A CENTRAL LIMIT THEOREM FOR m-DEPENDENT VARIANCES

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1. Summary. In this paper a central limit theorem is obtained for a sequence of m-dependent random variables with bounded γth moments $(\gamma > 2)$ and with the property that, for large n, n/s_n^2 is bounded, where s_n is the standard deviation of the nth partial sum of the sequence.

2. Preliminaries. In this section we explain some terms and notations and mention two theorems which we employ.

A sequence of random variables

$$\{X_i\} \qquad (i = 1, 2, \ldots) \tag{1}$$

is said to be m-dependent if the random variables $(X_1, \ldots, X_a)$ and $(X_b, \ldots, X_c)$ $(1 \leq a, b \leq c)$ are independent whenever $b - a > m$.

We define

$$S_n = X_1 + \ldots + X_n . \tag{2}$$

Then if the X_i have zero means and finite variances

$$s_n = \sqrt{E(S_n^2)} . \tag{3}$$

We write $\beta_{\gamma i}$ for the γth absolute moment of X_i. Thus

$$\beta_{\gamma i} = E\left|X_i^\gamma\right| \quad (i = 1, 2, \ldots) . \tag{4}$$

The two theorems which we use are the following:

Theorem 1 [2]. Let $\{W_n\}$ $(n = 1, 2, \ldots)$ be a sequence of random variables. Let $n = hk + r$ (h, k integers; $0 \leq r < k$). Suppose that, for each (n, k), there exist random variables $U_{n,k}$, $V_{n,k}$ and distribution functions $F_k(x)$, $F(x)$ such that

(a) $W_n = U_{n,k} + V_{n,k}$,

(b) the distribution function of $U_{n,k} \longrightarrow F_k(x)$ as $h \longrightarrow \infty$ (for all r and for $k \geq K$) if x is a continuity point of $F_k(x)$,

(c) $F_k(x) \longrightarrow F(x)$ as $k \longrightarrow \infty$ if x is a continuity point of $F(x)$, and

(d) $V_{n,k}$ converges in probability to zero uniformly in r as $h, k \longrightarrow \infty$.

Then the distribution function of $W_n \longrightarrow F(x)$ as $n \longrightarrow \infty$ if x is a continuity point of $F(x)$.

Theorem 2 [1]. Let (1) be a sequence of independent random variables with means zero and finite γth absolute moments for some $\gamma > 2$. Suppose, as $n \longrightarrow \infty$,

$$\frac{1}{s_n^\gamma} \sum_{i=1}^{n} \beta_{\gamma i} \longrightarrow 0 . \tag{5}$$

Then the distribution function of $S_n/s_n \longrightarrow$ the standardized normal distribution function .

3. The central limit theorem. We shall prove the following.

Theorem 3. Let (1) be a sequence of m-dependent random variables with means zero and γth absolute moments bounded for some $\gamma > 2$. Suppose, as $n \longrightarrow \infty$,

$$s_n^2/n > a^2 > 0 . \tag{6}$$

Then the distribution function of $S_n/s_n \longrightarrow$ the standardized normal distribution function.

Let h, k, r be as in Theorem 1. Define

$$\left.\begin{aligned} Y_i &= X_{(i-1)k+1} + \dots + X_{ik-m} \\ Z_i &= X_{ik-m+1} + \dots + X_{ik} \end{aligned}\right\} \quad (i = 1,\dots,h), \tag{7}$$

$$\left.\begin{aligned} & s_{n,k} = \sqrt{E(Y_1^2 + \dots + Y_h^2)}, \quad U_{n,k} = (Y_1 + \dots + Y_h)/s_{n,k} , \\ & V_{n,k;1} = (Z_1 + \dots + Z_h)/s_n , \quad V_{n,k;2} = (S_n - S_{hk})/s_n , \\ & V_{n,k;3} = (Y_1 + \dots + Y_h)\left(\frac{1}{s_n} - \frac{1}{s_{n,k}}\right) , \end{aligned}\right\} \tag{8}$$

$$V_{n,k} = V_{n,k;1} + V_{n,k;2} + V_{n,k;3}\ , \quad W_n = U_{n,k} + V_{n,k} \quad . \tag{9}$$

Then
$$W_n = S_n / s_n \quad . \tag{10}$$

Lemma 1. For large fixed k, as $h \longrightarrow \infty$,

$$L \equiv \frac{1}{s_{n,k}^{\gamma}} \sum_{i=1}^{h} E \left| Y_i^{\gamma} \right| \longrightarrow 0 \quad . \tag{11}$$

Proof.
$$E \left| Y_i^{\gamma} \right| = E \left| \left\{ X_{(i-1)k+1} + \dots + X_{ik-m} \right\}^{\gamma} \right|$$
$$\leq (k-m)^{\gamma-1} \sum_{g=1}^{k-m} E \left| X_{(i-1)k+g}^{\gamma} \right|$$
$$\leq k^{\gamma-1} \sum_{g=1}^{k} E \left| X_{(i-1)k+g}^{\gamma} \right| < k^{\gamma} b^{\gamma} \ , \tag{12}$$

where, by data of Theorem 3,

$$E \left| X_i^{\gamma} \right| < b^{\gamma} < \infty \quad . \tag{13}$$

Also, from (8) ,

$$s_{n,k}^2 = E \left\{ S_{hk} - (Z_1 + \dots + Z_h) \right\}^2 \quad . \tag{14}$$

From (13), since $\gamma > 2$, we have that

$$E(X_i^2) < b^2 \quad . \tag{15}$$

Hence, from (7),

$$E(Z_i^2) < m^2 b^2 \tag{16}$$

so that, from (14),

$$s_{n,k}^2 = s_{hk}^2 + \theta(2 s_{hk} \sqrt{h}\, mb + hm^2 b^2) \quad , \tag{17}$$

where $|\theta| \leq 1$. Thus, by (6),

$$\frac{s_{n,k}^2}{s_{hk}^2} = 1 + \theta \left\{ \frac{2mb}{a\sqrt{k}} + \frac{m^2 b^2}{a^2 k} \right\} \longrightarrow 1 \tag{18}$$

as $k \longrightarrow \infty$ uniformly in h and in r. Hence, for large fixed k,

$$s_{n,k}^2 > \frac{1}{2} s_{hk}^2 > \frac{1}{2} hka^2 \tag{19}$$

From (11), (12) and (19),

$$L < \frac{hk^{\gamma}b^{\gamma}}{(\frac{1}{2}hka^2)^{\gamma/2}} \longrightarrow 0 \tag{20}$$

as $h \longrightarrow \infty$, for k fixed (and large). The lemma is thus proved.

Lemma 2. For large k, the distribution function of $U_{n,k} \longrightarrow$ the standardized normal distribution function $(F_k(x) = F(x)$, say) as $h \longrightarrow \infty$.

Proof. If $k \geq 2m$ (as would happen for large k) the Y_i are independent. Hence we can apply Theorem 2 making use of Lemma 1. This gives us Lemma 2.

Lemma 3. $V_{n,k}$ converges in probability to zero uniformly in r as $h, k \longrightarrow \infty$.

Proof. We have, from (8), (6), (16), (15)

$$E(V^2_{n,k;1}) \leq hm^2b^2/ha^2 \leq m^2b^2/ka^2 \longrightarrow 0 \tag{21}$$

and

$$E(V^2_{n,k;2}) \leq (2m+1)rb^2/ha^2 \leq (2m+1)b^2/ha^2 \longrightarrow 0 \tag{22}$$

uniformly in r as $h, k \longrightarrow \infty$. Also

$$E(V^2_{n,k;3}) = s^2_{n,k}\left(\frac{1}{s_n} - \frac{1}{s_{n,k}}\right)^2 = \left(\frac{s_{n,k}}{s_{hk}} \cdot \frac{s_{hk}}{s_n} - 1\right)^2 . \tag{23}$$

Now, from (8) and (3),

$$\frac{s_{hk}^2}{s_n^2} = E\left\{\frac{S_{hk}^2}{s_n^2}\right\} = E\left\{\frac{S_n}{s_n} - V_{n,k;2}\right\}^2$$

$$= E\left\{\frac{S_n^2}{s_n^2}\right\} + \theta\left\{2\sqrt{E\left(\frac{S_n^2}{s_n^2}\right)}\sqrt{E\,(V_{n,k;2}^2)} + E\,(V_{n,k;2}^2)\right\} \longrightarrow 1 \qquad (24)$$

uniformly in r as h,k ⟶ oo. Using (24) and (18) in (23) we get

$$E(V_{n,k;3}^2) \longrightarrow 0 \qquad (25)$$

uniformly in r as h, k ⟶ oo. From (21), (22), (25) and (9) the lemma follows.

We can now complete the proof of Theorem 3 using Lemmas 2, 3 and Theorem 1 and noting (10), the conditions of Theorem 1 being all satisfied.

References

[1] Cramér, H. Random variables and probability distributions (Cambridge, 1937), p. 60.

[2] Dananda, P. H. "Some probability limit theorems with statistical applications," Proc. Camb. Phil. Soc. 49 (1953), 239-46.