ON THE RELATION BETWEEN ESTIMATING EFFICIENCY[1] AND THE POWER OF TESTS

by

R. M. Sundrum
University of Rangoon
Institute of Statistics, University of North Carolina

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12. In this paper a condition is derived for the validity of the assumption that a statistic which has a high efficiency in estimating an unknown parameter also gives a powerful test of hypotheses about that parameter. An example from genetics is given of a failure of this assumption. A presumption in favour of using the more efficient estimator is established by showing that the above condition fails only in situations where the power of the resulting test is less than $\frac{1}{2}$.

On the Relation Between Estimating Efficiency and the Power of Tests

by

R. M. Sundrum
University of Rangoon and
Institute of Statistics, University of North Carolina

It appears to be generally assumed that a statistic which has a high efficiency in estimating an unknown parameter also gives a powerful test of hypotheses about that parameter. For example, a common criterion for comparing distribution-free tests with corresponding distributional tests and with other distribution-free tests is the asymptotic relative efficiency first proposed by Pitman (1948). Recently, Alan Stuart (1954) has shown that this is equivalent to using the estimating efficiency of these statistics to compare their performance when used in tests of significance. In fact, it was by this argument that this measure of efficiency was earlier arrived at by Hotelling and Pabst (1936) in their study of the Spearman rank correlation coefficient and by Cochran (1937) in his study of two tests of the mean and of the correlation coefficient of normally distributed variables. In this note, we derive a condition for the validity of this assumption in the case when the statistics have normal distributions, a case for which the idea of estimating efficiency has most relevance.

Let t_1 and t_2 be two statistics which are normally distributed, unbiased estimators of a population parameter θ with variances $\sigma_1^2(\theta_0)$ and $\sigma_2^2(\theta_0)$ respectively under the null hypothesis H_0: $\theta = \theta_0$; and $\sigma_1^2(\theta_1)$ and $\sigma_2^2(\theta_1)$ respectively under the alternative hypothesis

H_1: $\Theta = \Theta_1 > \Theta_0$. Let $\sigma_1^2(\Theta_0) \leq \sigma_2^2(\Theta_0)$; $\sigma_1^2(\Theta_1) \leq \sigma_2^2(\Theta_1)$, the inequality holding at least once, so that t_1 may be called the more efficient estimator.

If λ_α is defined by

$$\Phi(\lambda_\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\lambda_\alpha}^{\infty} e^{-\frac{1}{2}y^2}\, dy = \alpha$$

the one-sided critical region of size α based on t_1 is given by

$$t_1 > \Theta_0 + \lambda_\alpha \sigma_1(\Theta_0) \tag{1}$$

and for t_2 by

$$t_2 > \Theta_0 + \lambda_\alpha \sigma_2(\Theta_0). \tag{2}$$

The power of the critical region (1) is then

$$P_1 = \Phi\left\{\frac{\Theta_0 + \lambda_\alpha \sigma_1(\Theta_0) - \Theta_1}{\sigma_1(\Theta_1)}\right\} \tag{3}$$

and of the critical region (2) is

$$P_2 = \Phi\left\{\frac{\Theta_0 + \lambda_\alpha \sigma_2(\Theta_0) - \Theta_1}{\sigma_2(\Theta_1)}\right\}. \tag{4}$$

As $\Phi(x)$ is a monotonically decreasing function of x, the test based on t_1 (the more efficient estimator) is more powerful if

$$\frac{\theta_0 - \theta_1 + \lambda_\alpha \sigma_1(\theta_0)}{\sigma_1(\theta_1)} < \frac{\theta_0 - \theta_1 + \lambda_\alpha \sigma_2(\theta_0)}{\sigma_2(\theta_1)}$$

i.e., if
$$\theta_1 - \theta_0 > \lambda_\alpha \left\{ \frac{\sigma_1(\theta_0)\sigma_2(\theta_1) - \sigma_2(\theta_0)\sigma_1(\theta_1)}{\sigma_2(\theta_1) - \sigma_1(\theta_1)} \right\}$$

$$= \lambda_\alpha \left(\frac{V_1 - V_0}{V_1 - 1} \right) \sigma_1(\theta_0) \quad \text{where } V_i = \frac{\sigma_2(\theta_i)}{\sigma_1(\theta_i)} \quad (i=1,2). \qquad (5)$$

If $V_0 \geq V_1$, then the right-hand side of (5) is less than or equal to 0 and the inequality is satisfied. But if $V_1 > V_0 \geq 1$, so that the relative efficiency of t_1 is greater under the alternative than under the null hypothesis, then $\left(\frac{V_1 - V_0}{V_1 - 1}\right) > 0$. As α, the size of the critical region is always taken less than $\frac{1}{2}$ in practical applications, λ_α is always positive and can be chosen so large that

$$\lambda_\alpha \left(\frac{V_1 - V_0}{V_1 - 1} \right) \sigma_1(\theta_0) > \theta_1 - \theta_0$$

and the inequality (5) is reversed.

An example from genetics is given by Fisher [1950, pp. 314-5] concerning a test for linkage in inheritance of two factors. Given the frequencies of four combinations in a sample as a, b, c, and d with

$$a + b + c + d = n \qquad (6)$$

and the corresponding probabilities of occurrence as

$$\frac{1}{4}(2 + \theta), \quad \frac{1}{4}(1 - \theta), \quad \frac{1}{4}(1 - \theta), \quad \frac{1}{4}\theta$$

the problem is one of estimating θ, when $\sqrt{\theta}$ is the recombination ratio. The maximum likelihood method gives a statistic t_1 as the positive solution of the equation

$$n\theta^2 - (a - 2b - 2c - d)\,\theta - 2d = 0 \qquad (7)$$

with a sampling variance

$$\sigma^2(t_1) = \frac{2\theta(1 - \theta)(2 + \theta)}{(1 + 2\theta)\,n} \qquad (8)$$

Another statistic t_2 may be defined by

$$t_2 = \frac{a - b - c + 5d}{2n} \qquad (9)$$

with expectation θ and sampling variance

$$\sigma^2(t_2) = \frac{1 + 6\theta - 4\theta^2}{4n} \qquad (10)$$

It is easily verified that $\sigma^2(t_1) \leq \sigma^2(t_2)$.

Now consider a test of H_0: $\theta = \frac{1}{4}$ (no linkage) against $H_1 : \theta = \theta_1 > \frac{1}{4}$. We find $V_0 = 1$ so that, assuming n large enough for a normal approximation to hold satisfactorily for the distributions of t_1 and t_2, and for the bias of the maximum likelihood estimator to be negligible, the inequality (5) is reversed for all θ_1 satisfying

$$(\theta_1 - \tfrac{1}{4}) < \frac{\lambda_\alpha \; 3}{4\sqrt{n}} \qquad (11)$$

However, this possibility is not very important in practice. For if λ_α is chosen so as to reverse the inequality (5), the power of the test will be reduced very much. In fact under the special assumptions of the above argument, the situation in which the more efficient estimator gives a less powerful test cannot arise, if the level of significance is chosen so as to make the power of that test greater than $\frac{1}{2}$. This can be seen as follows:

$$P_1 = \Phi\left\{\frac{\theta_0 + \lambda_\alpha\sigma_1(\theta_0) - \theta_1}{\sigma_1(\theta_1)}\right\} > \frac{1}{2} = \Phi(0)$$

if $\theta_0 - \theta_1 + \lambda_\alpha\sigma_1(\theta) < 0$

i.e., if $(\theta_1 - \theta_0) > \lambda_\alpha\sigma_1(\theta_0)$ (6)

From the condition $V_1 > V_0 \geq 1$, we have

$$0 < \frac{V_1 - V_0}{V_1 - 1} \leq 1 \qquad (7)$$

From (6) and (7) the inequality (5) follows.

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