SEVERAL NEW PROBLEMS OF THE THEORY OF MARKOFF CHAINS

by

V. Romanovsky

Institute of Statistics
Mimeo. Series No. 102
May, 1954

Paris 1938

QUELQUES PROBLEMES NOUVEAUX DE LA THEORIE DES CHAINES DE MARKOFF

Par

V. Romanovsky

ACTUALITIES SCIENTIFIQUES ET INDUSTRIELLES

No. 737, pp. 43-66, 1938, Paris, France

Translated by Dan Teichrow under the supervision of Imogene Riddick, Instructor of Modern Languages at North Carolina State College of the University of North Carolina, Raleigh, N. C. at the request of the Department of Experimental Statistics, North Carolina State College. (Edited Jointly by the Translation Service and the Department of Experimental Statistics.)

May 17, 1954

Paris 1938

SEVERAL NEW PROBLEMS OF THE THEORY OF MARKOFF CHAINS

V. Romanovsky

The object of my article is to indicate several new directions which have been pursued in the theory of Markoff chains and to state several results which I have obtained in my research in these directions. I am going to consider correlated chains, cyclic chains, and several statistical problems which are related to probabilities in a chain.

I. Correlated Chains

1. So far as I know, no one has yet considered sequences of random variables which are correlated and subject, at the same time, to a law of probability in a chain. But the problems which arise in the case of such sequences are not less important than the other problems of probabilities in chains and they must be dealt with in order to complete the theory of the former.

To specify and simplify my article, I shall consider only the following case. Let

$$u_o, u_1, u_2, \ldots,$$

be an infinite sequence of random variables connected in a simple Markoff chain and associated with experiments number 0, 1, 2, ... respectively. We shall suppose that, in the experiment number h, the corresponding variable n_h can take only one of the different values $x_1, x_2, \ldots, x_n$ which remains the same for all the experiments and have the probabilities $p_{01}, p_{02}, \ldots, p_{on}$, $(\sum_\alpha p_{0\alpha} = 1)$ in the initial experiment, number 0, and the transition probabilities $p_{\alpha\beta}$ are defined by the equations

$$p_{\alpha\beta} = P(u_{h+1} = x_\beta \mid u_h = x_\alpha)$$

$$(\alpha, \beta = 1, 2, \ldots, u; \; h = 1, 2, \ldots),$$

where the symbols on the right hand side designate the probability that the variable u_{k+1} takes the value of x_β when it is known that u_h has taken the value x_α.

Thus the matrix $\Phi = \|\phi_{\alpha\beta}\|$ represents the law of the chain being considered, more briefly of the chain C_n which connects the variable u_h.

Designate ([1]) further by $p_{k/\beta}$ the probability $P(u_k = x_\beta)$ calculated under the hypothesis that the values of all the other variables $u_0, u_1, \ldots, u_{k-1}, u_{k+1}, u_{k+2}, \ldots$, are indeterminate and consider a second infinite sequence of chance variables $v_0, v_1, v_2, \ldots$, associated to the same experiments and connected to the variables u_h by a correlation which is defined by the equations

$$r_{k\mid\beta\gamma} \equiv P(u_k = x_\beta, \; v_k = y_r) = (p_{k|\beta})(\psi_{\beta r}) \tag{1}$$
$$(\beta = 1, 2, \ldots, n; \; \gamma = 1, 2, \ldots, m; \; k = 0, 1, 2, \ldots),$$

where $\psi_{\beta\gamma}$ are the conditional probabilities, the same for all the experiments of the equations $v_k = y_\gamma$, the condition being that $u_k = x_\beta$, and where $y_1, y_2, \ldots, y_m$ are the values which can be taken by each of the variables $v_0, v_1, v_2, \ldots$, in the experiments considered.

Let $\Psi = \|\psi_{\beta\gamma}\|$. The matrices Φ and Ψ define the law of correlation of the sequences (u_h) and (v_h) or of the chain C_{uv}. We shall study several of the most important properties of this correlated chain.

2. First of all we can propose the following problem: what is the nature of the chain C_v which connects the variables v_h?

For greater clarity we shall consider on the simplest and most important case where Φ is a primitive and indecomposable matrix. In this case its characteristic equation

$$\Phi(\lambda) = |\lambda E - \Phi| = 0, \tag{2}$$

([1]) I will use the notation of my article in Acta Mathematica, Vol. 66, which will be cited below as A.M.

has a $\psi\lambda_0 = 1$ as a simple root, the absolute values of all the other roots of (2) are < 1 and the probabilities $p_{k/\beta}$ have for $k \to \infty$ the determinate limits p_β which represent the limiting probabilities of the equation $u_k = x_\beta$(1).

Let

$$q_{k|\gamma} = P(v_k = y_\gamma),$$

then obviously

(3)
$$q_{k|\gamma} = \sum_\beta p_{k|\beta} \psi_{\beta\gamma}.$$

Likewise, if we designate by $\phi^{(k)}_{\alpha\beta}$ the probability of the equation $u_{h+k} = x_\beta$ when it is known only that $u_h = x_\alpha$ we shall have

(4)
$$\Psi^{(k)}_{\alpha\beta} \equiv P(v_{h+k} = y_\gamma \mid u_h = x_\alpha) = \sum_\beta \phi^{(k)}_{\alpha\beta} \psi_{\beta\gamma}.$$

Further, let $\pi^{(k)}_{\beta\gamma}$ be the transition probability of the equation $v_{k-1} = y_\beta$ to the equation $v_k = y_\gamma$. They satisfy the relation

(5)
$$q_{k|\gamma} = \sum_\beta q_{k-1|\beta} \pi^{(k)}_\beta,$$

and can be obtained by the following procedure.

We have equations

$$P_1 \equiv P(u_{k-1} = x_\alpha \mid v_{k-1} = y_\beta) = \frac{P_{k-1|\alpha} \psi_{\alpha\beta}}{\sum_\alpha P_{k-1|\alpha} \psi_{\alpha\beta}} = \frac{q_{k-1|\alpha} \psi_{\alpha\beta}}{q_{k-1|\beta}}$$

$$P_2 \equiv P(u_k = x_\sigma \mid u_{k-1} = x_\alpha) = \phi_{\alpha\sigma},$$

$$P_3 \equiv P(v_k = y_\gamma \mid u_k = x_\sigma) = \psi_{\sigma\gamma},$$

of which the first is obtained by Bayes Theorem.

---

(1) A.M., pp. 183-185.

It is clear that

$$\pi_{\beta\gamma}^{(k)} \equiv P(v_k = y_\gamma | v_{k-1} = y_\beta) = \sum_{\alpha,\sigma} P_1 P_2 P_3,$$

whence

$$\pi_{\beta\gamma}^{(k)} = \sum_{\alpha,\sigma} \frac{p_{k-1|\alpha} \psi_{\alpha\beta}}{q_{k-1|\beta}} \phi_{\alpha\sigma} \psi_{\sigma\gamma}. \tag{6}$$

The probabilities $\pi_{\beta\gamma}^{(k)}$ define the law of chain C_v. This chain is not homogeneous and multiple, because $\pi_{\beta\gamma}^{(k)}$ depends on k and the probability P_1 changes its value according to what is known about the values of $u_{k-2}, u_{k-3}, \ldots$ and, consequently of the sequence, $v_{k-2}, v_{k-3}, \ldots$ It becomes homogeneous and remains multiple if the chain C_u is stabilized because then ([1])

$$p_{k|\alpha} = p_\alpha \quad \text{therefore} \quad q_{k-1|\beta} \equiv q_\beta = \sum_\alpha p_\alpha \psi_{\alpha\beta},$$

for all k and the probabilities

$$\pi_{\beta\gamma}^{(k)} \equiv \pi_{\beta\gamma} = \sum_{\alpha,\sigma} \frac{p_\alpha}{q_\beta} \psi_{\alpha\beta} \phi_{\alpha\sigma} \psi_{\sigma\gamma}, \tag{7}$$

do not depend on k, but the information on the values of the $v_{k-2}, v_{k-3}, \ldots$ can be such that certain values of the u_{k-1} become impossible and the sum on the right hand side changes.

3. For $k \to \infty$ the correlation of the sequences (u_h) and (v_h) has for a limit a well determined limiting correlation. In fact we have in our case of an indecomposable and primitive matrix, Φ:

$$\lim p_{k|\beta} = p_\beta, \ \lim q_{k|\gamma} = q_\gamma = \sum_\beta p_\beta \psi_{\beta\gamma},$$

$$\lim r_{k|\beta\gamma} = r_{\beta\gamma} = p_\beta \psi_{\beta\gamma},$$

([1]) A.M., p. 221.

and the various moments of chain C_{uv} have for their limits the corresponding moments of this limiting correlation, which remains a correlated chain.

4. Let us pass to the important problem of the limiting law of the probability of the sums

$$\sum_{h=0}^{s-1} u_h \quad \text{and} \quad \sum_{h=0}^{s-1} v_h .$$

We have for $k \to \infty$:

$$Eu_k = \sum_\beta p_{k|\beta} x_\beta \to \sum_\beta p_\beta x_\beta = x_0,$$

$$E\, v_k = \sum_\gamma q_{k|\gamma}\, y_\gamma \to \sum_\gamma q_\gamma y_\gamma = y_0,$$

and we shall consider the sums

$$\sum u_h' = \sum_{h=0}^{s-1} (u_h - x_0) \quad \text{and} \quad \sum v_h' = \sum_{h=0}^{s-1} (v_h - y_0).$$

We can calculate the moments of the sums $\sum u_h'$ by the method which is explained in detail in A.M., and having the moments of the sum $\sum v_h'$, we notice that

(7)
$$\begin{aligned} q_{k|\gamma} &= \sum_\beta p_{k|\beta} \psi_{\beta\gamma} = \sum_{\alpha,\beta} p_{0\alpha} \phi_{\alpha\beta}^{(k)} \psi_{\beta\gamma} \\ &= \sum_\beta p_\beta \psi_{\beta\gamma} + \sum_{\alpha,\beta} p_{0\alpha} \Psi_{\alpha\beta}^{(k)} \psi_{\beta\gamma} \\ &= q_\gamma + \sum_\alpha p_{0\alpha}\, X_{\alpha\gamma}^{(k)}, \end{aligned}$$

where

$$\begin{aligned} X_{\alpha\gamma}^{(k)} &= \sum_\beta \Psi_{\alpha\beta}^{(k)} \psi_{\beta\gamma} = \sum_{g=1}^{n-1} \lambda_g^k \sum_\beta \psi_{\alpha\beta}^{(g)} \psi_{\beta\gamma} \\ &= \sum_g \lambda_g^k \chi_{\alpha\gamma}^{(g)}, \quad \chi_{\alpha\gamma}^{(g)} = \sum_\beta \psi_{\alpha\beta}^{(g)} \psi_{\beta\gamma} \quad , \end{aligned}$$

and $\Psi^{(g)}_{\alpha\beta}$ are the coefficients of the development of $\phi^{(k)}_{\alpha\beta}$ according to powers of $\lambda_0^k = 1, \lambda_1^k, \dots, \lambda_{n-1}^k$ of the roots of the equation $\Phi(\lambda) = 0$, supposed for the greatest simplicity, to be all distinct. We see by these relations that the moments of the quantities v_h' are obtainable by beginning with the moments corresponding to the u_h' and replacing $p_\beta, \Psi^{(k)}_{\alpha\beta}$ and $\psi^{(g)}_{\alpha\beta}$ by q_γ, $\chi^{(k)}_{\alpha\gamma}$ and $\chi^{(g)}_{\alpha\gamma}$ respectively.

We see further that

$$N^{(k_1 k_2 \dots k_r)}_{m_1 m_2 \dots m_r} = E(v'^{\,m_1}_{k_1} v'^{\,m_2}_{k_1+k_2} \cdots v'^{\,m_r}_{k_1+k_2+\dots+k_r})$$

$$= \sum_{\alpha,\beta_1,\dots,\beta_r} p_{0\alpha}\, \phi^{(k_1)}_{\alpha\beta_1} \psi_{\beta_1\gamma_1} t^{m_1}_{\gamma_1} \cdot \phi^{(k_2)}_{\beta_1\beta_2} \psi_{\beta_2\gamma_2} t^{m_2}_{\gamma_2} \cdots \phi^{(k_r)}_{\beta_{r-1}\beta_r} \psi_{\beta_r\gamma_r} t^{m_r}_{\gamma_r},$$

where $t = y - y_0$. For the effective calculation of these moments we can apply the procedures given in A.M. Their general form shows us at once that the structure of the moments $N^{(k_1 k_2 \dots k_r)}_{m_1 m_2 \dots m_r}$ and $N_{m_1 m_2 \dots m_r} = \sum_{k_1 \dots k_r} N^{(k_1 k_2 \dots k_r)}_{m_1 m_2 \dots m_r}$ is the same as the structure of the moments, $M^{(k_1 k_2 \dots k_r)}_{m_1 m_2 \dots m_r}$ and $M_{m_1 m_2 \dots m_r}$ studied in A.M. In particular their asymptotic properties are identical.

We find, for example

$$(9) \qquad E(\sum v_h')^2 \sim s\mu_{02} + 2s \sum_{g=1}^{n-1} \frac{\lambda_g}{1-\lambda_g} \mu_{02}(\lambda_g),$$

where

$$(10) \qquad \mu_{02} = \sum_{\beta,\gamma} p_\beta \psi_{\beta\gamma} t^2_\gamma ,$$

$$\mu_{02}(\lambda^g) = \sum_{\beta_1,\beta_2,\gamma_1,\gamma_2} p_{\beta_1} \psi^{(g)}_{\beta_1\beta_2} \psi_{\beta_1\gamma_1} \psi_{\beta_2\gamma_2} t_{\gamma_1} t_{\gamma_2}.$$

This value is entirely equivalent to that of the dispersion of $\sum u_h'$:

(11) $$E(\sum u_h')^2 \sim s\mu_{20} + 2s \sum_g \frac{1-\lambda_g}{\lambda_g} \mu_{20}(\lambda_g),$$

$$\mu_{20} = \sum_\beta p_\beta z_\beta^2, \quad \mu_{20}(\lambda_g) = \sum_{\beta,\gamma} p_\beta \psi_{\beta\gamma}^{(g)} z_\beta z_\gamma$$

$$(z = x - x_0).$$

The calculation of the moments of the products of integral and positive powers of the sums $\sum u_h'$ and $\sum v_h'$ is naturally more complicated, but does not present any difficulties in principal. We cite only the moment for the product $\sum u_h' \sum v_h'$:

(12) $$E(\sum u_h' \sum v_h') = s\mu_{11} + \sum_g \frac{1-\lambda_g^s}{1-\lambda_g} \mu_{11}(\lambda_g)$$

$$+ \sum_g v_1(\lambda_g) [\mu_{11}'(\lambda_g) + \mu_{11}''(\lambda_g)]$$

$$+ \sum_{g,h} v_2(\lambda_g, \lambda_h) [\mu_2(\lambda_g, \lambda_h) + \mu_2'(\lambda_g, \lambda_h)]$$

$$\sim s\mu_{11} + s \sum \frac{\lambda_g}{1-\lambda_g} [\mu_{11}'(\lambda_g) + \mu_{11}''(\lambda_g)] ,$$

where

$$\mu_{11}'(\lambda_g) = \sum p_{\beta_1} \psi_{\beta_1\beta_2}^{(g)} \psi_{\beta_2\gamma} z_{\beta_1} t_\gamma ,$$

$$\mu_{11}''(\lambda_g) = \sum p_{\beta}, \psi_{\beta_1\beta_2}^{(g)} \psi_{\beta_1\gamma} z_{\beta_2} t_\gamma ,$$

(13) $$\mu_2(\lambda_g, \lambda_h) = \sum p_{0\alpha} \psi_{\alpha\beta_1}^{(g)} \psi_{\beta_1\beta_2}^{(h)} \psi_{\beta_2\gamma} z_{\beta_1} t_\gamma ,$$

$$\mu_2'(\lambda_g, \lambda_h) = \sum p_{0\alpha} \psi_{\alpha\beta_1}^{(g)} \psi_{\beta_1\beta_2}^{(h)} \psi_{\beta,\gamma} z_{\beta_2} t_\gamma ,$$

$$v_1(\lambda_g) = \frac{s\lambda_g}{1-\lambda_g},$$

(13) continued

$$v_2(\lambda_g, \lambda_h) = \frac{\lambda_h(1-\lambda_g{}^s)}{(1-\lambda_h)(1-\lambda_g)} - \frac{\lambda_h(\lambda_h{}^s - \lambda_g{}^s)}{(1-\lambda_h)(\lambda_h-\lambda_g)},$$

the sums on the right hand sides being taken over all the values of the greek letters under the corresponding $\sum$ sign.

With the aid of the formulae written above we can easily calculate the correlation coefficient of the sums $\sum u'_h$ and $\sum v'_h$ according to the general relation

$$r = \frac{E(\sum u'_h \sum v'_h)}{E(\sum u'_h)^2 \; E(\sum v'_h)^2}.$$

Its final value, for $k \to \infty$, is also obtained immediately.

5. It is not difficult to write the generator of all the moments of the sums $\sum u'_h$ and $\sum v'_h$ it is

$$(14) \quad \phi_s = \sum p_{0\alpha} \psi_{\alpha\gamma} \phi_{\alpha\beta_1} \psi_{\beta_1\gamma_1} \cdots \phi_{\beta_{s-2}\beta_{s-1}} \psi_{\beta_{s-1}\gamma_{s-1}} e^{i\theta(z_\alpha + \sum z_{\beta h}) + i\tau(t_\gamma + \sum t_\gamma)}$$

and satisfies the recurrence relation

$$(15) \quad \phi_s = \psi_1 \phi_{s-1} + \psi_2 \phi_{s-2} + \cdots + \psi_{s-2} \phi_2 + \psi_{s-1} \phi_2' \phi + \psi'_s \phi, 0$$

where

$$(16) \quad \psi_h = \sum q_{\beta_{s-h}} \Psi^{(1)}_{\beta_{s-h}\beta_{s-h+1}} \psi_{\beta_{s-h+1}\gamma_{s-h+1}} \cdots \Psi^{(1)}_{\beta_{s-2}\beta_{s-1}} \psi_{\beta_{s-1}\gamma_{s-1}}$$

$$\cdot\, e^{i\theta(z_{\beta_{s-h}} + \cdots + z_{\beta_{s-1}}) + i\tau(t_{\gamma_{s-h}} + \cdots + t_{\gamma_{s-1}})}$$

$$\psi_s' = \sum p_{0\alpha}\psi_{\alpha\gamma}\Psi^{(1)}_{\alpha\beta_1}\psi_{\beta 1\,\gamma 1}\cdots\Psi^{(1)}_{\beta_{s-2}\beta_{s-1}}\psi_{\beta_{s-1}\gamma_{s-1}}$$

(16) cont'ed $\quad \cdot\, e^{i\theta\,(z_\alpha + \sum z_{\beta h}) + i\tau\,(t_\gamma + \sum t_{\gamma h})}$

$$\phi_1' = \sum p_{0\alpha}\psi_{\alpha\gamma}e^{i\theta z_\alpha + i\tau t\gamma}, \phi_0 = 1 \quad (i = \sqrt{-1}).$$

The relation (15) gives us

$$s\left[\frac{\phi\, s-1}{\phi\; s} - 1\right] = s(\Psi_1 - 1) + \frac{\phi\, s-2}{\phi\, s-1}\; s\,\Psi_2 + \cdots$$

and we can show [1] that

$$\lim_{S\to\infty} s\left[\frac{\phi\, s}{\phi\, s-1} - 1\right] = \lim_{S\to\infty}\log\phi_s = -\frac{1}{2}(\theta^2 + 2r\theta\tau + \tau^2),$$

if we set in (14)

$$z = \frac{x - x_0}{\sqrt{sB}},\; B = \mu_{20} + 2\sum_g \frac{\lambda_g}{1-\lambda_g}\mu_{20}(\lambda_g),$$

$$t = \frac{y - y_0}{\sqrt{sB_1}},\; B_1 = \mu_{02} + 2\sum_g \frac{\lambda_g}{1-\lambda_g}\mu_{02}(\lambda_g).$$

Therefore,

(17) $$\lim \phi_s = e^{-\frac{1}{2}(\theta^2 + 2\theta\tau r + \tau^2)}$$

and we see that the correlation of the sums $\sum u_h'$ and $\sum v_h'$ has for its limit the normal correlation; r is the limiting value of the correlation coefficient defined above.

---

[1] My proof is not yet rigorous and it is a little long, therefore, I have omitted it here.

6. We have considered only the simplest case of correlated chains which is characterized, in particular, by the fact that chain C_u is independent of chain C_v. But we can form chains where $P(u_{k+1} = x_\beta)$ depends not only on the value taken by u_k, but also on the value of v_k.

A still more general case is obtained when we suppose that the probabilities of the simultaneous equations $u_{k+1} = x_\alpha$, $v_{k+1} = y_\beta$ depend on the values taken by u_k and v_k simultaneously. I have calculated the fundamental formulae equivalent to formulae (1) - (17) which are shown above for the most general case of correlated chains, but I omit them here because of their complexity. They do not present any essentially new particulars.

II. Cyclic Chains

1. We call a cyclic chain of index k a chain in which the law by an identical permutation of lines and columns can be put in the cyclic form

$$(1) \qquad \Phi = \begin{pmatrix} 0 & L_{12} & 0 & \cdots & 0 \\ 0 & 0 & L_{23} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & L_{k-1,k} \\ L_{k1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $L_{12}, L_{23}, \ldots, L_{k1}$ are the non zero sub-matrices, all other sub-matrices $L_{\alpha\beta}$ being zero ($L_{11}, L_{22}, \ldots, L_{kk}$ should be square matrices). The matrix is said in this case to be a cyclic matrix of index k.

Let there be given a stochastic indecomposable matrix. Then in order to say that it is cyclic of index k, it is necessary and sufficient that we be able to put it in the form (1) (A.M., p.165). The simplest means of demonstrating this theorem consists in considering the cycles of the indecomposable matrix in question, that is, the sequence of non-zero elements,

$$\phi_{\alpha\beta_1},\ \phi_{\beta_1\beta_2},\ \dots ,\ \phi_{\beta_{s-1}\alpha}$$

It is evident that, for the matrix Φ presented in the form (1), all the cycles have their orders divisible by k. The order of a cycle is the number of elements which form the cycles. Inversely, if all the cycles of an indecomposable matrix have their orders divisible by k, we can state that it is cyclic of index k.

This last assertion is demonstrated in A.M. by the aid of the characteristic equation of the matrix considered. But we can give a direct and simple procedure for reducing a matrix to cyclic form. We shall describe this procedure because it is new and useful.

It is simplest to consider an example of the procedure in question, which will not diminish the generality.

Let us take for example, the following matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

where the non-zero elements have, for simplicity, been replaced by unity. To simplify the writing we shall write the cycles and merely indicate the indices of their elements. For example we shall write the cycle

$$a_{12}, a_{23}, a_{31}$$

in the form

$$12, 23, 31.$$

We can see without difficulty that the cycles of A are the following:

12, 23, 31	17, 73, 31
12, 23, 34, 47, 75, 51	17, 73, 34, 42, 25, 51
12, 23, 36, 67, 75, 51	17, 73, 36, 62, 25, 51
12, 25, 51	17, 75, 51
12, 25, 54, 47, 73, 31	17, 75, 54, 42, 23, 31
12, 25, 56, 67, 73, 31	17, 75, 56, 62, 23, 31

All these cycles have orders divisible by k = 3 and we shall separate their first indices into three groups

I: 1, 4, 6
II: 2, 7
III: 3, 5.

In group I, one finds the indices 1, 4, 6 - the first indices of the first and fourth pairs of indices in the table above; in group II one finds the first indices of the second and fifth pairs, and so on in the sequence. It is now evident how to proceed in the general case.

Notice that to exhaust all the indices 1, 2, 7, in the groups I, II, III, it is sufficient to apply the procedure described in the three first cycles. Therefore, we can diminish very considerably the labor of searching for the cyclic form of a matrix if we write at the same time as the cycle, the partition of the first indices in groups, the number of groups being defined by any means whatever.

Take now the permutation of lines and columns of the matrix defined by the substitution

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 6 & 2 & 7 & 3 & 5 \end{pmatrix}$$

Then the matrix A will be put in the desired cyclic form;

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Let $A_1, A_2, \dots, A_n$ be independent events which we can observe in an infinite sequence of experiments connected in a simple cyclic chain of index k and let the matrix $\mathfrak{Q}$ of the equation (1) be their law. Let also $\ell_1, \ell_2, \dots, \ell_k$ be the number of lines of the $L_{12}, L_{23}, \dots, L_{k1}$ of $\mathfrak{Q}$. These conditions being fulfilled, one can divide the events A_α into k groups,

$$\begin{aligned} &B_1 = (A_1, A_2, \dots, A_{\ell_1}), \\ (2) \qquad &B_2 = (A_{\ell_1 + 1}, A_{\ell_1 + 2}, \dots, A_{\ell_1 + \ell_2}), \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &B_k = (A_{\ell_1 + \ell_2 + \dots + \ell_{k-1} + 1}, \dots, A_n) \end{aligned}$$

of which the fundamental property is the following. The initial experiment can lead us to any of these groups, say B_α, but the succeeding experiments must necessarily and consecutively lead us into the groups,

$$(3) \qquad B_{\alpha+1}, B_{\alpha+2}, \dots, B_k, B_1, B_2, \dots, B_k, B_1, \dots,$$

the succession $B_1, B_2, \ldots, B_k$ being repeated without end. One can say that the succession (3) of groups (2) is entirely determined by the results of one initial experiment.

Now we can construct the cyclic law such that the events A_α can still be divided into groups, the succession of these groups not being entirely determined by the initial experiment. Such is for example, the law

$$\phi_1 = \begin{pmatrix} 0 & L_{12} & 0 & L_{14} & 0 & 0 \\ 0 & 0 & L_{23} & 0 & 0 & 0 \\ L_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{56} \\ L_{61} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let ℓ_1, ℓ_2, ℓ_3, ℓ_4, ℓ_5, and ℓ_6, be the numbers of lines of the $L_{12}, L_{23}, L_{31}, L_{45}, L$ and L_{61}. We shall now divide events A_α into 6 groups $B_1, B_2, \ldots, B_6$ including $\ell_1, \ell_2, \ldots, \ell_6$, events A_α taken in the order of their indices as above.

Consider now the experiments subjected to the law ϕ_1. Suppose that the initial experiment leads us into group B_1. It is clear that the following experiment can lead us either into group B_2 or group B_4. In the first case the experiments number 2 and 3 will necessarily lead us to groups B_3 and B_1 and in the second case the three experiments numbered 2, 3, 4, have us pass through the succession B_6, B_6, B_1. Thus we shall have either the cycle

$$B_1, B_2, B_3, B_1$$

or the cycle

$$B_1, B_4, B_5, B_6, B_1$$

In an infinite number of experiments these cycles will be repeated without end, but in an order regulated by chance.

We notice immediately that group B_1 has a particular property; from this group, we can pass either to group B_2 or to group B_4, whereas all the other groups B always lead us into a single determined group. Therefore, group B_1 is a critical group, a group of indetermination, a group of branching of cycles.

It is the existence of cyclic laws similar to Φ_1, the polycyclic laws, which I should like to draw to the attention of those who are concerned with the theory of Markoff chains. These laws can be very complicated and show us how complex the phenomena of the real world can be, a priori. We can expect that their study will aid us in the understanding of these phenomena.

3. Of the numerous problems which are raised by the notion of a polycyclic law of probabilities in a chain, we shall stop only for the following two:

(a) To recognize whether a given law is polycyclic;

(b) To study the behavior of the probabilities $p_{k/\beta}$ and $\phi_{\alpha\beta}^{(k)}$ for $k \to \infty$.

To simplify and shorten the article we shall consider only a few examples of these problems which will nevertheless be sufficient to show the general methods for their solution.

(a) Let us take an example of the first problem. Take as given the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where to simplify the writing, the unit element is substituted for all non-zero elements. To determine whether it is cyclic, we shall look for its cycles. These cycles are

1,3	3,2	2,9	9,1	1,6	6,7	7,1
1,3	3,10	10,9	9,1	1,8	8,7	7,1
1,4	4,2	2,9	9,1	5,6	6,7	7,1
5,3	3,2	2,9	9,5	5,8	8,7	7,1

etc.; as above we indicate only the indices of the elements of these cycles. We see two groups of cycles: of three and of four elements. Therefore, we can suppose that A is bicyclic and take as the first indices of the cycles the six following groups:

I	II	III	IV	V	VI
1,5	3,4	2,10	9	6,8	7

Now, by making in A the simultaneous permutation of lines and columns defined by the transformation

$$S = \begin{pmatrix} 1 & 5 & 3 & 4 & 2 & 10 & 9 & 6 & 8 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

we obtain for A the form

$$A^* = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which places in evidence its bicylic nature.

The events $A_1, A_2, \dots, A_{10}$, subject to a law ϕ which has the form of A^*, are divided into groups

$$B_1 = (A_1, A_2),\ B_2 = (A_3, A_4),\ B_3 = (A_5, A_6),$$
$$B_4 = (A_7),\ B_5 = (A_8, A_9),\ B_6 = (A_{10})$$

which will succeed each other in two cycles B_1, B_2, B_3, B_4, B_1 and B_1, B_5, B_6, B_1 (1).

(b) Let us pass to the second problem and consider only bicyclic laws. For such a law events $A_1, A_2, \dots, A_n$ are divided in general into groups

$$B_1, B_2, \dots, B_m$$

which, in the experiments succeed each other in two cycles

(4) $$B_1, B_2, \dots, B_k, B_1$$

(1) We can still utilize the examination of the characteristic equation of the chain considered for the solution of the first problem. If, for example, the chain is bycyclic and its cycles in B have orders k and ℓ its characteristic equation is necessarily of the form

$$\phi(1) = \lambda^r \sum C_{g,h}\lambda^{gk+h\ell} = 0,$$

where r = Constant and g, h are integral, positive numbers such that $gk + h\ell \leq n - r$. Therefore, if the exponents of the characteristic equation of the chain considered have the form $r + gk + h\ell$, we can conclude that we have a chain which is bicyclic and whose cycles in B have the orders k and ℓ. That does away with the necessity of looking for all the cycles formed by the elements $\phi_{\alpha\beta}$, for us.

and

(5) $$B_1, B_{k+1}, B_{k+2}, \ldots, B_m, B_1,$$

for example, the orders k and $\ell = m - k + 1$. If k and ℓ, or, more generally, the orders of the most numerous cycles (when a law is polycyclic) and similar to cycles (4) and (5), do not have a common divisor and if the matrix corresponding to Φ is not decomposable, we easily see that Φ does not have roots with absolute value 1 other than $\lambda_0 = 1$. Therefore, in this case, the final probabilities of Φ, $p_\beta = \lim p_{k/\beta}$, will exist and be determinate, their limits for the transition probabilities $\phi_{\alpha\beta}^{(k)}$ depending only on β. We can prove this proposition by using the well known formulae and representing the probabilities $p_{k/\beta}$ and $\phi_{\alpha\beta}^{(k)}$ as functions of the roots of Φ.

On the other hand, if the orders of the cycles (4) and (5) have a common divisor, say v, the matrix Φ will necessarily have among its roots the roots of the equation $\lambda^v - 1 = 0$ and the probabilities $p_{k/\beta}$ and $\phi_{\alpha\beta}^{(k)}$ will not have, for $k \to \infty$, a unique limit, but will have different limits depending on the initial probabilities $p_{0\alpha}$ and the values taken by k.

For example, for the bicyclic law

$$\Phi = \begin{pmatrix} 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{matrix} a + b = 1, \\ 0 < a < 1, \end{matrix}$$

we have $\Phi(\lambda) = \lambda^2(\lambda - 1)\ (\lambda^3 + \lambda^2 + \lambda + a)$ and

$$\phi_{\alpha\beta}^{(k)} = \frac{\Phi_{\alpha\beta}(1)}{\Phi'(1)} + D\left[\frac{\lambda^k \Phi_{\alpha\beta}(\lambda)}{(\lambda - 1)\ (\lambda^3 + \lambda^2 + \lambda + a)}\right]_{\lambda = 0} + \sum_{i=1}^{3} \frac{\Phi_{\alpha\beta}(\lambda_i)}{\Phi'(\lambda_i)},$$

where λ_1, λ_2, λ_3, are the roots of the equations

$$\lambda^3 + \lambda^2 + \lambda + a = 0$$

By calculating the minor $\Phi_{\alpha\beta}(\lambda)$, we shall verify at once that the limiting values of $\phi_{\alpha\beta}^{(k)}$ are

$$\phi_{\alpha 1}^{(\infty)} = \frac{1}{1 + 3a + 2b};$$

$$\phi_{\alpha 2}^{(\infty)} = \phi_{\alpha 3}^{(\infty)} = \phi_{\alpha 4}^{(\infty)} = \frac{a}{1 + 3a + 2b};$$

$$\phi_{\alpha 5}^{(\infty)} = \phi_{\alpha 6}^{(\infty)} = \frac{b}{1 + 3a + 2b};$$

$$(\alpha = 1, 2, \dots, 6)$$

The lim $p_{k/\beta}$ has the same values.

Again let

$$\Phi = \begin{pmatrix} 0 & a & b & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{matrix} a + b = 1, \\ 0 < a < 1, \end{matrix}$$

We now have

$$\Phi(\lambda) = \lambda(\lambda^4 - a\lambda^2 - b) = \lambda(\lambda^2 - 1)\ (\lambda^2 + b)$$

and

$$\phi_{\alpha\beta}^{(k)} = \frac{\Phi_{\alpha\beta}(1)}{\Phi'(1)} + \frac{(-1)^k\ \Phi_{\alpha\beta}(-1)}{\Phi'(-1)} + D\ \frac{\lambda^k \Phi_{\alpha\beta}(\lambda)}{(\lambda^2 - 1)\ (\lambda^2 + b)}\Bigg|_{\lambda = 0} + \sum_{i=1}^{2} \frac{\lambda_i^{\,2}\Phi_{\alpha\beta}(\lambda_i)}{\Phi'(\lambda_i)},$$

where λ_1 and λ_2 are the roots of the equation $\lambda^2 + b = 0$.

Therefore,

$$\phi_{\alpha\beta}^{(k)} \sim \frac{\Phi_{\alpha\beta}(1)}{\Phi'(1)} + \frac{(-1)^k \Phi_{\alpha\beta}(-1)}{\Phi'(-1)} = \pi_{\alpha\beta}^{(k)}$$

and the $\pi_{\alpha\beta}^{(k)}$ have the values given in the table below where

$$\theta_k = \frac{1 + (-1)^k}{1 + a + 3b} \quad \text{and} \quad \theta_k' = \frac{1 - (-1)^k}{1 + a + 3b}$$

Table of Values of $\pi_{\alpha\beta}^{(k)}$

α \ β	1	2	3	4	5
1	θ_k	$a\theta_k'$	$b\theta_k'$	$b\theta_k$	$b\theta_k'$
2	θ_k'	$a\theta_k$	$b\theta_k$	$b\theta_k'$	$b\theta_k$
3	θ_k'	$a\theta_k$	$b\theta_k$	$b\theta_k'$	$b\theta_k$
4	θ_k	$a\theta_k'$	$b\theta_k'$	$b\theta_k$	$b\theta_k'$
5	θ_k'	$a\theta_k$	$b\theta_k$	$b\theta_k'$	$b\theta_k$

It is evident that, in the present case, the values of the limits of the $\phi_{\alpha\beta}^{(k)}$ for $k \to \infty$ are different according to whether k passes through even or odd or indeterminate values when k takes all the integral values. Besides, the two systems of limits of $\phi_{\alpha\beta}^{(k)}$ for k increasing by even or odd values depends on α, therefore the asymptotic values of the $p_{k/\beta}$ which are equal to

$$\sum_{\alpha} p_{0\alpha} \pi_{\alpha\beta}^{(k)}$$

are oscillating and depend on the initial probabilities.

III. Statistical Problems

1. The statistical problems connected with the Markoff chain have not been considered up to the present though they appear quite naturally when dependent experiments are studied. In fact, the simplest hypothesis about the nature of the dependence which regulates the experiments considered, consists of supposing that it represents a simple Markoff chain.

The first of these problems is to establish the law of the assumed chain, that is, finding the unknown probabilities $\phi_{\alpha\beta}$ of the chain Φ by observing the frequencies of the events $A_1, A_2, \ldots, A_n$ in the experiments examined. This can be done in two ways.

a) We can observe N series of which each one consists of s experiments. Any one series, number h, will be divided into partial series

$$S^{(h)}_{A_1}, \quad S^{(h)}_{A_2}, \ldots, \quad S^{(h)}_{A_r}$$

of observations "after A_1", "after A_2" ... , "after A_n".

Let $s^{(h)}_{\alpha}$ be the number of observations which constitute the series $S^{(h)}_{A_\alpha}$ and $s^{(h)}_{\alpha\beta}$, the repetition of the event A_β in this series. Then $\frac{s^{(h)}_{\alpha\beta}}{s^{(h)}_{\alpha}}$ is an empirical value of the probability $\phi_{\alpha\beta}$. The most probable value of $\phi_{\alpha\beta}$ is defined by the approximate equality

$$\phi_{\alpha\beta} \approx \frac{\sum_h s^{(h)}_{\alpha\beta}}{\sum_h s^{(h)}_{\alpha}} = \phi'_{\alpha\beta}.$$

We well know how the precision in this equality is estimated and we shall not insist on this point. We only note that we can judge the quality of the adjustment obtained by comparing the numbers $s^{(h)}_{\alpha\beta}$, $\beta = 1, 2, \ldots, n$, with the numbers $s_\alpha \phi'_{\alpha\beta}$, $\beta = 1, 2, \dots, n$, $s_\alpha = \sum_h s_\alpha^{(h)}$, with the aid of the test of K. Pearson

$$\chi^2 = \sum_\beta \frac{(s_{\alpha\beta}^{(h)} - s_\alpha \phi'_{\alpha\beta})^2}{s_\alpha \phi'_{\alpha\beta}}$$

b) The first procedure gives us series $S_{A_\alpha}^{(h)}$ with different lengths, which is inconvenient in practice. To avoid this inconvenience, we can observe N series such that in each of them, every partial series $S_{A_\alpha}^{(h)}$ is of the same length, s, which we can obtain, for example, by keeping each series $S_{A_\alpha}^{(h)}$ with only s observations. The series having been obtained, we proceed as above.

Let us say several words on the determination of the limiting probabilities $p_\beta = \lim p_{k/\beta}$. Suppose N = KL series, consisting of K partial series with L sequences of observations each have been observed. Designate these partial series by S'_k, $k = 1, 2, \dots, K$. In S'_k, we find L sequences each consisting of s observations; let $n_\beta^{(k)}$ be the number of cells of those sequences which terminate by the observation of the event A_β; we have $\sum_\beta n_\beta^{(k)} = L$.

$\frac{n_\beta^{(k)}}{L}$ is evidently an empirical value approaching the probability $p_{k/\beta}$; the most probable value of this latter is

$$\frac{\sum_k n_\beta^{(k)}}{KL} .$$

2. An important problem which can be set for a series of experiments which are supposed to be in a chain is to recognize whether the chain is simple or not. We can solve it as follows.

Let us designate by $p^{(\gamma)}_{k/\alpha\beta}$ the probability of events A_α, A_γ, A_β in the experiments number k, k + 1, k + 2 respectively; evidently

(1) $$p^{(\gamma)}_{k/\alpha\beta} = p_{k/\alpha}\,\phi_{\alpha\gamma}\,\phi_{\gamma\beta}.$$

Let us now consider table I in which

Table I

$A_\alpha \backslash A_\beta$	$A_1 \quad \cdots \quad A_n$	$p^{(\gamma)}_{k\mid\alpha.}$
A_1	$p^{(\gamma)}_{k\mid 11} \cdots p^{(\gamma)}_{k\mid 1n}$	$p^{(\gamma)}_{k\mid 1.}$
$\vdots$	$\cdots\cdots\cdots\cdots\cdots\cdots$	$\vdots$
A_n	$p^{(\gamma)}_{k\mid n1} \cdots p^{(\gamma)}_{k\mid nn}$	$p^{(\gamma)}_{k\mid n.}$
$p^{(\gamma)}_{k\mid .\beta}$	$p^{(\gamma)}_{k\mid .1} \cdots p^{(\gamma)}_{k\mid .n}$	$p^{(\gamma)}_{k\mid ..}$

(2) $$\begin{aligned} p^{(\gamma)}_{k\mid\alpha.} &= \sum_\beta p^{(\gamma)}_{k\mid\alpha\beta} = p_{k\mid\alpha}\,\phi_{\alpha\gamma}, \\ p^{(\gamma)}_{k\mid .\beta} &= \sum_\alpha p^{(\gamma)}_{k\mid\alpha\beta} = p_{k+1\mid\gamma}\,\phi_{\gamma\beta}, \\ p^{(\gamma)}_{k\mid ..} &= \sum_{\alpha,\beta} p^{(\gamma)}_{k\mid\alpha\beta} = p_{k+1\mid\gamma}\,. \end{aligned}$$

We know that, in a simple chain, events A_α and A_β should be independent when A_γ is given. Therefore, the coefficient of contingency of table I,

(3) $$K = \frac{1}{1-n}\sum_{\alpha,\beta}\delta^{(\gamma)}_{k\mid\alpha\beta},$$

where

(4) $$\delta^{(\gamma)}_{k\mid\alpha\beta} = \frac{\left(\dfrac{p^{(\gamma)}_{k\mid\alpha\beta}}{p^{(\gamma)}_{k\mid ..}} - \dfrac{p^{(\gamma)}_{k\mid\alpha.}}{p^{(\gamma)}_{k\mid ..}}\,\dfrac{p^{(\gamma)}_{k\mid .\beta}}{p^{(\gamma)}_{k\mid ..}}\right)^2}{\dfrac{p^{(\gamma)}_{k\mid\alpha.}}{p^{(\gamma)}_{k\mid ..}}\,\dfrac{p^{(\gamma)}_{k\mid .\beta}}{p^{(\gamma)}_{k\mid ..}}},$$

should reduce to zero for a simple chain, which we verify without difficulty for $\delta_{k/\alpha\beta}^{(\gamma)} \equiv 0$ in consequence of (1) and (2).

We now see how we would proceed to verify the simplicity of a chain from the observations. We must construct s series each one consisting of three observations in the experiments number k, k + 1 and k + 2; let us call such a series a k-triad. Of these sk-triads, there are $S^{(\gamma)}$ which may have A_γ in the (k + 1)st experiment and among these there may be $s_{\alpha\beta}^{(\gamma)}$ which give the sequence A_α, A_γ, A_β. Therefore, for s large, we can write the approximate equalities

$$\frac{s^{(\gamma)}}{s} \approx p_{k+1|\gamma}, \qquad \frac{s_{\alpha\beta}^{(\gamma)}}{s} \approx p_{k|\alpha\beta}^{(\gamma)},$$

$$s_{\alpha\cdot}^{(\gamma)} = \sum_{\beta} s_{\alpha\beta}^{(\gamma)} \approx s p_{k|\alpha\cdot}^{(\gamma)},$$

$$s_{\cdot\beta}^{(\gamma)} = \sum_{\alpha} s_{\alpha\beta}^{(\gamma)} \approx s p_{k|\cdot\beta}^{(\gamma)},$$

from which we deduce:

$$\bar{\delta}_{k|\alpha\beta}^{(\gamma)} \equiv \frac{\dfrac{1}{s^{(\gamma)}}\left(s_{\alpha\beta}^{(\gamma)} - \dfrac{s_{\alpha\cdot}^{(\gamma)} s_{\cdot\beta}^{(\gamma)}}{s^{(\gamma)}}\right)^2}{\dfrac{s_{\alpha\cdot}^{(\gamma)} s_{\cdot\beta}^{(\gamma)}}{s^{(\gamma)}}} \tag{5}$$

$$\approx \delta_{k|\alpha\beta}^{(\gamma)}$$

Therefore, in the case of a simple chain, we should have approximately

$$\bar{\delta}_{k|\alpha\beta}^{(\gamma)} \approx 0$$

from which also

$$\bar{K} = \frac{1}{n-1} \sum_{\alpha,\beta} \bar{\delta}_{k|\alpha\beta}^{(\gamma)} \approx 0,$$

where K is the empirical coefficient of contingency constructed for our observations. By the well known formulae of the theory of contingency we can estimate the precision of equations (6) and, in this manner, render our conclusions on the nature of the chain considered more exact and more solid.

We can proceed further. The observations which we have described just now can be summarized by table II which correspond to table I. Now if the chain considered is simple, the

Table II

A_α \ A_β	$A_1 \quad \cdots \quad A_n$	$s^{(\gamma)}_{\alpha\cdot}$
A_1	$s^{(\gamma)}_{11} \quad \cdots \quad s^{(\gamma)}_{1n}$	$s^{(\gamma)}_{1\cdot}$
$\vdots$	$\cdots\cdots\cdots$	$\vdots$
A_n	$s^{(\gamma)}_{n1} \quad \cdots \quad s^{(\gamma)}_{nn}$	$s^{(\gamma)}_{n\cdot}$
$s^{(\gamma)}_{\cdot\beta}$	$s^{(\gamma)}_{\cdot 1} \quad \cdots \quad s^{(\gamma)}_{\cdot n}$	$s^{(\gamma)}$

rows and columns in table I are proportional as we can verify immediately. Therefore, if our observations relate to such a chain the rows and columns of the table II should be approximately proportional. This proportionality can be estimated as follows.

We construct the sequence

$$\bar{s}^{(\gamma)}_{\alpha\beta} = \frac{s^{(\gamma)}_{\alpha\beta}\, s^{(\gamma)}}{s^{(\gamma)}_{\alpha\cdot}}, \quad \beta = 1, 2, \ldots, n;$$

of which the sum is $s^{(\gamma)}$. In this case the criterion

$$\chi^2_\alpha = \sum_\beta \frac{\left(\bar{s}^{(\gamma)}_{\alpha\beta} - s^{(\gamma)}_{\cdot\beta}\right)^2}{s^{(\gamma)}_{\cdot\beta}}$$

and the table of Elderton of the corresponding probability P (we take $n' = n$ to enter this table in our case) willallow us to estimate the degree of accordance of the sequences $\bar{s}_{\alpha\beta}^{(\gamma)}$ and $s_{\cdot\beta}^{(\gamma)}$, $\beta = 1, 2, \dots, n$, for any α.

If we calculate

$$\chi^2 = \sum_{\alpha} \chi_{\alpha}^2$$

and enter this table of Elderton with

$$n' = n(n - 1) + 1$$

we whall find the probability P which estimates the simultaneous discrepancy for all the sequences

$$\bar{s}_{\alpha\beta}^{(\gamma)},\ \beta = 1, 2, \dots, n;\ \alpha = 1, 2, \dots, n$$

and

$$s_{\cdot\beta}^{(\gamma)},\ \beta = 1, 2, \dots, n$$

(See R.A. Fisher, Statistical Methods for Research Workers, 1934, p.104).

Notice that for a chain which we can suppose stable, the triads can be formed by the results of the experiments numbers 1, 2, 3; 2, 3, 4; and so on. In the case of a non-stable sequence, the number k for the k triads can be any whatsoever, but is fixed by the entirety of the triads considered.

Finally we can apply a third method. Among the consecutive triads, for example formed from results of the experiments 1, 2, 3; 2, 3, 4; etc, we will choose those which contain A_γ, γ being fixed, and will determine the empirical probability $\bar{\phi}_{\alpha\beta}^{(\gamma)}$ of the A_β, $\beta = 1, 2, \dots, n$, in the last element of our triad and for A_α, $\alpha = 1, 2, \dots, n$, in the first element. Then, if the observed chain is simple, the quantities

$$\bar{\phi}_{1\beta}^{(\gamma)},\ \bar{\phi}_{2\beta}^{(\gamma)},\ \dots,\ \bar{\phi}_{n\beta}^{(\gamma)}$$

should be approximately equal. The probabilities $\bar{\phi}_{\alpha\beta}^{(\gamma)}$ will be more precise if we determine them by the means

$$\phi_{\alpha\beta}^{(\gamma)} = \frac{1}{\ell} \sum_{i} \bar{\phi}_{\alpha\beta i}^{(\gamma)},$$

the $\bar{\phi}_{\alpha\beta i}^{(\gamma)}$ being calculated to divide the partial sequences of triads. In this case we must have ℓn^2 in such partial sequences. The agreement of results obtained with the hypothesis that the chain in question is simple is verified as follows. The numbers

$$m_{\alpha\beta}^{(\gamma)} = \sum_{i} m \, \bar{\phi}_{\alpha\beta i}^{(\gamma)}, \quad \alpha = 1, 2, \ldots, n,$$

where m is the length of the partial sequences supposed equal, should be approximately equal; by estimating their agreement by one of the well known methods, we will judge if our hypothesis is admissable or not.

3. Let us consider, in conclusion, a new notion which can be useful in the theoretical or empirical examination of Markoff chains, the notion of the rigidity of a given chain.

We call rigidity or the coefficient of rigidity of the chain $\Phi = \|\phi_{\alpha\beta}\|$ the number

$$K_k = \frac{1}{n-1} \sum_{\alpha,\beta} \frac{(p_{k|\alpha\beta} - p_{k|\alpha}\, p_{k+1|\beta})^2}{p_{k|\alpha}\, p_{k+1|\beta}} \tag{7}$$

which can also be written in the form

$$K_k = \frac{1}{n-1} \left[\sum_{\alpha,\beta} \frac{p_{k|\alpha}\, \phi_{\alpha\beta}^2}{p_{k+1|\beta}} - 1 \right]. \tag{8}$$

This number depends, in general, on the number k, that is, on the place in the infinite chain considered; they are not equally rigid throughout, but they become more and more uniformly rigid for $k \to \infty$, if the chain considered possesses determined limiting probabilities. If the chain is stabilized, its rigidity is constant and equal to

$$K = \frac{1}{n-1} \left[\sum_{\alpha,\beta} \frac{p_\alpha \phi_{\alpha\beta}^2}{p_\beta} - 1 \right]. \tag{9}$$

The name given to the number K_k is justified by its following properties.

(a) For the independent events $A_1, A_2, \dots, A_n$ subject to the law Φ, we have

$$K_k = 0.$$

We can verify in this case this equality, either directly or a consequence of the general theory of contingence, because we perceive that K_k is nothing else than the coefficient of contingence of events A_α in the experiments number k and k + 1.

(b) We always have $K_k \leq 1$, and $K_k = 1$ is possible only for the A_α connected by an absolutely rigid chain, that is, one such when the initial experiment, which is regulated only by chance, is made and given us for example, A_α; we have necessarily in the following experiments

$$A_{\beta_1}, A_{\beta_2}, \dots, A_{\beta_{n-1}}, A_\alpha, A_{\beta_1}, \dots,$$

the events $A_1, A_2, \dots, A_n$ succeeding themselves in an entirely defined order.

In fact, $K_k \leq 1$, because in general

$$\sum_{\alpha,\beta} \frac{p_{k|\alpha}\phi^2_{\alpha\beta}}{p_{k+1|\beta}} < \sum_{\alpha,\beta} \frac{p_{k|\alpha}\phi_{\alpha\beta}}{p_{k+1|\beta}} = \sum_{\beta} \frac{p_{k+1|\beta}}{p_{k+1|\beta}} = n,$$

therefore, $K_k < 1$. But if the chain is cyclic of index n (in this case it is evidently absolutely rigid) we can put the law Φ in the form where

$$\phi_{12} = \phi_{23} = \dots = \phi_{n-1,\, n} = \phi_{n1} = 1$$

and all the other $\phi_{\alpha\beta}$ are zero. Then

$$K_k = \frac{1}{m-1}\left[\sum_{\alpha\beta} \frac{P_{k|\alpha}\psi^2_{\alpha\beta}}{P_{k+1|\beta}} - 1\right]$$

$$= \frac{1}{m-1}\left[\frac{P_{k+1|2}}{P_{k|1}} + \cdots \frac{P_{k+1|1}}{P_{k|n}}\right] - \frac{1}{n-1}$$

$$= \frac{n}{n-1} - \frac{1}{n-1} = 1$$

because $P_{k+1|\alpha+1} = 1$, if in the experiment number k we have A_α, and $P_{k+1|1} = 1$, if we have A_n in the experiment number k, therefore

$$P_{k+1|2} = P_{k|1}, \ldots, P_{k+1|1} = P_{k|n}.$$

(c) Conversely, if K = 0, the events A are independent and if K = 1, the chain Φ is necessarily cyclic of indice n, therefore absolutely rigid, as we can easily verify.

In this way the value of the number K_k shows us, the more or less strong dependence of events $A_1, A_2, \ldots, A_n$ in the experiments connected in a chain. At the same time K_k shows also a more or less apparent, more or less temporary succession of these events and represents in this way an essential characteristic of the processes described by the chain considered.