

TWO-STAGE PROCEDURES FOR ESTIMATING THE DIFFERENCE BETWEEN MEANS*

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Mimeograph Series No. 80
August, 1953

* Work supported by the U. S. Air Force under Contract AF 18(600)-83.

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Introduction and summary. Given two populations P_i ($i = 1, 2$) with unknown

means θ_i and variances σ_i^2 , we wish to estimate the difference $\theta_1 - \theta_2$. Let

$t_i(n)$ be the mean $(X_{i,1} + X_{i,2} + \dots + X_{i,n})/n$ of a sample from P_i . Then

$t_1(n_1) - t_2(n_2)$ is an unbiased estimate of $\theta_1 - \theta_2$, with variance $\sum_i (\sigma_i^2/n_i)$.

Assuming the cost of sampling to be a known linear function of the number of observations, the cost of taking n_1 observations from P_1 and n_2 from P_2 is

$a_1 n_1 + a_2 n_2 + a_3$. If there is a prescribed upper bound A_0 to the cost of

sampling, n_1 and n_2 are subject to the restriction

$$(0.1) \quad a_1 n_1 + a_2 n_2 \leq A = A_0 - a_3.$$

The quantity $\sum_i (\sigma_i^2/n_i)$, equal to the variance of $t_1(n_1) - t_2(n_2)$ for

integer values of the n_i , is minimized for continuous $n_i > 0$ subject to

(0.1) by taking $n_i = n_i^0$, where

$$(0.2) \quad n_i^0 = (A/a_i) a_i^{1/2} \sigma_i / \sum_j a_j^{1/2} \sigma_j ;$$

the minimum value being equal to

$$(0.3) \quad V^0(A) = \sum_i (\sigma_i^2/n_i^0) = A^{-1} (\sum_i a_i^{1/2} \sigma_i)^2.$$

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When the ratio σ_2/σ_1 on which the optimum values (0.2) depend is not known, we can use a two-stage procedure for estimating $\theta_1 - \theta_2$, first taking a sample of $m_1 + m_2$ observations, m_1 from P_1 , and then using estimates of σ_1 obtained from this preliminary sample to distribute the remaining observations between the P_i . We shall investigate the performance of this estimation procedure.

For previous work done on problems of this kind, reference may be made to Putter [1] and the literature cited in that paper. Putter considers the problem of estimating the mean of a population composed of a known number of normally distributed strata whose relative proportions are known. See also Robbins [2; p. 528].

In section 1, we assume the P_i to be normal and evaluate the variance of the two-stage estimate. In section 2, we show that as m_1, m_2 and $A \rightarrow \infty$ in a certain way, the ratio of this variance to the minimum variance $V^0(A)$ tends to unity, and we also prove the asymptotic result for more general populations.

1. Normal populations. When the P_i are known to be normal, we choose positive integers m_i such that $a_1 m_1 + a_2 m_2 < A$ and take m_i observations from P_i . Let

$$(1.1) \quad s_i^2(m_i) = \left\{ \sum_{j=1}^{m_i} x_{i,j}^2 - m_i t^2(m_i) \right\} / (m_i - 1) = \text{estimated variance of } P_i^*,$$

* Putter [1] uses $s_i^2(m_i)(m_i - 1)/(m_i - 2)$ instead of $s_i^2(m_i)$ because it minimizes an expression σ_D^2 which is the variance of the estimate obtained by ignoring the fact that the sample-sizes prescribed by the two-stage procedure are truncated short of the extreme limits possible under (0.1).

$$(1.2) \quad u(m_1, m_2) = a_1^{1/2} s_1(m_1) / \sum_i a_i^{1/2} s_i(m_i);$$

$$(1.3) \quad n_1^* = \begin{cases} m_1 & \text{if } Au(m_1, m_2)/a_1 < m_1, \\ (A - a_2 m_2)/a_1 & \text{if } (A - a_2 m_2)/a_1 < Au(m_1, m_2)/a_1, \\ Au(m_1, m_2)/a_1 & \text{if } m_1 \leq Au(m_1, m_2)/a_1 \leq (A - a_2 m_2)/a_1; \end{cases}$$

$$(1.4) \quad n_2^* = (A - a_1 n_1^*)/a_2;$$

and

$$(1.5) \quad \tilde{n}_1 = \lfloor n_1^* \rfloor,$$

where $\lfloor x \rfloor$ is the largest integer in x . Having computed \tilde{n}_1 , we take $(\tilde{n}_1 - m_1)$ more observations $(X_{1,j}, j = m_1 + 1, \dots, \tilde{n}_1)$ from P_1 , and estimate $\theta_1 - \theta_2$ by

$$(1.6) \quad t_1(\tilde{n}_1) - t_2(\tilde{n}_2) = \sum_{j=1}^{\tilde{n}_1} X_{1,j}/\tilde{n}_1 - \sum_{j=1}^{\tilde{n}_2} X_{2,j}/\tilde{n}_2.$$

Let

$$(1.7) \quad \tilde{V}(A) = \text{Var} \left\{ t_1(\tilde{n}_1) - t_2(\tilde{n}_2) \right\}.$$

Now,

$$(1.8) \quad t_i(\tilde{n}_i) = m_i t_i(m_i)/\tilde{n}_i + (\tilde{n}_i - m_i) t_i(\tilde{n}_i - m_i)/\tilde{n}_i,$$

$$\text{where } t_i(\tilde{n}_i - m_i) = \sum_{j=m_i+1}^{\tilde{n}_i} X_{i,j}/(\tilde{n}_i - m_i).$$

Since the \tilde{n}_i depend only on the $s_i(m_i)$, for fixed s_i the random variables $t_1(m_1)$, $t_2(m_2)$, $t_1(\tilde{n}_1 - m_1)$, $t_2(\tilde{n}_2 - m_2)$ are mutually independent, the

conditional distributions of $t_1(m_1)$ and $t_1(\tilde{n}_1 - m_1)$ being respectively $\mathcal{N}\{\theta_1, \sigma_1^2/m_1\}$ and $\mathcal{N}\{\theta_1, \sigma_1^2/(\tilde{n}_1 - m_1)\}$. Hence for fixed s_1 the conditional distribution of $t_1(\tilde{n}_1) - t_2(\tilde{n}_2)$ is $\mathcal{N}\{\theta_1 - \theta_2, \sum_1 (\sigma_1^2/\tilde{n}_1)\}$. Consequently,

$$(1.9) \quad E \left\{ t_1(\tilde{n}_1) - t_2(\tilde{n}_2) \right\} = \theta_1 - \theta_2, \text{ and } \tilde{V}(A) = E \left\{ \sum_1 (\sigma_1^2/\tilde{n}_1) \right\}.$$

Let

$$(1.10) \quad F(u) = \Pr \left\{ u(m_1, m_2) \leq u \right\}.$$

Then from (1.5), (1.9) and (1.10), we have

$$(1.11) \quad \begin{aligned} \tilde{V}(A) = & \left\{ \sigma_1^2 m_1^{-1} + \sigma_2^2 \lfloor (A - a_1 m_1)/a_2 \rfloor^{-1} \right\} F(a_1 m_1/A) \\ & + \left\{ \sigma_1^2 \lfloor (A - a_2 m_2)/a_1 \rfloor^{-1} + \sigma_2^2 m_2^{-1} \right\} \left\{ 1 - F(1 - a_2 m_2/A) \right\} \\ & + \int_{a_1 m_1/A}^{1 - a_2 m_2/A} \left\{ \sigma_1^2 \lfloor Au/a_1 \rfloor^{-1} + \sigma_2^2 \lfloor A(1 - u)/a_2 \rfloor^{-1} \right\} dF(u). \end{aligned}$$

In what follows we shall denote by $V^*(A)$ the expression obtained by dropping the square brackets in (1.11).

Let

$$(1.12) \quad \begin{cases} \rho = \sigma_2/\sigma_1, & c = (a_2/a_1)^{1/2}, & b_1 = (A - a_1 m_1)/(a_2 m_1), \\ b_2 = a_1 m_2/(A - a_2 m_2), & r_1 = (m_1 - 1)/2, & \text{and } q = r_2/r_1. \end{cases}$$

Then from (0.3) we have

$$V^0(A) = A^{-1} a_1 \sigma_1^2 (1 + \rho c)^2,$$

and we can reduce the expression for $V^*(A)$ to

$$\begin{aligned}
 (1.13) \quad A \{ V^*(A) - V^0(A) \} &= a_1 \sigma_1^2 (\rho - cb_1)^2 b_1^{-1} F(a_1 m_1 / A) \\
 &+ a_1 \sigma_1^2 (\rho - cb_2)^2 b_2^{-1} \left\{ 1 - F(1 - a_2 m_2 / A) \right\} \\
 &+ \int_{a_1 m_1 / A}^{1 - a_2 m_2 / A} a_1 \sigma_1^2 \left\{ (1 - u) u^{-1} + \rho^2 c^2 (1 - u)^{-1} u \right. \\
 &\left. - 2\rho c \right\} dF(u).
 \end{aligned}$$

Finally, making the substitution

$$\begin{aligned}
 (1.14) \quad w &= \rho^2 c^2 u^2 / \left\{ \rho^2 c^2 u^2 + q(1 - u)^2 \right\} = (m_1 - 1) (s_1^2 / \sigma_1^2) / \Sigma (m_1 - 1) (s_i^2 / \sigma_i^2) \\
 &= x_{m_1 - 1}^2 / (x_{m_1 - 1}^2 + x_{m_2 - 1}^2),
 \end{aligned}$$

so that w has the density function

$$(1.15) \quad f(w) = \left\{ 1/B(r_1, r_2) \right\} w^{r_1 - 1} (1 - w)^{r_2 - 1}, \quad 0 \leq w \leq 1$$

we reduce (1.13) to

$$\begin{aligned}
 (1.16) \quad V^*(A)/V^0(A) &= 1 + (1 + \rho c)^{-2} \left\{ \rho c I_1 + (\rho - cb_1)^2 b_1^{-1} I_2 \right. \\
 &\left. + (\rho - cb_2)^2 b_2^{-1} I_3 \right\},
 \end{aligned}$$

where

$$(1.17) \quad \left\{ \begin{aligned}
 I_1 &= \int_{\beta_1}^{\beta_2} \left\{ (qw)^{1/2} (1 - w)^{-1/2} + (qw)^{-1/2} (1 - w)^{1/2} - 2 \right\} f(w) dw, \\
 I_2 &= \int_0^{\beta_1} f(w) dw, \quad I_3 = \int_{\beta_2}^1 f(w) dw, \quad \beta_i = \rho^2 / \left\{ \rho^2 + c^2 q b_i^2 \right\}.
 \end{aligned} \right.$$

Hence, V^*/V^0 can be computed by means of tables of the incomplete β -function. We have done this for $a_1 = a_2 = a$, $N = A/a = 30, 50$, $m_1 = m_2 = m = (0.2)N$, $(0.3)N$, $(0.4)N$ and various values of ρ . The results are given in the table.

For $a_1 = a_2$, the usual procedure for estimating $\theta_1 - \theta_2$ consists in taking $n_1 = n_2 = N/2$ and using the estimate $t_1(N/2) - t_2(N/2)$. The variance is

$$V' = 2(\sigma_1^2 + \sigma_2^2)/N.$$

For comparing V^* with V' , the values of V'/V^0 are given in the last row of the table.

TABLE
Comparison of V^* with V^0 and V' for normal populations.

	m/N \ ρ	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
V^*/V^0 for $N = 30$	0.2	1.064	1.062	1.058	1.054	1.049	1.044	1.039	1.034	1.030
	0.3	1.034	1.032	1.028	1.023	1.018	1.016	1.016	1.018	1.022
	0.4	1.017	1.014	1.012	1.014	1.025	1.039	1.056	1.075	1.094
V^*/V^0 for $N = 50$	0.2	1.032	1.031	1.031	1.029	1.027	1.025	1.022	1.019	1.017
	0.3	1.021	1.019	1.017	1.014	1.011	1.010	1.008	1.011	1.016
	0.4	1.013	1.009	1.007	1.010	1.021	1.036	1.055	1.074	1.094
V'/V^0		1.000	1.012	1.040	1.074	1.111	1.148	1.184	1.218	1.250

The two-stage procedure seems to effect considerable improvement over the usual one-stage procedure for values of ρ away from 1; and the performance seems to be best for m/N in the neighbourhood of $\sigma_1/(\sigma_1 + \sigma_2)$ if $\sigma_1 < \sigma_2$.

2. Asymptotic efficiency. The idea of substituting $s_i(m_i)$ for σ_i in (0.2) is based on the belief that as $m_i \rightarrow \infty$, the ratio \hat{n}_1/\hat{n}_2 will approach the optimum value n_1^0/n_2^0 and \tilde{V}/V^0 will approach unity. We prove that this is the case when the populations are normal, and then we shall prove a similar result which is true also for other populations.

THEOREM 1. Let P_i be normal, and consider $V^*(A)/V^0(A)$ as given by (1.16). Let a_1, a_2 , and ρ remain fixed while m_1, m_2 and A become infinite in such a way that

$$(2.1) \quad \begin{aligned} &0 < h \leq m_1/m_2 \leq h' < \infty, \text{ where } h, h' \text{ are fixed,} \\ &\text{and } m_i/A \rightarrow 0, \quad i = 1, 2. \end{aligned}$$

Then

$$(2.2) \quad V^*(A)/V^0(A) \rightarrow 1.$$

PROOF. In (1.16),

$$\begin{aligned} I_1 &\leq q^{1/2} \beta(r_1+1/2, r_2-1/2) / \beta(r_1, r_2) + q^{-1/2} \beta(r_1-1/2, r_2+1/2) / \beta(r_1, r_2) - 2 \\ &= \left\{ q^{1/2} \Gamma(r_1+1/2) \Gamma(r_2-1/2) + q^{-1/2} \Gamma(r_1-1/2) \Gamma(r_2+1/2) \right\} / \left\{ \Gamma(r_1) \Gamma(r_2) \right\} \end{aligned}$$

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which converges to zero, since

$$(2.3) \quad x^h \Gamma(x-h) / \Gamma(x) \rightarrow 1 \text{ as } x \rightarrow \infty.$$

$$\begin{aligned} I_2 &\leq \Pr \left\{ w < \rho^2 / (\rho^2 + c^2 q b_1^2) \right\} \leq \Pr \left\{ w^{-1} > c^2 q b_1^2 / \rho^2 \right\} \\ &\leq E \left\{ w^{-1} \right\} \rho^2 / (c^2 q b_1^2), \end{aligned}$$

and

$$I_3 \leq \Pr \left\{ w > \rho^2 / (\rho^2 + c^2 q b_2^2) \right\} \leq \Pr \left\{ (1 - w)^{-1} > \rho^2 / (c^2 q b_2^2) \right\} \\ \leq E \left\{ (1 - w)^{-1} \right\} c^2 q b_2^2 / \rho^2.$$

Now,

$$E \left\{ w^{-1} \right\} = \Gamma(r_1 - 1) \Gamma(r_1 + r_2) / \left\{ \Gamma(r_1) \Gamma(r_1 + r_2 - 1) \right\},$$

and

$$E \left\{ (1 - w)^{-1} \right\} = \Gamma(r_2 - 1) \Gamma(r_1 + r_2) / \left\{ \Gamma(r_2) \Gamma(r_1 + r_2 - 1) \right\},$$

both of which remain bounded on account of (2.1) and (2.3). Since $b_1 \rightarrow \infty$ and $b_2 \rightarrow 0$, we have

$$(\rho - c b_1)^2 b_1^{-1} I_2 \rightarrow 0 \quad \text{and} \quad (\rho - c b_2)^2 b_2^{-1} I_3 \rightarrow 0.$$

Hence (2.2) is proved.

From the expression (1.11) for $\tilde{V}(A)$, it follows that

$$\tilde{V}(A) / V^0(A) \rightarrow 1.$$

Next, we remove the restriction that the P_i be normal, assuming only that they have finite variances σ_i^2 and that we know functions f_i such that the statistics

$$(2.4) \quad s_i(n) = f_i(X_{i,1}, \dots, X_{i,n}; n), \quad i = 1, 2,$$

satisfy conditions (I)-(III) below.

$$\text{Let } F_i(s; n) = \Pr \left\{ s_i(n) \leq s \right\}, \quad \text{and } C_i(\epsilon) = \left[\sigma_i - \epsilon, \sigma_i + \epsilon \right].$$

We assume:

(I) There exists an $\alpha > 1$ such that for every fixed $\epsilon > 0$,

$$n^\alpha \Pr \left\{ s_i(n)/C_i(\epsilon) \right\} \text{ is bounded for all } n > 0, i = 1, 2;$$

(II) There exists an $\epsilon > 0$ and $\delta < \min(\sigma_1, \sigma_2)$ such that $n \left\{ \int_{C_i(\epsilon)} s^k dF_1(s; n) \right.$

$$\left. - \sigma_1^k \right\} \text{ is bounded for all } n > 0 \text{ and } k = 2, -2;$$

(III) Either $t_i(n)$ and $s_i(n)$ from the same sample are a pair of mutually independent random variables, or P_i has a finite fourth moment.

We shall follow the two-stage procedure given by (1.2) - (1.6) with $s_i(m_i)$ as given by (2.4) instead of (1.1). Then we have

THEOREM 2. Let a_1, a_2 and ρ remain fixed while m_1, m_2 and A become infinite in such a way that

$$(2.5) \quad \left\{ (2.1) \text{ holds, and } A/m_1^{(1+\alpha)/2} \text{ is bounded.} \right.$$

Then

$$(2.6) \quad E \left\{ t_1(\tilde{n}_1) - t_2(\tilde{n}_2) \right\} \longrightarrow \theta_1 - \theta_2 ,$$

$$(2.7) \quad \tilde{V}(A)/V^0(A) \longrightarrow 1.$$

PROOF. We shall first prove the following statements:

(2.8) Assumption (II) also holds for $k = 1$ and -1 ;

$$(2.9) \quad Am_i \left\{ E(\tilde{n}_i^{-k}) - (n_i^0)^{-k} \right\} \text{ is bounded for } k = 1, 2;$$

$$(2.10) \quad A^2 m_i^2 E(\tilde{n}_i^{-4}) \longrightarrow 0.$$

To prove (2.8) for $k = -1$, we note that

$$\int_{C_i(\epsilon)} s^{-1} dF_i(s; n) \leq \left\{ \int_{C_i(\epsilon)} s^{-2} dF_i(s; n) \right\}^{1/2} = \sigma_i^{-1} \{ 1 + o(n^{-1}) \};$$

and since $s_i^{-1} = \sigma_i^{-1} \{ 1 + (s_i^2 \sigma_i^{-2} - 1) \}^{-1/2} \geq \sigma_i^{-1} \{ 1 - (1/2)(s_i^2 \sigma_i^{-2} - 1) \}$

for $s_i \in C_i(\epsilon)$, we have

$$\begin{aligned} \int_{C_i(\epsilon)} s^{-1} dF_i(s; n) &\geq \sigma_i^{-1} \left\{ (3/2) \int_{C_i(\epsilon)} dF_i(s; n) - (1/2) \int_{C_i(\epsilon)} s^2 \sigma_i^{-2} dF_i(s; n) \right\} \\ &= \sigma_i^{-1} \{ 1 + o(n^{-1}) \}, \text{ by (I) and (II).} \end{aligned}$$

Hence (2.8) is proved for $k = -1$, and in the same way for $k = 1$. We need only prove (2.9) and (2.10) for $i = 1$, since the proof for $i = 2$ is similar. Moreover, from the definitions of \tilde{n}_i and n_i^* , it is evident that (2.9) and (2.10) hold if and only if they hold with \tilde{n}_i replaced by n_i^* .

Let $m = (m_1, m_2)$ and set

$$(2.11) \quad v_m = s_2(m_2)/s_1(m_1),$$

and

$$(2.12) \quad v_m^* = \begin{cases} N_m & \text{if } v_m > N_m = (A - a_1 m_1)/c a_1 m_1, \\ \delta_m & \text{if } v_m < \delta_m = a_2 m_2 / (cA - c a_2 m_2), \\ v_m & \text{if } \delta_m \leq v_m \leq N_m. \end{cases}$$

Then

$$(2.13) \quad \delta_m = o(m_1 A^{-1}) \longrightarrow 0 \text{ and } N_m = o(m_1^{-1} A) \longrightarrow \infty ;$$

and

$$(2.14) \quad (n^*)^{-1} = a_1(1 + cv_m^*)/A.$$

Consequently, to prove (2.9), we need only show that

$$(2.15) \quad m_1 \left\{ E(v_m^*)^k - \rho^k \right\} \text{ is bounded for } k = 1, 2.$$

Let us choose an ϵ to satisfy (II); we can, by (2.13), choose m_1, m_2, A large enough so that

$$\delta_m \leq (\sigma_2 - \epsilon)/(\sigma_1 + \epsilon) \quad \text{and} \quad N_m \geq (\sigma_2 + \epsilon)/(\sigma_1 - \epsilon),$$

and hence such that

$$\left\{ s_i(m_i) \in C_i(\epsilon), \quad i = 1, 2 \right\} \implies \delta_m \leq v_m \leq N_m.$$

Under these circumstances, we have from (2.12)

$$(2.16) \quad 0 \leq E(v_m^*)^k - \int_{C_1(\epsilon)} \int_{C_2(\epsilon)} s_1^{-k} s_2^k dF_2(s_2; m_2) dF_1(s_1; m_1) \leq N_m^k p(m_1, m_2),$$

$$\text{where } p(m_1, m_2) = 1 - \prod_{i=1,2} \Pr \left\{ s_i(m_i) \in C_i(\epsilon) \right\} = O(m^{-\alpha}).$$

By (II), the second term in the middle in (2.16) is $\rho^k + O(m^{-1})$, and by (2.13) the last term is $O(A^k m^{-k} m^{-\alpha}) = O(m^{-1})$ for $k = 1, 2$. Thus, we have (2.15) and hence (2.9). We can prove (2.10) similarly.

Now, let

$$(2.17) \quad \begin{cases} T_1 = m_1 t_1(m_1)/\tilde{n}_1 - m_2 t_2(m_2)/\tilde{n}_2, \\ T_2 = (\tilde{n}_1 - m_1) t_1(\tilde{n}_1 - m_1)/\tilde{n}_1 - (\tilde{n}_2 - m_2) t_2(\tilde{n}_2 - m_2)/\tilde{n}_2 \end{cases}$$

where $(\tilde{n}_i - m_i)t_i(\tilde{n}_i - m_i) = \sum_{j=m_i+1}^{\tilde{n}_i} X_{i,j}$.

Then

$$(2.18) \quad E \left\{ t_1(\tilde{n}_1) - t_2(\tilde{n}_2) \right\} = E \left\{ T_1 \right\} + E_{\tilde{n}_1} E \left\{ T_2 \mid \tilde{n}_1 \right\}.$$

Since \tilde{n}_i depends only on $\{X_{i,j}, j = 1, 2, \dots, m_i\}$ and $t_i(\tilde{n}_i - m_i)$, for fixed \tilde{n}_1 , only on $\{X_{i,j}, j = m_i+1, \dots, \tilde{n}_i\}$, we have

$$(2.19) \quad \Pr \left\{ t_i(\tilde{n}_i - m_i) \leq x_i, i = 1, 2 \mid \tilde{n}_i = n_i \right\} = \prod_{i=1,2} \Pr \left\{ t_i(n_i - m_i) \leq x_i \right\}.$$

Therefore,

$$E \left\{ T_2 \mid \tilde{n}_1 \right\} = (\tilde{n}_1 - m_1)\theta_1/\tilde{n}_1 - (\tilde{n}_2 - m_2)\theta_2/\tilde{n}_2;$$

so that

$$E_{\tilde{n}_1} E \left\{ T_2 \mid \tilde{n}_1 \right\} = \theta_1 - \theta_2 - m_1\theta_1 E(1/\tilde{n}_1) + m_2\theta_2 E(1/\tilde{n}_2)$$

$$\rightarrow \theta_1 - \theta_2 \text{ by (2.5) and (2.9).}$$

Moreover,

$$m_i E \left\{ t_i(m_i)/\tilde{n}_i \right\} \leq m_i \left\{ E t_i^2(m_i) E(1/\tilde{n}_i^2) \right\}^{1/2} = A^{-1} m_i O(1) \left\{ A^2 E(1/\tilde{n}_i^2) \right\}^{1/2}$$

$$\rightarrow 0 \text{ by (2.5) and (2.9).}$$

Therefore, $ET_1 \rightarrow 0$ and hence, we have (2.6).

Finally,

$$(2.20) \quad A\tilde{V}(A) = A \text{ Var}(T_1 + T_2) = A \text{ Var } T_1 + A \text{ Var } T_2 + 2A \text{ Cov } (T_1 \ T_2).$$

But

$$\begin{aligned}
 (2.21) \quad A \text{ Var } T_1 &= A \text{ Var } \left\{ m_1 t_1(m_1)/\tilde{n}_1 - m_2 t_2(m_2)/\tilde{n}_2 \right\} \\
 &\leq 2Am_1^2 \text{ Var } \left\{ t_1(m_1)/\tilde{n}_1 \right\} + 2Am_2^2 \text{ Var } \left\{ t_2(m_2)/\tilde{n}_2 \right\} \\
 &\leq 2 \sum_i Am_i^2 \text{ Var } \left\{ \left[t_i(m_i) - \theta_i \right] / \tilde{n}_i + \theta_i / \tilde{n}_i \right\} \\
 &\leq 4 \sum_i Am_i^2 E \left\{ \left[t_i(m_i) - \theta_i \right]^2 / \tilde{n}_i^2 \right\} \\
 &\quad + 4 \sum_i Am_i^2 \theta_i^2 \text{ Var } (1/\tilde{n}_i).
 \end{aligned}$$

From (2.5) and (2.9), we know that $Am_i^2 \text{ Var } (1/\tilde{n}_i) \rightarrow 0$. As for the other term on the right hand side in (2.21), if $t_i(m_i)$ and $s_i(m_i)$ are independent, we have

$$\begin{aligned}
 Am_i^2 E \left\{ \left[t_i(m_i) - \theta_i \right]^2 / \tilde{n}_i^2 \right\} &= Am_i^2 E \left\{ \left[t_i(m_i) - \theta_i \right]^2 \right\} E(1/\tilde{n}_i^2) \\
 &= Am_i^2 \sigma_i^2 E(1/\tilde{n}_i^2) \\
 &\rightarrow 0 \text{ by (2.9)}.
 \end{aligned}$$

If $t_i(m_i)$ and $s_i(m_i)$ are not independent, we still have

$$\begin{aligned}
 Am_i^2 E \left\{ \left[t_i(m_i) - \theta_i \right]^2 / \tilde{n}_i^2 \right\} &\leq Am_i^2 \left\{ E \left[t_i(m_i) - \theta_i \right]^4 E(1/\tilde{n}_i^4) \right\}^{1/2} \\
 &= \left\{ m_i^2 E \left[t_i(m_i) - \theta_i \right]^4 A^2 m_i^2 E(1/\tilde{n}_i^4) \right\}^{1/2} \\
 &\rightarrow 0 \text{ by (III) and (2.10)}.
 \end{aligned}$$

Hence, $A \text{ Var}(T_1) \rightarrow 0$.

We shall see below that $A \text{ Var}(T_2)$ is bounded, so that

$$A \text{ Cov } (T_1, T_2) \leq \left\{ A \text{ Var } (T_1) A \text{ Var } (T_2) \right\}^{1/2} \rightarrow 0.$$

Therefore,

$$\begin{aligned} (2.22) \quad \lim \tilde{A} \tilde{V}(A) &= \lim A \text{ Var } T_2 = \lim A E_{\tilde{n}_1} \text{ Var}(T_2 | \tilde{n}_1) + \lim A \text{ Var } E(T_2 | \tilde{n}_1) \\ &= \lim A E_{\tilde{n}_1} \text{ Var} \left\{ (\tilde{n}_1 - m_1)t_1(\tilde{n}_1 - m_1)/\tilde{n}_1 \right. \\ &\quad \left. - (\tilde{n}_2 - m_2)t_2(\tilde{n}_2 - m_2)/\tilde{n}_2 \right\} \\ &\quad + \lim A \text{ Var} \left\{ (\tilde{n}_1 - m_1)\theta_1/\tilde{n}_1 - (\tilde{n}_2 - m_2)\theta_2/\tilde{n}_2 \right\} \\ &= \lim A E_{\tilde{n}_1} \left\{ \sum_i (\tilde{n}_i - m_i)\sigma_i^2/\tilde{n}_i^2 \right\} + \lim A \text{ Var} \left\{ m_2\theta_2/\tilde{n}_2 - m_1\theta_1/\tilde{n}_1 \right\} \\ &= \lim \sum_i \sigma_i^2 A E(1/\tilde{n}_i) - \lim \sum_i A m_i \sigma_i^2 E(1/\tilde{n}_i^2) + \lim A \text{ Var} \left\{ m_2\theta_2/\tilde{n}_2 - m_1\theta_1/\tilde{n}_1 \right\}, \end{aligned}$$

The last two terms are zero on account of (2.5) and (2.9); and from (2.9), we see that the first term on the right hand side of (2.22) is the required limit in (2.7).

If we use sample sizes $n'_i = (A/a_i)a_i^{1/2}/\sum_j a_j^{1/2}$, which by (0.2) is what we would be led to do if we thought that $\sigma_1 = \sigma_2$, the variance of the estimate of $\theta_1 - \theta_2$ would be

$$v' = v^0 \left\{ 1 + c(1 - \rho)^2/(1 + \rho c)^2 \right\} > v^0 \text{ for } \rho > 1.$$

Hence, asymptotically, the two-stage procedure is more efficient than this one stage procedure if $\rho > 1$.

EXAMPLES. (1) If the P_i are normal, the conditions of Theorem 2 are satisfied

for every $\alpha > 0$, and hence in (2.5) A may increase as any power of m_i . We have seen in Theorem 1 that it is actually not necessary to restrict A to be of the order of a power of m_i .

(2) If the P_i are Poisson, $\sigma_i^2 = \theta_i$, and $s_i^2(n) = t_i(n)$. From the fact that $nt_i(n)$ has a Poisson distribution, it can be seen that the conditions of Theorem 2 are satisfied for every $\alpha > 0$.

(3) If the P_i are binomial, with $\Pr \{ X_i = 1 \} = \theta_i$ and $\Pr \{ X_i = 0 \} = 1 - \theta_i$, we have $\sigma_i^2 = \theta_i(1 - \theta_i)$ and $s_i^2(n) = t_i(n) \{ 1 - t_i(n) \}$. Using the fact that $nt_i(n)$ is a binomial variate, we can show that the conditions of the theorem are satisfied for every $\alpha > 0$.

(4) If we do not know the forms of P_i , we would use the estimate $s_i(m_i)$ of σ_i given by (1.1). If we know that P_i has a sufficient number, say 8, of moments finite, we can show that Theorem 2 is true for the procedure given by (1.1) - (1.6).

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