \sum_{i=1}^{3} n_{ij}\alpha_i$	-51.8512	+23.2227	+28.6285			
$\frac{\Upsilon}{\mu}(k\beta_j + \gamma\beta_j')$	-13.0474	+ 8.9084	+ 4.1390			

In the marginal rows and columns of this matrix we calculate some auxiliary quantities with the help of which formula (4.30) for estimating treatment effects with recovery of inter-block information reduces to

$$t_{ij}^{(\ell)} = \frac{1}{2} \left[P_{ij}^{(\ell)} + \frac{\Upsilon}{\mu} \left\{ k(\alpha_i + \beta_j) + \gamma(\alpha_i' + \beta_j') \right\} \right]$$

Thus, for treatment number 1, we get the estimate as

$$\frac{1}{2} \left[- 85.9960 - 4.7539 - 13.0474 \right] = - 51.90 \quad .$$

The estimates for all the 15 treatments were calculated in this way and are given in column (8) of Table 2.

7. By adding the general mean 283.1 to the values of treatment effects given in columns (5) and (8) we get the adjusted treatment means without and with recovery of inter-block information. These values are given in Table 4 alongside unadjusted treatment means.

Table 4. Treatment Means

Treatment Number	Unadjusted Mean	Adjusted, with inter-block information: Not Recovered	Adjusted, with inter-block information: Recovered
1	238.0	228.73	231.20
2	292.5	283.23	285.70
3	281.5	296.81	293.25
4	286.0	301.31	297.75
5	263.5	284.33	279.34
6	297.5	264.25	272.23
7	295.0	286.33	288.27
8	327.5	318.83	320.77
9	326.0	322.85	323.36
10	269.5	266.35	266.86
11	293.5	281.33	284.39
12	273.5	261.33	264.39
13	259.0	271.42	268.44
14	266.5	284.44	280.03
15	277.0	294.94	290.53

8. For calculating the lowest significant difference among the values in the last column of Table 4, we have to calculate the variance of the difference between every pair of them. These pairs fall into six groups:

(1) Those occurring in the same cell of the u x u matrix. For example, 1 and 2. There are six pairs of them, each having a variance of difference equal to

$$\frac{1}{w} = 702.73 \tag{7.80}$$

(2) Those occurring in different cells of the same row or column. Example 1 and 3. There are just 48 such pairs eachhaving a variance of difference equal to

$$\frac{1}{w}(1 + \frac{k\gamma}{\mu}) = 702.73\ (1 + 5 \times 0.02964) \tag{7.81}$$

$$= 806.8746$$

(3) Those not occurring together in the same block. They fall into four sub-types.

(i) Those for which the quadrant difference d is + 1. Example 1 and 7. There are 24 such pairs.

(ii) Those for which the quadrant difference d is - 1. Example 1 and 13. There are 12 such pairs.

(iii) Those for which the quadrant difference d is +2. Example 1 and 9. There are 12 such pairs.

(iv) Those for which the quadrant difference d is -2. Example 5 and 6. There are 3 such pairs.

The variance for each of these four sub-types is given by the common expression

$$(7.82)\qquad \frac{1}{w}\left[\, 1 + \frac{2k\gamma + d\gamma^2}{\mu} \,\right] = 702.73 \left[\, 1 + 10 \times 0.02964 + 0.02150d \,\right]$$

where d takes the values + 1, -1, +2, -2 respectively.

It will be convenient and perhaps sufficient for all practical purposes if we calculate a common variance for these four sub-types by using the weighted mean of d in the above expression, namely,

$$(7.83)\qquad \bar{d} = \frac{24(+1) + 12(-1) + 12(+2) + 3(-2)}{51} = \frac{10}{17}$$

The approximate variance for comparisons between treatments not occurring together in the same block will therefore be

$$(7.84)\qquad 702.73 \left[\, 1 + 10 \times 0.02964 + \frac{10}{17} \times 0.02150 \,\right] = 919.9066$$

The mean variance of all the 105 differences between pairs of treatment effects is the weighted average of (7.80), (7.81) and (7.84), the weights being 6, 48 and 51 respectively. This mean variance is

$$(7.85)\qquad \frac{1}{w}\left[\, 1 + \frac{50\gamma + 2\gamma^2}{7\mu} \,\right] = 855.8247$$

9. If the experiment is analyzed as consisting of two complete randomized blocks, the error variance (and, in this case of two replications, the mean variance for all pairs of treatment differences) will be

$$(7.90)\qquad \frac{(v - k)\frac{1}{w'} + v(k - 1)\frac{1}{w}}{k(v - 1)} = \frac{1}{w}\left[\, 1 + \frac{2\gamma}{7(1 - \gamma)} \,\right] = 1232.9096$$

This variance may be calculated directly from the following table of analysis of variance:

Source of Variation	d.f.	S.S.	M.S.
Replication	1	8101.6	
Treatments	14	15914.2	
Error	14	17260.9	1232.9
Total	29	41276.7	

10. The efficiency of the incomplete block design is given by the ratio

$$\frac{1232.9}{855.8} = 1.44$$

Hence, in this example, the incomplete block design is 44 percent more efficient than the complete block design if the inter-block information is recovered from the former design.

11. If it is desired to calculate the lowest significant differences among the intra-block estimates given in the third column of Table 4 we have only to substitute $\gamma = 1$ in the formulae for variances given in paragraph 8 above, remembering that $\mu = 25 - \gamma^2$ in those formulae and have to be replaced by 24.

The mean variance of all the 105 differences between pairs of intra-block estimates of treatment effects follows, by substituting $\gamma = 1$, $\mu = 24$ in (7.85):

$$\frac{1}{w} \left[1 + \frac{50 + 2}{7 \times 24} \right] = 702.73 \times \frac{55}{42}$$

$$= 920.2417$$

The efficiency of the incomplete block design without recovery of inter-block information is given by the ratio

$$\frac{1232.9}{920.2} = 1.34$$

Hence, in this particular example, even if we had not gone through all the trouble of recovering inter-block information, the efficiency of the incomplete block design would be 34 percent more than the corresponding complete block design. In fact, the recovery of inter-block information brought only 10 percent additional efficiency to the design.

VIII. List of Designs

Let us list all the designs for which $k \leq 10$.

No.	u	r	λ	p	q	k	v	
1	2	2	2	1	0	2	4	S.S.L.
2	2	2	2	2	0	4	8	E.S.L.
3	2	2	2	3	0	6	12	E.S.L.
4	2	2	2	4	0	8	16	E.S.L.
5	2	2	2	5	0	10	20	E.S.L.
6	3	2	1	1	0	2	6	S.R.L.
7	3	2	1	1	1	3	9	S.S.L.
8	3	2	1	1	2	4	12	
9	3	2	1	1	3	5	15	
10	3	2	1	1	4	6	18	
11	3	2	1	1	5	7	21	
12	3	2	1	1	6	8	24	
13	3	2	1	1	7	9	27	
14	3	2	1	1	8	10	30	
15	3	2	1	2	0	4	12	E.R.L.
16	3	2	1	2	1	5	15	
17	3	2	1	2	2	6	18	E.S.L.

VIII. List of Designs (continued)

No.	u	r	λ	p	q	k	v	
18	3	2	1	2	3	7	21	
19	3	2	1	2	4	8	24	
20	3	2	1	2	5	9	27	
21	3	2	1	2	6	10	30	
22	3	2	1	3	0	6	18	E.R.L.
23	3	2	1	3	1	7	21	
24	3	2	1	3	2	8	24	
25	3	2	1	3	3	9	27	E.S.L.
26	3	2	1	3	4	10	30	
27	3	2	1	4	0	8	24	E.R.L.
28	3	2	1	4	1	9	27	
29	3	2	1	4	2	10	30	
30	3	2	1	5	0	10	30	E.R.L.
31	4	3	2	1	0	3	12	S.R.L.
32	4	3	2	1	1	4	16	S.S.L.
33	4	3	2	1	2	5	20	
34	4	3	2	1	3	6	24	
35	4	3	2	1	4	7	28	
36	4	3	2	1	5	8	32	
37	4	3	2	1	6	9	36	
38	4	3	2	1	7	10	40	
39	4	3	2	2	0	6	24	E.R.L.

VIII. List of Designs (continued)

No.	u	r	λ	p	q	k	v	
40	4	3	2	2	1	7	28	
41	4	3	2	2	2	8	32	E.S.L.
42	4	3	2	2	3	9	36	
43	4	3	2	2	4	10	40	
44	4	3	2	3	0	9	36	E.R.L.
45	4	3	2	3	1	10	40	
46	5	4	3	1	0	4	20	S.R.L.
47	5	4	3	1	1	5	25	S.S.L.
48	5	4	3	1	2	6	30	
49	5	4	3	1	3	7	35	
50	5	4	3	1	4	8	40	
51	5	4	3	1	5	9	45	
52	5	4	3	1	6	10	50	
53	5	4	3	2	0	8	40	E.R.L.
54	5	4	3	2	1	9	45	
55	5	4	3	2	2	10	50	E.S.L.
56	6	5	4	1	0	5	30	S.R.L.
57	6	5	4	1	1	6	36	S.S.L.
58	6	5	4	1	2	7	42	
59	6	5	4	1	3	8	48	
60	6	5	4	1	4	9	54	
61	6	5	4	1	5	10	60	
62	6	5	4	2	0	10	60	E.R.L.

VIII. List of Designs (continued)

No.	u	r	λ	p	q	k	v	
63	7	6	5	1	0	6	42	S.R.L.
64	7	6	5	1	1	7	49	S.S.L.
65	7	6	5	1	2	8	56	
66	7	6	5	1	3	9	63	
67	7	6	5	1	4	10	70	
68	8	7	6	1	0	7	56	S.R.L.
69	8	7	6	1	1	8	64	S.S.L.
70	8	7	6	1	2	9	72	
71	8	7	6	1	3	10	80	
72	9	8	7	1	0	8	72	S.R.L.
73	9	8	7	1	1	9	81	S.S.L.
74	9	8	7	1	2	10	90	
75	10	9	8	1	0	9	90	S.R.L.
76	10	9	8	1	1	10	100	S.S.L.
77	11	10	9	1	0	10	110	S.R.L.
78	7	3	1	1	0	3	21	
79	7	3	1	0	1	4	28	
80	7	3	1	2	0	6	42	
81	7	3	1	0	2	8	56	
82	7	3	1	2	1	10	70	
83	13	4	1	1	0	4	52	
84	13	4	1	0	1	9	117	

VIII. List of Designs (continued)

No.	u	r	λ	p	g	k	v
85	13	4	1	2	0	8	104
86	21	5	1	1	0	5	105
87	21	5	1	2	0	10	210
88	31	6	1	1	0	6	186
89	57	8	1	1	0	8	456
90	73	9	1	1	0	9	657
91	91	10	1	1	0	10	910
92	11	5	2	1	0	5	55
93	11	5	2	0	1	6	66
94	11	5	2	2	0	10	110
95	16	6	2	1	0	6	96
96	16	6	2	0	1	10	160
97	37	9	2	1	0	9	333
98	15	7	3	1	0	7	105
99	15	7	3	0	1	8	120
100	25	9	3	1	0	9	225
101	31	10	3	1	0	10	310
102	19	9	4	1	0	9	171
103	19	9	4	0	1	10	190

S.S.L. = Simple Square Lattice [$v = u^2$, $k = u$, $r = 2$, $b = 2u$]

S.R.L. = Simple Rectangular Lattice [$v = u(u-1)$, $k = (u-1)$, $r = 2$, $b=2u$]

E.S.L. = Extended Square Lattice [$v = u^2p$, $k = up$, $r = 2$, $b = 2u$]

E.R.L. = Extended Rectangular Lattice [$v = u(u-1)p$, $k = 9u-1)p$, $r=2, b=2u$]

We have 103 designs for $k \leq 10$ out of which 9 each are simple square lattices and simple rectangular lattices. If arranged according to magnitude of v and k, we have the following scheme

v =	4	6	8	9	12	15	16	18	20	21	24	25	27	28	30	32	35	36	40
k =	2	2	4	3	3	5	4	6	4	3	6	5	9	4	5	8	7	6	8
					4	5	8	6	5	7	6		9	7	6	8		9	8
					4			6	10	7	8		9	7	10			9	10
					6					7	8		9		10			9	10
											8				10				10
											8				10				
															10				

v =	42	45	48	49	50	52	54	55	56	60	63	64	66	70	72	80	81
k =	6	9	8	7	10	4	9	5	7	10	9	8	6	10	8	10	9
	6	9			10				8	10				10	9		
	7								8								

v =	90	96	100	104	105	110	117	120	160	171	186	190	210	225	310	333
k =	9	6	10	8	5	10	9	8	10	9	6	10	10	9	10	9
	10				7	10										

v =	456	657	910
k =	8	9	10

2. The simple and extended square lattices and the simple and extended rectangular lattices are p.b.i.b. designs with two, three, four and five associate classes respectively. The number of different variances to be calculated for treatment differences in these cases is 2, 3, 4 and 5 respectively.

Out of the remaining designs listed above we have found that those designs derived from the symmetrical balanced incomplete block design

$$u = s^2 + s + 1$$

$$r = s + 1$$

$$\lambda = 1$$

and for which $p > 0$ and $q = 0$, so that

$$k = p(s + 1)$$

$$v = p(s + 1)(s^2 + s + 1)$$

are p.b.i.b. designs with four associate classes.

The parameters are:

$$v = p(s + 1)(s^2 + s + 1) \qquad r = 2$$

$$k = p(s + 1) \qquad b = 2(s^2 + s + 1)$$

$$\lambda_1 = 2 \qquad \lambda_2 = 1 \qquad \lambda_3 = 0 \qquad \lambda_4 = 0$$

$$n_1 = (p-1) \qquad n_2 = 2ps \qquad n_3 = 2ps^2 \qquad n_4 = ps^3$$

$$p'_{jk} = \begin{pmatrix} (p-2) & 0 & 0 & 0 \\ 0 & 2ps & 0 & 0 \\ 0 & 0 & 2ps^2 & 0 \\ 0 & 0 & 0 & ps^3 \end{pmatrix}$$

$$p_{jk}^2 = \begin{pmatrix} 0 & (p-1) & 0 & 0 \\ (p-1) & p(s-1) & ps & 0 \\ 0 & ps & ps(s-1) & ps^2 \\ 0 & 0 & ps^2 & ps^2(s-1) \end{pmatrix}$$

$$p_{jk}^3 = \begin{pmatrix} 0 & 0 & (p-1) & 0 \\ 0 & p & p(s-1) & ps \\ (p-1) & p(s-1) & ps & 2ps(s-1) \\ 0 & ps & 2ps(s-1) & ps(s-1)^2 \end{pmatrix}$$

$$p_{jk}^4 = \begin{pmatrix} 0 & 0 & 0 & (p-1) \\ 0 & 0 & 2p & 2p(s-1) \\ 0 & 2p & 2p(s-1) & 2p(s-1)^2 \\ (p-1) & 2p(s-1) & 2p(s-1)^2 & p(s-1)(s^2-s+1) \end{pmatrix}$$

If $p = 1$, the design becomes a p.b.i.b. design with three associate classes. The parameters are:

$v = (s+1)(s^2+s+1)$ $\qquad$ $r = 2$

$k = (s + 1)$ $\qquad$ $b = 2(s^2+s+1)$

$\lambda_1 = 1$ $\qquad$ $\lambda_2 = 0$ $\qquad$ $\lambda_3 = 0$

$n_1 = 2s$ $\qquad$ $n_2 = 2s^2$ $\qquad$ $n_4 = s^3$

$$p_{jk}^{1} = \begin{pmatrix} (s-1) & s & 0 \\ s & s(s-1) & s^2 \\ 0 & s^2 & s^2(s-1) \end{pmatrix} \quad p_{jk}^{2} = \begin{pmatrix} 1 & (s-1) & s \\ (s-1) & s & 2s(s-1) \\ s & 2s(s-1) & s(s-1)^2 \end{pmatrix}$$

$$p_{jk}^{3} = \begin{pmatrix} 0 & s^2 & 2(s-1) \\ 2 & 4(s-1) & 2(s-1)^2 \\ 2(s-1) & 2(s-1)^2 & (s-1)(s^2-s+1) \end{pmatrix}$$

3. Among the remaining designs it is likely that some are p.b.i.b. designs. For instance, design No. 79 is a p.b.i.b. design with the following parameters.

$$v = 28 \qquad k = 4 \qquad r = 2 \qquad b = 14$$

$$\lambda_1 = 1 \qquad \lambda_2 = 0 \qquad \lambda_3 = 0 \qquad \lambda_4 = 0$$

$$n_1 = 6 \qquad n_2 = 3 \qquad n_3 = 12 \qquad n_4 = 6$$

$$p_{jk}^{1} = \begin{pmatrix} 2 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 2 & 2 & 4 & 4 \\ 0 & 0 & 4 & 2 \end{pmatrix} \qquad p_{jk}^{2} = \begin{pmatrix} 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \\ 4 & 0 & 4 & 4 \\ 0 & 2 & 4 & 0 \end{pmatrix}$$

$$p_{jk}^{3} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 6 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix} \qquad p_{jk}^{4} = \begin{pmatrix} 0 & 0 & 4 & 2 \\ 0 & 1 & 2 & 0 \\ 4 & 2 & 4 & 2 \\ 2 & 0 & 2 & 1 \end{pmatrix}$$

It follows from this that design No. 81 is a p.b.i.b. design with five associate classes.

4. Designs with both p and q unequal and different from zero will not be p.b.i.b. designs.

References

Bose, R. C. and Nair, K. R. Partially balanced incomplete block designs. Sankhya 4, 337-372, 1939.

Bose, R. C. Partially balanced incomplete block designs with two associate classes involving only two replications. Calcutta Statistical Assoc. Bull. 3, 120-125, 1951.

Nair, K. R. and Rao, C. R. A note on partially balanced incomplete block designs. Science and Culture 7, 568-569, 1942.

Nair, K. R. The recovery of inter-block information in incomplete block designs. Sankhya 6, 383-390, 1944.

Nair, K. R. Partially balanced incomplete block designs involving only two replications. Calcutta Statistical Assoc. Bull. 3, 83-86, 1950.

Nair, K. R. Some two-replicate partially balanced designs. Ibid., 3, 174-176, 1951.

Robinson, H. F., Rigney, J. A. and Harvey, P. H. Investigations in plot technique with peanuts. North Carolina State College; Agricultural Experiment Station, Technical Bulletin No. 86, 1948.

Yates, F. Incomplete randomized blocks. Annals of Eugenics, 7, 121-140, 1936.