

ITERATED LIMITS AND THE CENTRAL LIMIT THEOREM  
FOR DEPENDENT VARIABLES

by

GEORGE MARSAGLIA

Special report to the Office of Naval Research  
of work at Chapel Hill under Project NR 042 031  
for research in probability and statistics.

Institute of Statistics  
Mimeograph Series No. 93  
February 1954

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by  
George Marsaglia  
University of North Carolina

1. Introduction. Section 2 of this paper gives some results on iterated limits which may be considered generalizations of well-known results [1, p. 254]. Section 3 applies these results to give easy proofs of some central limit theorems for m-dependent variables.

2. Iterated Probability Limits. Here, we use the strong sense of an iterated limit: for constants  $a_{ij}$ ,  $i, j = 1, 2, \dots$ ,

$\lim_j \lim_i a_{ij} = a$  means

$$(1) \quad \lim_{j \rightarrow \infty} \left( \overline{\lim}_{i \rightarrow \infty} |a_{ij} - a| \right) = 0.$$

We note that (1) holds if, and only if, for each  $\epsilon > 0$  there exist integers  $M, N_1, N_2, \dots$  such that if the pair  $(i, j)$  satisfies  $j > M$ ,  $i > N_j$ , then  $|a_{ij} - a| < \epsilon$ .

DEFINITION 1. Let  $f, f_{ij}, i, j = 1, 2, \dots$  be random variables.

Then

$$\text{plim}_j \text{plim}_i f_{ij} = f$$

means, for every  $\epsilon > 0$ ,

$$\lim_j \lim_i P(|f_{ij} - f| > \epsilon) = 0$$

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1. Work sponsored by the Office of Naval Research under Contract No. N7-onr-28402.

THEOREM 1. Let  $h_{ij}, g_{ij}, i, j = 1, 2, \dots$  be random variables.

Let  $G$  be a function such that at each of its continuity points  $x$

$$\lim_j \lim_i P(g_{ij} \leq x) = G(x),$$

and suppose

$$\text{plim}_j \text{plim}_i h_{ij} = 0.$$

Then

$$\lim_j \lim_i P(g_{ij} + h_{ij} \leq x) = G(x).$$

Let  $\epsilon = 8\delta > 0$  and a continuity point  $x$  of  $G$  be given. We shall exhibit integers  $M, N_1, N_2, \dots$  such that

$$(2) \quad |P(g_{ij} + h_{ij} \leq x) - G(x)| < \epsilon$$

if  $j > M$  and  $i > N_j$ . First, choose  $\beta$  so that  $G$  is continuous at  $x + \beta$ , at  $x - \beta$ , and so that

$$(3) \quad |G(x + \beta) - G(x - \beta)| < \delta.$$

Then choose  $M, N_1, N_2, \dots$  so that, simultaneously,

$$(4) \quad P(|h_{ij}| > \beta) < \delta$$

$$(5) \quad |P(g_{ij} \leq x) - G(x)| < \delta$$

$$(6) \quad |P(g_{ij} \leq x - \beta) - G(x - \beta)| < \delta$$

$$(7) \quad |P(g_{ij} \leq x + \beta) - G(x + \beta)| < \delta$$

whenever  $(i, j)$  satisfies  $j > M$  and  $i > N_j$ . Then for such a pair  $(i, j)$ ,

let  $F(x) = P(g_{ij} + h_{ij} \leq x)$ ,  $H(x, \beta) = P(g_{ij} + h_{ij} \leq x, |h_{ij}| \leq \beta)$ ,

$L(x, \beta) = P(g_{ij} \leq x, |h_{ij}| \leq \beta)$  and  $Q(x) = P(g_{ij} \leq x)$ . We have

$$|F(x) - G(x)| \leq |F(x) - H(x, \beta)| + |H(x, \beta) - L(x, \beta)| + |L(x, \beta) - Q(x)| + |Q(x) - G(x)|.$$

Now by (4) and (5), each of the terms on the right except the second

is bounded by  $\delta$ , and since  $L(x - \beta, \beta) \leq H(x, \beta) \leq L(x + \beta, \beta)$  and

$$L(x - \beta, \beta) \leq L(x, \beta) \leq L(x + \beta, \beta),$$

$$|H(x, \beta) - L(x, \beta)| \leq |L(x + \beta, \beta) - L(x - \beta, \beta)| < 5\delta$$

by (4), (7), (3) and (6), since

$$\begin{aligned} |L(x + \beta, \beta) - L(x - \beta, \beta)| &\leq |L(x + \beta, \beta) - Q(x + \beta)| + |Q(x + \beta) - G(x + \beta)| + |G(x + \beta) - G(x - \beta)| \\ &\quad + |G(x - \beta) - Q(x - \beta)| + |Q(x - \beta) - L(x - \beta, \beta)| \end{aligned}$$

Hence  $|F(x) - G(x)| < 8\delta = \epsilon$ , which is condition (2).

**THEOREM 2.** Under the conditions of Theorem 1, if there exist constants  $a_{ij}$  such that  $\lim_j \lim_i a_{ij} = a > 0$ , and if  $G$  is continuous

at  $x/a$  then  $\lim_j \lim_i P(a_{ij} g_{ij} \leq x) = G(x/a)$ .

Using the artifice, for suitable  $i, j, \gamma$ ,

$$\left| P(g_{ij} \leq \frac{x}{a_{ij}}) - G\left(\frac{x}{a}\right) \right| \leq \left| P(g_{ij} \leq \frac{x}{a_{ij}}) - P(g_{ij} \leq \frac{x}{a}) \right| + \left| P(g_{ij} \leq \frac{x}{a}) - G\left(\frac{x}{a}\right) \right|$$

$$\left| P(g_{ij} \leq \frac{x}{a_{ij}}) - P(g_{ij} \leq \frac{x}{a}) \right| \leq \left| P(g_{ij} \leq \frac{x}{a-\gamma}) - P(g_{ij} \leq \frac{x}{a+\gamma}) \right|,$$

the proof is routine. The details are omitted.

3. Applications to partitioned sequences of m-dependent random variables. Let  $x_1, x_2, \dots$  be an m-dependent sequence of random variables with zero means. For each pair  $(n, k)$  with  $2m < k \leq n$ , define

$$y_i = x_{ik-k+1} + \dots + x_{ik-m} \quad i = 1, 2, \dots$$

$$g_{nk} = \sum_1^{\left[ \frac{n}{k} \right]} y_i, \quad t_{nk}^2 = E(g_{nk}^2)$$

$$s_n^2 = E(x_1 + \dots + x_n)^2, \quad h_{nk} = \frac{1}{s_n} \left( \sum_1^n x_i - g_{nk} \right)$$

Since we shall be dealing with  $\lim_k \lim_n$  relations,  $g_{nk}$ ,  $n < k$ , may be

defined indifferently.

According to Theorems 1 and 2, if

$$(8) \quad \lim_k \lim_n \frac{t_{nk}}{s_n} = 1$$

$$(9) \quad \text{plim}_k \text{plim}_n h_{nk} = 0$$

$$(10) \quad \lim_k \lim_n P\left(\frac{g_{nk}}{t_{nk}} \leq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}t^2} dt$$

then

$$(11) \quad \lim_n P\left(\frac{x_1 + \dots + x_n}{s_n} \leq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}t^2} dt.$$

The following theorems give conditions which imply (8), (9), and (10).

**THEOREM 3.** If there exist constants  $\alpha > 2$ ,  $B > 0$  such that

$$(12) \quad \frac{n}{s_n^2} < B, \quad n = 1, 2, \dots$$

$$(13) \quad E(x_n^2) < B, \quad n = 1, 2, \dots$$

$$(14) \quad \lim_n \frac{\left(\frac{n}{s_n} \sum_{i=1}^n E(|x_i|^\alpha)\right)^{1/\alpha}}{s_n} = 0$$

then condition (11) holds.

We first establish (8) and (9). One readily finds, for  $2m < k \leq n$ ,

$$(15) \quad |s_n^2 - t_{nk}^2| < \left(\left[\frac{n}{k}\right] + k^2\right) 8m^2 B.$$

and

$$(16) \quad E(h_{nk}^2) < \frac{1}{s_n^2} \left(\left[\frac{n}{k}\right] + k^2\right) 8m^2 B.$$

But, using (12),

$$(17) \quad \lim_k \lim_n \frac{\left[\frac{n}{k}\right] + k^2}{s_n^2} = \lim_k \lim_n \frac{\left[\frac{n}{k}\right]}{s_n^2} = \lim_k \left(\frac{1}{k} \overline{\lim_n \frac{n}{s_n^2}}\right) = 0.$$

Relations (15), (16) and (17) imply (8) and (9).

Condition (10) will be true, by Liapounoff's Theorem [4, p. 284] if, for large k,

$$\lim_n \frac{\left( \sum_{i=1}^{\lfloor \frac{k}{n} \rfloor} E(|y_i|^\alpha) \right)^{1/\alpha}}{t_{nk}} = 0.$$

Now  $E(|y_i|^\alpha) \leq k^\alpha \sum_{j=ik-k+1}^{ik-m} E(|x_j|^\alpha)$ , so that

$$\lim_n \frac{\left( \sum_{i=1}^{\lfloor \frac{k}{n} \rfloor} E(|y_i|^\alpha) \right)^{1/\alpha}}{t_{nk}} \leq \lim_n \frac{k \left( \sum_{i=1}^n E(|x_i|^\alpha) \right)^{1/\alpha}}{s_n} \cdot \frac{s_n}{t_{nk}} = 0.$$

by (14) and (8), if k is large.

THEOREM 4. If  $x_1, x_2, \dots$  is a stationary m-dependent sequence with zero means, then (11) holds.

For in that case, (12) holds, and, since the variances are bounded, (8) and (9) are established as above. (10) holds, since, for each  $k > 2m$ , the sequence  $y_1, y_2, \dots$  is stationary and independent.

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