A NOTE ON THE NORMAL INTEGRAL

by

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Let

$$(1) \qquad v = \int_0^x (2\pi)^{-1/2} e^{-t^2/2}\, dt, \qquad x \geq 0.$$

G. Polya [1] and J. D. Williams [2] proved independently that

$$(2) \qquad v \leq \tfrac{1}{2}(1 - e^{-2x^2/\pi})^{1/2} .$$

Two simple questions follow naturally. 1. Is it possible to replace the constant $2/\pi$ in (2) by a smaller quantity without breaking the inequality? 2. Does there exist a lower bound, in a similar form, for v? We find the following answer.

If for all $x \geq C$, the integral v given by (1) satisfies

$$(3) \qquad \tfrac{1}{2} \cdot (1 - e^{-ax^2})^{1/2} \leq v \leq \tfrac{1}{2} (1 - e^{-bx^2})^{1/2},$$

then it is necessary and sufficient that $a \leq 1/2$ and $b \geq 2/\pi$.

The proof of this statement is simple. First

$$\lim_{x \longrightarrow 0} v^2/(1 - e^{-bx^2}) = (2\pi b)^{-1} .$$

Hence if (3) is true, $b \geq 2/\pi$. On the other hand, $x^2/[-\log(1 - 4v^2)] \leq 1/a$ for all real x. Since the limit of this ratio, as $x \longrightarrow \infty$, is 2. We have

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$a \leq 1/2$. Finally, it is easy to prove that

$$\tfrac{1}{2}(1 - e^{-x^2/2})^{1/2} \leq v.$$

Further, it follows from [1] and [2] that as x varies from 0 to oo,

$$2v/(1 - e^{-2x^2/\pi})^{1/2}$$

decreases steadily from 1 to its minimum value .9930 at x = 1.6 (approximately), then increases steadily. Using a similar method used by G. Polya, it can be shown that

$$2v/(1 - e^{-x^2/2})^{1/2} \tag{4}$$

is a steadily decreasing function of x for all $x \geq 0$. This is because the derivative of this ratio (4) has the same sign as

$$2(e^{x^2/2} - 1) - x\, e^{x^2/2} \int_0^x e^{-t^2/2}\, dt,$$

which is non-positive since

$$e^{x^2/2} \int_0^x e^{-t^2/2}\, dt = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{1 \cdot 3 \ldots (2n-1)} .$$

As a consequence, we obtain that the ratio (4) has an upper bound $(4/\pi)^{1/2}$.

REFERENCES

[1] G. Polya, Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, 1949, pp. 63-78.

[2] J. D. Williams, "An approximation to the normal integral," Ann. Math. Stat. Vol. 17 (1946), pp. 363-365.