ON BOUNDS FOR THE NORMAL INTEGRAL

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1. Summary. Bounds are derived for the normal integral. Some comparisons are made with several known results.

2. Let

(1) $$v = \int_0^x (2\pi)^{-\frac{1}{2}} e^{-t^2/2} dt \quad , \qquad x \geq 0.$$

G. Pólya and J. D. Williams proved independently that

(2) $$v \leq \tfrac{1}{2} (1 - e^{-2x^2/\pi})^{\frac{1}{2}} .$$

Two simple questions follow naturally. 1. Is it possible to replace the constant $2/\pi$ in (2) by a smaller quantity without breaking the inequality? 2. Does there exist a lower bound, in a similar form, for v? We find the following answer.

If for all $x \geq 0$, the integral v given by (1) satisfies

(3) $$\tfrac{1}{2}(1 - e^{-ax^2})^{\frac{1}{2}} \leq v \leq \tfrac{1}{2}(1 - e^{-bx^2})^{\frac{1}{2}},$$

then it is necessary and sufficient that $0 \leq a \leq \frac{1}{2}$ and $b \geq 2/\pi$.

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2. Revised and enlarged from "A note on the normal integral", Institute of Statistics Mimeograph Series No. 114.

The proof of this statement is simple. First,

$$\lim_{x \to 0} v^2/(1 - e^{-bx^2}) = (2\pi b)^{-1}.$$

Hence if (3) is true, $b \geq 2/\pi$. On the other hand, $x^2/[-\log(1-4v^2)] \leq 1/a$ for all real x. Since the limit of this ratio, as $x \to \infty$, is 2, we have $a \leq 1/2$. Finally

$$4v^2 = \int_{-x}^{x} \int_{-x}^{x} (2\pi)^{-1} e^{-(s^2+t^2)/2} \, ds \, dt \geq \int_{0}^{2\pi} \int_{0}^{x} (2\pi)^{-1} e^{-r^2/2} \, r \, dr \, d\theta .$$

Therefore

(4) $$v \geq \tfrac{1}{2}(1 - e^{-x^2/2})^{\frac{1}{2}}.$$

Pólya showed that as x varies from 0 to ∞, the ratio of the LHS (Left hand side) of (2) to the RHS decreases steadily from 1 to a minimum value and then increases steadily. Williams' calculations indicate that, approximately, the minimum value .9930 is taken at $x = 1.6$. Using a similar method to that of Pólya, it can be shown that the ratio of the LHS of (4) to the RHS is a steadily decreasing function of x for all $x \geq 0$; for the derivative of this ratio has the same sign as

$$2(e^{x^2/2} - 1) - x\, e^{x^2/2} \int_{0}^{x} e^{-t^2/2} \, dt,$$

which is non-positive since

(5) $$e^{x^2/2} \int_{0}^{x} e^{-t^2/2} \, dt = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{1.3\ldots(2n-1)} .$$

As a consequence, we obtain that this ratio (of the LHS of (4) to the RHS) has an upper bound $2/\pi^{\frac{1}{2}}$.

3. A different lower bound for v can be obtained easily from a result proved by Chu and Hotelling. There we showed that for all $x \geq 0$,

$$x^2(1 - 4v^2)/4v^2 \leq \pi/2. \tag{6}$$

Hence it follows that

$$v \geq \frac{1}{2} \left[2x^2/(\pi + 2x^2) \right]^{\frac{1}{2}}. \tag{7}$$

For easy reference, we will give here a proof of (6). Let

$$g_0(x) = x^2(1-4v^2)/(4v^2), \tag{8}$$

then $\lim_{x \to 0} g_0(x) = \pi/2$. We will show that $g_0(x)$ is decreasing. Let "'" denote differentiation with respect to x. Then,

$$g_0'(x) = (x/2v^3)\ g_1(x), \tag{9}$$

where

$$g_1(x) = v(1-4v^2) - xv'. \tag{10}$$

(11) $g_1'(x) = g_2(x)v'$, where $g_2(x) = x^2-12v^2$, $g_2'(x) = (12/\pi)\ e^{-x^2} g_3(x)$, where

$$g_3(x) = (\pi/6)xe^{x^2} - e^{x^2/2} \int_0^x e^{-t^2/2} dt.$$

From (5), we have $g_3(x) = \sum_{n=0}^{\infty} \{\frac{\pi}{6n!} - \frac{1}{1.3...(2n+1)}\} x^{2n+1}$. It can be shown, by a similar argument used by Pólya for a similar purpose, that

$$(12) \qquad g_3(x) = x^3(a_0x^{-2} + a_1 + a_2x^2 + \dots),$$

where $a_0 < 0$ and $a_i > 0$, $i = 1, 2, \dots$. Hence there exists an $x_0 > 0$ such that $g_3(x) \leq 0$ if $0 \leq x \leq x_0$ and $g_3(x) \geq 0$ if $x \geq x_0$. So as x increases from 0 to ∞, $g_2(x)$ decreases steadily from 0 to a minimum and then increases steadily to ∞. Consequently $g_1(x)$ first decreases steadily and then increases steadily. As $\lim_{x \to 0} g_1(x) = \lim_{x \to \infty} g_1(x) = 0$, it becomes clear that $g_1(x) \leq 0$ for all $x \geq 0$. Therefore $g_0(x)$ is a decreasing function of x. Hence we have (6).

Comparison can be made easily of the two lower bounds for v given by (4) and (7). For simplicity, they will be denoted by a(x) and b(x) respectively. Now $a(x) \gtreqless b(x)$ according as $c(x) = e^{x^2/2} - 2x^2/\pi - 1 \gtreqless 0$. As x varies from 0 to ∞, $c'(x)$, the derivative of c(x), changes sign from negative to positive. So does c(x). If $x = x_0$ is the solution of $c(x) = 0$, then $x_0 = 1$ approximately (the exact value is slightly smaller). Therefore, the lower bound in (7) is closer to v than that in (4) if $0 \leq x \leq 1$ (approximately) and less close if $x \geq 1$.

Further, the following statement is of similar nature to the one made in § 2: If for all $x \geq 0$,

$$(13) \qquad v \geq \frac{1}{2} [ax^2/(1 + ax^2)]^{\frac{1}{2}},$$

then it is necessary and sufficient that $0 \leq a \leq 2/\pi$. On the other hand, for no finite a can the RHS of (13) be, for all $x \geq 0$, an upper bound for v.

The above statement can be shown easily by considering the limit, as $x \longrightarrow 0$, of the ratio of v to the RHS of (13); and the limit, as $x \longrightarrow \infty$, of $(1-4v^2)(1+ax^2)$.

4. Several authors have derived inequalities for Mills' ratio. Their results can be written in the form of bounds for the normal integral. For example, in our notation, Gordon's inequalities are equivalent to

$$(14) \quad \frac{1}{2} - \frac{1}{x}(2\pi)^{-\frac{1}{2}} e^{-x^2/2} \leq v \leq \frac{1}{2} - \frac{x}{x^2+1}(2\pi)^{-\frac{1}{2}} e^{-x^2/2}, \quad \text{for } x > 0.$$

Birnbaum improved Gordon's upper bound in (14) and obtained

$$(15) \quad v \leq \frac{1}{2} - \frac{(4+x^2)^{\frac{1}{2}} - x}{2}(2\pi)^{-\frac{1}{2}} e^{-x^2/2}, \quad \text{for } x \geq 0.$$

More recently, Tate showed what amounts to

$$(16) \quad \left(\frac{1}{4} + \frac{e^{-x^2}}{2\pi x^2}\right)^{\frac{1}{2}} - \frac{e^{-x^2/2}}{x(2\pi)^{\frac{1}{2}}} \leq v \leq \frac{1}{2}(1-e^{-x^2}), \text{ for } x \geq 0.$$

We will now compare briefly (2) and (4) with (14), (15), and (16). The upper bound in (16) is obviously not so good as that in (2). The lower bound in (16) is non-negative for all real x. It is $\gtreqless$ the RHS of (4) according as $h(x) = x^2 - (8/\pi)(1 - e^{-x^2/2}) \gtreqless 0$. As x varies from 0 to ∞, h(x) decreases steadily from 0 to a minimum and then increases steadily to ∞; and vanishes at $x = 1.01$ approximately (the exact value is slightly smaller). Therefore the lower bound in (16) is closer to v than that in (4) if and only if $x \geq 1.01$ approximately.

The lower bound in (14) is an increasing function of x for all $x > 0$. It is non-negative when $x \geq .65$ (approximately); and in this case it is $\lesseqgtr$ the RHS of (4) according as $g(x) = 2(2/\pi)^{\frac{1}{2}} x - x^2 - (2/\pi) e^{-x^2/2} \gtreqless 0$. As x varies from 0 to ∞, g(x) increases steadily from $-2/\pi$ to a maximum and then decreases steadily to $-\infty$. The two roots of $g(x) = 0$ are approximately $x = .5$ and $x = 1.45$. Hence the lower bound in (14) is closer to v than that of (4) if and only if $x \geq 1.45$ approximately.

Finally we point out that, for values of x close to 0, the upper bound in (2) is better than those in (14) and (15); while for large values of x, the latter two are better. No detailed comparison is attempted.

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