# UNBIASED ESTIMATION OF FUNCTIONS OF THE BINOMIAL PROPORTION

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1. Summary. Explicit characterisations are given for the class of estimable functions (EPF) of the parameter p in the Binomial Population and the subclass G of EPF's possessing uniformly minimum variance unbiased estimators (UMVUE). For EPF's not possessing UMVUE, the explicit expression for the unbiased estimator with minimum variance at some $p' \in \Theta$ (the parameter space) is given. The subclass G* of G of EPF's possessing unreasonable UMVUE has been considered when $\Theta = [0,1]$. Such estimators are necessarily inadmissible in the Waldian sense, with any loss function which increases with the absolute error. No attempt is however made either to prove or disprove the admissibility of reasonable UMVUE (in general) in such a case except for f(p) = p with loss function proportional to squared error, when it is shown that the reasonable UMVUE for p is admissible also.

2. Introduction.

In a parametric point estimation problem, we are usually given a sample space X, a family of probability measures $\{\mu_\theta\}$ $\theta \in \Theta$, defined over a field $\mathcal{F}_x$ of sets in X and a single valued function $f(\theta)$ defined on $\Theta$ whose value is desired to be estimated on the basis of a random observation $\underline{x} \in X$.

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An estimator $t(\underline{x})$ of $f(\Theta)$ is a mapping $t(\underline{x})$ of X onto a space T (usually a subset of $\Omega_f$, where $\Omega_f$ is the image of $\Omega$ under the transformation f), specifying for each $\underline{x} \epsilon X$, the number $t(\underline{x}) \epsilon T$, which will be taken as an estimate of $f(\Theta)$, when $\underline{x}$ is observed

The estimator $t(\underline{x})$ is said to be unbiased if

(2.1) $$E \{ t(\underline{x}) \mid \Theta \} = f(\Theta) \quad \text{for all } \Theta \epsilon \Omega .$$

$t(\underline{x})$ is said to be a locally minimum variance unbiased estimator $(LMVUE)_{\Theta'}$ of $f(\Theta)$ if $t(\underline{x})$ is unbiased and if, for some $\Theta' \epsilon \Omega$,

(2.2) $$E \{ (t(\underline{x}) - f(\Theta'))^2 \mid \Theta' \} \leq E \{ (t'(\underline{x}) - f(\Theta'))^2 \mid \Theta' \} ,$$

for all other unbiased estimators $t'(\underline{x})$ of $f(\Theta)$.

If the inequality (2.2) holds for all $\Theta' \epsilon \Omega$, $t(\underline{x})$ is said to be a uniformly minimum variance unbiased estimator (UMVUE) of $f(\Theta)$.

An estimator $t(x)$ of $f(\Theta)$ may be said to be reasonable [3] if, excluding a set of probability measure zero (w.r.t. $\mu_\Theta$ for all $\Theta \epsilon \Omega$ ], T is a subset of $\Omega_f$.

Any function $f(\Theta)$ is said to be estimable if an unbiased estimator $t(x)$ of $f(\Theta)$ exists. Such a function will be called an estimable parametric function (EPF).

3. <u>The Class of EPF for the Binomial Case</u>. Let $(x_1, x_2, \ldots, x_n)$ be n independent observations on the chance variable x, where

(3.1) $$\begin{aligned} \text{Prob} \{ x = 1 \mid p \} &= p, \\ \text{Prob} \{ x = 0 \mid p \} &= 1-p, \end{aligned}$$

and where $p \in \Omega$ [a subset of [0,1]]. It is well known that $r = \sum_{i=1}^{n} x_i$ is a sufficient statistics for p and also that

$$(3.2) \qquad \text{Prob}\left\{r = k \mid p\right\} = \binom{n}{k} p^k (1-p)^{n-k} \quad , \quad k = 0, 1, \ldots, n.$$

Hence, for any EPF $_n f(p)$ (the subscript n is introduced to indicate its possible dependence on the sample size n), there should exist a function t(r) of r satisfying

$$(3.3) \qquad E\left\{t(r) \mid p\right\} = \sum_{k=0}^{n} t(k) \binom{n}{k} p^k (1-p)^{n-k} = f(p) \text{ for } \underline{\text{all}}\ p \in \Omega .$$

It follows from the completeness of the family of measures (3.2) when $\Omega$ consists of (n+1) or more distinct points [Lehmann and Scheffe[4]] that the relation (3.3) should be an identity in p, that is f(p) should be uniquely expressible in the form (3.3) or f(p) should be a ploynomial in p of degree $\leq n$. Conversely, any polynomial in p of degree $\leq n$ is uniquely expressible as

$$f(p) = \sum_{i=0}^{n} C_i \binom{n}{i} p^i (1-p)^{n-i}$$

and admits an unbiased estimator $t(r) = C_r$. Incidentally it also follows from the completeness of the family (3.2) that this estimate is the unique UMVUE.

When $\Omega$ consists of only $m \leq n$ distinct points, (3.3) is equivalent to solving m equations in (n + 1) unknowns; hence any arbitrary function of p is estimable and there are more than one function of the suffieient statistics r, each of which is an unbiased estimator of f(p).

4. The Class G of $EPF_n$ admitting UMVUE. When $\Omega$ consists of $m \geq (n + 1)$ distinct points we have seen in section 3 that any $EPF_n$ admits UMVUE. The only real interest is for the case where $m \leq n$. We shall prove

Theorem 4.1.

For a $\Omega$ which consists of $m \leq n$ distinct points (a) if $\Omega$ excludes the two and points 0 and 1, then G consists only of real constants, (b) if $\Omega$ includes 1 but excludes 0, then G consists only of functions of the form $C_0 + C_1 p^n$, (c) if $\Omega$ includes 0 but excludes 1, then G consists only of functions of the form $C_0 + C_2(1 - p)^n$ and (d) if $\Omega$ includes both 0 and 1, then G consists only of functions of the form $C_0 + C_1 p^n + C_2(1-p)^n$, where $C_0$, $C_1$, $C_2$ are real constants.

Proof of (a) Let the distinct points of $\Omega$ be denoted by $p_0, p_1, \ldots p_{m-1}$. It is given that $0 < p_i < 1$, for $i = 0, 1, \ldots, m - 1$. Let $V_r^0$ be the class of all functions $Z(r)$ of r such that

$$E\{Z(r) \mid p_i\} = 0, \text{ for } i = 0, 1, \ldots, m-1, \tag{4.1}$$

or

$$\sum_{k=0}^{n} Z(k) \binom{n}{k} p_i^k (1 - p_i)^{n-k} = 0, \text{ for } i = 0, 1, \ldots, m-1, \tag{4.2}$$

or

$$\sum_{k=0}^{n} Z(k) a_{ki} = 0 \text{ for } i = 0, 1, \ldots, m - 1, \tag{4.3}$$

where

$$a_{ki} = \binom{n}{k} \theta_i^k \text{ and } \theta_i = \frac{p_i}{1-p_i} . \tag{4.4}$$

Any $Z(r) \epsilon V_r^0$ may be obtained by assigning arbitrary values to $Z(m)$, $Z(m+1)$, ..., $Z(n)$ and solving for the equations

$$(4.5) \qquad \sum_{k=0}^{m-1} Z(k)a_{ki} = -\sum_{k=m}^{n} Z(k)\, a_{ki} \qquad i = 0, 1, \ldots, m-1.$$

It is easy to see that for $i \neq j$, $\Theta_i \neq \Theta_j$, since $p_i \neq p_j$. Hence we have the following expression for the determinant of the matrix of equations (4.5):

$$(4.6) \qquad |a_{ki}| = \prod_{k=0}^{m-1} \binom{n}{k} \prod_{i>j} (\Theta_i - \Theta_j) \neq 0.$$

Hence (4.5) always leads to unique solution, once $Z(m)$, ..., $Z(n)$ are assigned.

As an illustration we may take $Z_1(k) = 0$, for $k = (m+1), \ldots, n$ and $Z_1(m) = 1$. Then $Z_1(0)$, $Z_1(1)$, ... $Z_1(m-1)$ will be given by

$$(4.7) \qquad Z_1(k) = -\binom{n}{m} \sum_{i=0}^{m-1} \Theta_i^m \; a^{ki} \qquad k = 0, 1, \ldots, m-1,$$

where

$$(4.8) \qquad ((a^{ki})) = ((a_{ki}))^{-1}.$$

We shall now prove

<u>Lemma 4.1.</u> $Z_1(k) \neq 0$, for $k = 0, 1, 2, \ldots, m-1$.

Proof of Lemma: Suppose $Z_1(S) = 0$ where $S$ is an integer and $0 \leq S \leq m-1$. $Z_1(0)$, $Z_1(1)$, ..., $Z_1(S-1)$, $Z_1(S+1)$, ..., $Z_1(m-1)$ should therefore satisfy

$$(4.9) \qquad Z_1(0)a_{oi} + \ldots + Z_1(S-1)\, a_{(S-1)i} + Z_1(S+1)a_{(S+1)i} + \ldots + Z_1(m-1)a_{(m-1)i}$$
$$= -\binom{n}{m} a_{mi} \text{ for } i = 0, 1, \ldots, m-1.$$

Hence if A be the matrix $((a_{ki}))_{\substack{0 \leq k \leq m-1; k \neq S \\ 0 \leq i \leq m-1}}$ of equations (4.9) and B be the augmented matrix $((a_{ki}))_{\substack{0 \leq k \leq m; k \neq S \\ 0 \leq i \leq m-1}}$, then rank A should be the same as rank B. But it can be easily verified [2] that

$$(4.10) \qquad |B| = \prod_{\substack{k \neq S \\ 0 \leq k \leq m}} \binom{n}{k} \prod_{\substack{i > j \\ i,j=0,1,\ldots,m-1}} (\theta_i - \theta_j) \ \sigma_{n-S} ,$$

where $\sigma_{n-S}$ is the (n-S)th elementary symmetric function $\Sigma\, \theta_{i_1}, \theta_{i_2}, \ldots \theta_{i_{(n-S)}}$ of the $\theta$'s and where the summation extends over all possible combination $i_1, \ldots i_{(n-S)}$ over integral values $0 \leq i_1 < i_2 < \ldots < i_{(n-S)} \leq m - 1$. Since $\theta_i$'s are positive and distinct it follows that $\sigma_k > 0$ and

$$(4.11) \qquad |B| \neq 0.$$

$\therefore$ rank B = m, where as rank A can only be $\leq (m-1)$. Hence there cannot exist $Z_1(k)$'s satisfying (4.9) which proves lemma 4.1.

It is well known [3] that if t(r) be the UMVUE of its expectation and if $Z(r) \in V_r^0$, then $t(r)\ Z\ (r) \in V_r^0$. Therefore,

$$(4.12) \qquad \sum_{k=0}^{m-1} t(k) Z_1(k)\ a_{ki} = -t(m) a_{mi}, \qquad i = 0, 1, \ldots, m-1,$$

or

$$(4.13) \quad t(k)Z_1(k) = -t(m)\binom{n}{m} \sum_{i=0}^{m-1} \theta_i^m a^{ki}$$

$$= t(m)Z_1(k), \quad \text{for } k = 0, 1, \ldots, m - 1.$$

Since $Z_1(k) \neq 0$, it follows that

$$(4.14) \qquad t(k) = t(m), \quad \text{for } k = 0, 1, \ldots, m - 1.$$

Let us next take $Z_2(r) \in V_r^0$ , where $Z_2(k) = 0$ for $k = 0, (m + 2) \ldots n$ and $Z_2(m+1) = 1$ and when $Z_2(1), \ldots, Z_2(m)$ are the solution of

$$(4.15) \qquad \sum_{k=1}^{m} Z_2(k)\, a_{ki} = -a_{(m+1)i}, \quad \text{for } i = 0, 1, \ldots, m - 1.$$

In a similar way we can show that $Z_2(k) \neq 0$, for $k = 1, 2, \ldots, m$, and, using the fact that $t(r)\, Z_2(r) \in V_r^0$ , we can show that in order for $t(r)$ to be an UMVUE it is necessary that

$$t(k) = t\,(m+1), \qquad k = 1, 2, \ldots, m.$$

Proceeding step by step in an exactly analogous manner we can show that in order that $t(r)$ may be an UMVUE of its expectation, it is necessary that

$$t(0) = t(1) = \ldots = t(n) = C$$

or, in other words, $E\left\{t(r) \mid p_i\right\} = C$, $i = 0, 1, \ldots, m - 1$, which proves Theorem 4.1 (a).

<u>Proof</u> of (d). $\overline{\Omega}$ consists of 0, 1 and $(m - 2)$ other distinct points $p_i (i = \;, 1, \ldots, m - 2)$.

Notice that in this case in order that $Z(r) \epsilon V_r^0$, it follows from (4.2) that $Z(0) = Z(n) = 0$ and $Z(k)$, for $k = 1, \ldots, n - 1$, should satisfy

$$(4.16) \qquad \sum_{k=1}^{n-1} Z(k) \binom{n}{k} p_i^k (1-p_i)^{n-k} = 0 \qquad i = ., 1, \ldots, m - 2$$

Proceeding exactly as in proof of (a) we may show that in order that $t(r)$ may be a UMVUE of its expectation it is necessary that

$$(4.17) \qquad t(1) = t(2) = \ldots = t(n-1) = C_0,$$

whereas, since $Z(0) = Z(n) = 0$, we may assign arbitrary finite values to $t(0)$ and $t(n)$ and still satisfy the condition $t(r)\, Z(r) \epsilon V_r^0$.

If we put $t(0) = C_0 + C_2$ and $t(n) = C_0 + C_1$ we have

$$f(p) = E(t(r) \mid p) = C_0 + C_1 p^n + C_2 (1-p)^n,$$

which proves (d).

The proof of (b) and (c) is similar to that of (d) and is omitted for brevity.

5. <u>A Lower Bound to Variance of Unbiased Estimators of $EPF_n$ $f(p)$ at $p' \epsilon$ ( ) and the $LMVUE_{p'}$ of $f(p)$.</u> When ( ) consists of $m \geq (n + 1)$ points it can be easily verified that for any $EPF_n$ $f(p)$, and any unbiased estimator $t(r)$ of $f(p)$, we have

$$(5.1) \qquad Var\left\{t(r) \mid p'\right\} \geq \left(\frac{\partial f}{\partial p}\right)_{p'}^2 p'q'/n.$$

We shall therefore consider the case $m \leq n$.

Let $p_i$ , for i = 0, 1, ..., m - 1, be the m distinct points of $\Omega$ . From (3.3) we have, if $l_i$ for i = 0, 1, ..., m - 1 be m real constants,

$$(5.2) \qquad \sum_{k=0}^{n} t(k) \sum_{i=0}^{m-1} l_i \left\{ \text{Prob} \quad r = k \mid p_i \right\} = \sum_{i=0}^{m-1} l_i \, f(p_i).$$

or

$$(5.3) \qquad E \left\{ t(r) \;\; \Sigma \, l_i \, \frac{\binom{m}{r} p_i^r (1-p_i)^{n-r}}{\binom{n}{r} p_s^r (1-p_s)^{n-r}} \;\Big|\; p_s \right\} = \sum_{i=0}^{m-1} l_i f(p_i)$$

$0 \leq$ s an integer $\leq (m-1)$.

Hence, using Cauchy Schwartz's inequality, we have

$$(5.4) \qquad E \left\{ t^2(r) \mid p_s \right\} \geq \frac{\Sigma l_i l_j f(p_i) f(p_j)}{\Sigma l_i l_j C_{ij}} \quad ,$$

where

$$(5.5) \qquad C_{ij} = E \left\{ a_{is} (r) \; a_{js}(r) \mid p_s \right\} \; ,$$

and

$$(5.6) \qquad a_{is}(r) = p_i^r (1-p_i)^{n-r} / \; p_s^r (1-p_s)^{n-r} \; .$$

Therefore,

$$(5.7) \qquad E\{t^2(r) \mid p_s\} \geq \sup_{l} \frac{\Sigma l_i l_j f(p_i) f(p_j)}{\Sigma l_i l_j C_{ij}} = \Sigma f(p_i) f(p_j) C^{ij},$$

where $((C^{ij})) = ((C_{ij}))^{-1}$.

$$(5.8) \quad \mathrm{Var}\{t(r) \mid p_s\} \geq \Sigma f(p_i) f(p_j) C^{ij} - f^2(p_s).$$

The above method is essentially due to Bhattacharyya [1]

We shall now obtain the $LMVUE_{p_s}$ of $f(p)$ and show that the limit given by (5.8) is attained.

In (5.4) the sign of equality holds if $\exists$ $t(r)$ and $l_0, l_1, \ldots, l_{m-1}$ such that

$$(5.9) \qquad t(r) = \sum_{i=0}^{m-1} l_i a_{is}(r)$$

and $E\{t(r) \mid p_j\} = f(p_j)$, $j = 0, 1, \ldots, m-1$, that is,

$$(5.10) \qquad E\{\Sigma l_i a_{is}(r) \mid p_j\} = f(p_j), \qquad j = 0, 1, \ldots, m-1.$$

But $E\{a_{is}(r) \mid p_j\} = E\{a_{is}(r)\, a_{js}(r) \mid p_s\} = C_{ij}$ . Therefore,

$$(5.11) \qquad \sum_i l_i C_{ij} = f(p_j), \qquad j = 0, 1, \ldots, m-1,$$

or

$$(5.12) \qquad l_i = \sum_j f(p_j) C^{ij}.$$

Hence the $LMVUE_{p_s}$ of $f(p)$ is given by

(5.13) $$t(r) = \Sigma l_i a_{is}(r) = \sum_{i,j} a_{is}(r) f(p_j) C^{ij}.$$

It can be easily verified that the $LMVUE_{p_s}$ for $f(p)$ is unique.

6. Unreasonable UMVUE. We shall prove

Theorem 6.1. If $\Omega = [0,1]$, the necessary and sufficient condition that the UMVUE of any $EPF_n$ $f(p)$ of the form $\Sigma C_i \binom{n}{i} p^i (1-p)^{n-i}$ be reasonable is that either

(6.1) $$C_0 \leq C_i \leq C_n \qquad \text{or}$$

$$C_0 \geq C_i \geq C_n, \qquad \text{for all } i = 0, 1, 2, \ldots, n.$$

Proof: Sufficiency Let $C_0 \leq C_i \leq C_n$, for all $i = 0, 1, \ldots, n$.

The UMVUE of $f(p)$ is given by

$$t(r) = C_r .$$

We have

(6.2) $$C_0 \leq f(p) = E\{t(r) \mid p\} \leq C_n, \text{ for all } p \in [0, 1].$$

Moreover, we have $C_0 = f(0)$ and $C_n = f(1)$.

Note that $f(p)$ is continuous, and $C_0 = \inf_{p \in [0,1]} f(p)$, $C_1 = \sup_{p \in [0,1]} f(p)$

Hence from (6.2) it is clear that $t(r)$ is reasonable. The same thing could be shown in the other case also.

Necessity. Let there be some integer $j$, such that

$$(6.4) \qquad 0 < j < n, \quad C_j < C_0, \quad C_j < C_n, \quad C_j \leq C_i \text{ for } i = 1, 2, \ldots, n-1$$

$$\text{and } t(r) = C_r.$$

We shall show that $t(j) < \inf_{p \in (\bar{\ })} f(p)$ and hence $t(r)$ is unreasonable.

Since $f(p)$ is continuous, $\inf_{p \in (\bar{\ })} f(p)$ is attained either at 0 or 1 or at some $p^*$, $0 < p^* < 1$.

In the first two cases, we observe that since $f(0) = C_0$ and $f(1) = C_n$, we have

$$(6.5) \qquad t(j) = C_j < \inf_{p \in (\bar{\ })} f(p).$$

In the last case, since $0 < p^* < 1$ and $t(0) > C_j$, $t(n) > C_j$ and $t(r) \geq C_j$ for $r = 1, 2, \ldots, n-1$. we have $\inf_{p \in (\bar{\ })} f(p) = f(p^*) = E\{t(r) \mid p^*\} > C_j = t(j)$.

Hence $t(r)$ is unreasonable.

When $\exists$ integer $j$ such that

$$0 < j < n, \; C_j > C_0, \; C_j > C_n \text{ and } C_j \geq C_i \quad i = 1, 2, \ldots, n-1,$$

it can be shown similarly that

$$t(j) > \sup_{p \in (\bar{\ })} f(p)$$

and hence $t(r)$ is unreasonable.

It is evident therefore that in both the cases $t(r)$ could be uniformly improved (with any loss function which increases with the absolute error) if $t(r)$ is replaced by another estimator $t^*(r)$, where

$$t^*(r) = t(r) \quad \text{for } r \neq j, \text{ and}$$

$$t^*(j) = \underline{f} = \inf_{p \in \underline{(\bar{\ })}} f(p) \text{ in the former case}$$

$$= \bar{f} = \sup_{p \in \underline{(\bar{\ })}} f(p) \text{ in the latter case.}$$

$\therefore$ $t(r)$ is inadmissible in the Waldian sense.

<u>Example</u> (i) $f(p) = \binom{n}{i} p^i (1-p)^{n-i}$, $0 < i < n$.

In this case $t(r) = 0$ if $r \neq i$ $t(i) = 1$.

But $\sup_{p \in \underline{(\bar{\ })}} f(p) = \binom{n}{i}\left(\frac{i}{n}\right)^i \left(\frac{n-i}{n}\right)^{n-i} < 1$.

(ii) $f(p) = pq$, $q = 1-p$.

Here $t(r) = \frac{r(n-r)}{n(n-1)}$ and

$\sup_{p \in \underline{(\bar{\ })}} p\,q = \frac{1}{4}$ .

But $t\left(\frac{n+1}{2}\right) = \frac{n+1}{4n}$ if n is odd,

and $t\left(\frac{n}{2}\right) = \frac{n}{4(n-1)}$ if n is even.

In both the cases the estimates are $> \frac{1}{4}$ .

As an immediate corollary to the above theorem it follows that

Corollary 6.1 If, for some $\delta$, $0 < \delta < 1$, $(\ )$ excludes any one or both the segments $(0, \delta)$ and $(1-\delta, 1)$, the UMVUE of any $EPF_n$ is unreasonable.

7. Admissibility of the UMVUE for p. Here we prove that if $(\ ) = [0,1]$ and the loss function be proportional to the square of the error, the UMVUE for p, given by $t(r) = r/n$, is reasonable and also admissible [5]. That $t(r)$ is reasonable is apparent from the fact that $0 \leq r/n \leq 1$ for all r.

As is clear from the earlier discussions the class of admissible estimators is the subclass of all non-randomised estimators, based on the sufficient statistics r.

Without any loss of generality we may take the loss function to be

$$W(\delta,p) = (p-\delta)^2. \tag{7.1}$$

For any estimator $t(r)$ the corresponding risk function will be givan by

$$R_t(p) = \sum_{r=0}^{n} (p-t(r))^2 \binom{n}{r} p^r (1-p)^{n-r} \tag{7.2}$$

corresponding to any a-priori distribution function $\mu(p)$ of p the average risk is

$$\begin{aligned} R_t[\mu] &= \int_0^1 R_t(p)\,d\mu(p) \\ &= \sum_{r=0}^{n} f(r) \int_0^1 (p-t(r))^2 d\mu^*(p|r) \end{aligned} \tag{7.3}$$

where $f(r)$ is the marginal frequency function for r and $\mu^*(p \mid r)$ is the aposteriori distribution of p for any given r.

Clearly the Bayes solution $t_\mu(r)$ (the estimator for which $R_t[\mu]$ is a minimum) is given by

(7.4) $$t_\mu(r) = \int_0^1 p d\mu^*(p|r).$$

Now if under $\mu(p)$ the whole mass of the a-priori distribution is concentrated at $p = 0$ and $p = 1$, $f(r) = 0$ for all $r \neq 0$ or $n$ and hence from the point of view of minimising $R_t[\mu]$ it is immaterial how we define $t(r)$ for any $r \neq 0$ or $n$ and thus we have no unique Bayes solution. But if otherwise, then $f(r) > 0$ for all $r$ and in this case it is clear that there exists an unique Bayes solution given by (7.4).

Therefore, for any $\mu$ for which the entire mass of the distribution is not concentrated at 0 and 1, the Bayes solution (7.4) is admissible.

Corresponding to $d\mu_\delta(p) = \frac{1}{B(\delta,\delta)} p^{\delta-1}q^{\delta-1} dp$, $\delta > 0$ we have

(7.5) $$t_{\mu_\delta}(r) = t_\delta(r) = \frac{r+\delta}{n+2\delta} ,$$

(7.6) $$R_{t_\delta}(p) = \frac{\delta^2(p^2+q^2)+(n-2\delta^2)pq}{(n+2\delta)^2}$$

and

(7.7) $$R_{t_\delta}[\mu_\delta] = \frac{\delta}{2(n+2\delta)(1+2\delta)} = \frac{\delta}{2n} - 0(\delta^2).$$

The risk function generated by UMVUE $t_0(r) = \frac{r}{n}$ is

(7.8) $$R_{t_0}(p) = \frac{pq}{n} ,$$

and the corresponding average risk

$$(7.9) \qquad R_{t_0}[\mu_\delta] = \frac{\delta}{2n(1+2\delta)} = \frac{\delta}{2n} - O(\delta^2).$$

Now if possible let there exist an estimator t(r) uniformly better than $t_0(r)$, that is, let

$$(7.10) \qquad R_t(p) \leq R_{t_0}(p) \quad \text{for all } p \in (\ )$$

with strict inequality holding for at least one $p \in (\ )$ . Since $R_{t_0}(p) = 0$ at p = 0 and - = 1, it follows that the strict inequality holds at a point different from 0 or 1. Again from (7.2) it is clear that both $R_t(p)$ and $R_{t_0}(p)$ are continuous functions of p. Hence there exists $\epsilon > 0$ under $0 < a < b < 1$ such that

$$(7.11) \qquad R_t(p) < R_{t_0}(p) - \epsilon, \text{ whenever } a \leq p \leq b.$$

From (7.10)and (7.11) it follows that

$$(7.12) \qquad R_t[\mu_\delta] < R_{t_0}[\mu_\delta] - \epsilon \int_a^b d\mu\delta(p).$$

Now

$$(7.13) \qquad \int_a^b d\mu\delta(p) = \frac{\Gamma 2\delta}{(\Gamma\delta)^2} \int_a^b p^{\delta-1}(1-p)^{\delta-1}\,dp$$

$$= \frac{\delta^2}{2\delta}\frac{\Gamma 1+2\delta}{(\Gamma 1+\delta)^2} \int_a^b p^{\delta-1}(1-p)^{\delta-1}\,dp$$

$$= \frac{\delta}{2} \int_a^b [p(1-p)]^{-1} \, dp + O(\delta^2).$$

From (7.9), (7.12) and (7.13) it follows that

$$R_t[\mu_\delta] < \frac{\delta}{2n} - \frac{\delta\epsilon}{2} \int_a^b [p(1-p)]^{-1} \, dp + O(\delta^2).$$

$\therefore$ for sufficiently small $\delta$, $R_t[\mu_\delta] < R_{t_\delta}[\mu_\delta]$ which contradicts the fact that $t_\delta$ is the Bayes solution w.r.t. $\mu_\delta(p)$.

Hence $t_0(r) = \frac{r}{n}$ is admissible.

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