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SOME USES OF QUASI-RANGES

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1. Summary. Confidence intervals for, and tests of hypotheses about, the difference of two quantiles (of the same distribution) are obtained, using one or two properly chosen quasi-ranges. Consistency (of the estimates and tests) is proved. Applications are discussed to the standard deviation of a normal distribution and the parameter of a binomial distribution.

2. Introduction. Let a population (not necessarily continuous) be given with cdf (cumulative distribution function) $F(x)$. For a given p, $0 < p < 1$, any ξ_p satisfying

(1) $$F(\xi_p - 0) \leqq p \leqq F(\xi_p)$$

is called a quantile of order p (or p-quantile) of the given distribution. Usually one of them is chosen, by one way or another, as the p-quantile. Let ξ_q be the q-quantile, where $p < q < 1$. Suppose that a sample of size n is drawn and x_r and x_s are the r-th and s-th order statistics (in ascending order of magnitude). Intuitively, it seems obvious that if n is sufficiently large, and r is small compared with n while s is not much less than n, then $x_s - x_r$ is likely to be relatively large, consequently the event $x_s - x_r \geqq \xi_q - \xi_p$ occurs with probability close to 1. An interesting question then arises : for given $0 < p < q < 1$, and $0 < \alpha < 1$, what would be a nice way, assuming there exists at least one way, for choosing two order statistics x_r and x_s from a sample of given size n, such that

(2) $$P(x_s - x_r \geqq \xi_q - \xi_p) \geqq 1 - \alpha,$$

where P(E) denotes the probability of the event E ? One may also ask similar questions concerning

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(2') $$P(x_{s'} - x_{r'} \leqq \xi_q - \xi_p) \geqq 1 - \alpha \text{ , and}$$
$$P(x_{s'} - x_{r'} \leqq \xi_q - \xi_p \leqq x_s - x_r) \geqq 1 - 2\alpha.$$

These questions, of course, are rather loosely stated. For example, the meaning of the word " nice " is not specified. Besides, there are at least two different approaches to the problems, namely: parametric and non-parametric. Incidentally, in a sample of size n, $x_n - x_1$ is known as the range, and $x_{n-r+1} - x_r$, where $1 < r \leqq (n + 1) / 2$, a quasi-range or the r-th range, e.g., [4]. Here, however, any $x_s - x_r$, $s \geqq r$, will be called a quasi-range.

In this paper, an attempt is made to obtain some distribution-free methods, optimal in a certain sense, of choosing r,s,r',and s' such that (2) and (2') hold for given p,q,α, and n. In section 3, lower bounds (4), (5), (6), (14), and (15), are obtained for the probabilities given in (2) and (2'). Corresponding to the bounds (14) and (15), two integers k and k' are defined by (19) and they can be found by using [5]. If r, s, r', and s' are defined by (8) and (9) in terms of these k and k', then the corresponding order statistics satisfy (2) and (2'). When the parent distribution meets certain continuity requirements, we are able to show (section 3, Theorem) that both $x_s - x_r$ and $x_{s'} - x_{r'}$, chosen in the way just described, are consistent estimates of $\xi_q - \xi_p$ and provide consistent tests with respect to various hypotheses and alternatives concerning $\xi_q - \xi_p$.

For a normal parent distribution with variance σ^2 , if we choose $q = 1 - p$, then $\xi_q - \xi_p = 2a\sigma$, where $\Phi(-a) = p$, and $\Phi(x)$ is defined by (17). To obtain, e.g., a 100 (1 - 2α) per cent confidence interval for σ , one proceeds as follows: Take any p ($0 < p < 1$). Let

(3) $$w_r = (x_{n-r+1} - x_r) / 2a , \quad w_{r'} = (x_{n-r'+1} - x_{r'}) / 2a,$$

where $\Phi(-a) = p$ and r and r' are respectively the largest and smallest integers which satisfy (26) with $q = 1 - p$. Then $P(w_{r'} \leqq \sigma \leqq w_r) \geqq 1 - 2\alpha$. Here, a question follows naturally: p is taken arbitrarily, but, does the value of p effect the efficiencies of the corresponding estimates and tests ?; and if so, how to choose p to maximize the efficiencies ? To these questions, some answers are given in section 5. The efficiencies of w_r and $w_{r'}$ relative to S' of (29), are defined by (30) in the conventional way, i.e., as the ratios of the variance of S' to the variances of $w_r\sigma/E(w_r)$ and $w_{r'}\sigma/E(w_{r'})$ which are unbiased estimates of σ. However, exact values of the efficiencies of w_r and $w_{r'}$ are hard to obtain, because the variances of w_r and $w_{r'}$ usually cannot be found without very laborious computations. For large samples, a method of approximation is suggested for obtaining the expectations and variances of w_r and $w_{r'}$, and the efficiency of w_r (and similarly $w_{r'}$) is approximated by the asymptotic efficiency of $(x_{\nu'} - x_{\mu'})/2a'$, where $\mu' = [np'] + 1$, $\nu' = [nq'] + 1$, $(r - 1)/n < p' < r/n$, $q' = 1 - p'$, and $\Phi(-a') = p'$. Maximum efficiencies of w_r and $w_{r'}$ are about .65. A method is suggested of finding, for given n and α, the values of p which maximize respectively the efficiencies of w_r and $w_{r'}$. Table 2 in section 5 is given for illustration.

For samples of moderate sizes, we made no attempt to solve the problem of " how to find p ". There is evidence, however, that if p is properly chosen, the efficiencies of the corresponding w_r and $w_{r'}$ are reasonably high. We find, e.g., for sample sizes around 20 and 30, if p = .25, the efficiencies of the corresponding w_r and $w_{r'}$ are about .70 (Tables 3 and 4).

In section 6, we discuss some applications to the parameter θ of a binomial distribution $f(x) = \theta^{1-x}(1-\theta)^x$, $x = 0, 1$. Consistent tests are obtained for testing $p \leq \theta \leq q$ and $\theta = p$ with respect to the alternatives $\theta < p$ or $\theta > q$, and $\theta \neq p$. For testing $\theta = p$, our method improves a known one [7, p. 14], in making unique the choice of critical region and the test consistent with respect to $\theta \neq p$.

Finally we note that the idea is by no means new of using quasi-ranges, as well as similar statistics, in estimations and testing of hypotheses. In fact, much work has been done along these lines. For references, see [1], [4], and those cited there.

3. Consistency. Let a population be given with cdf $F(x)$. Suppose that for given $0 < p < q < 1$, the corresponding quantiles of orders p and q, ξ_p and ξ_q, are uniquely defined. Let $x_1 \leqq x_2 \leqq \dots \leqq x_n$ be the order statistics corresponding to a random sample of given size n. Then,

Lemma 1. If $1 \leqq r \leqq r' \leqq s' \leqq s \leqq n$ are integers, then

$$(4) \qquad L = 1 - P(x_s < \xi_q) - P(x_r > \xi_p) \leqq P(x_s - x_r \geqq \xi_q - \xi_p) \leqq P(x_s \geqq \xi_q) + P(x_r < \xi_p) = U ;$$

$$(5) \qquad L' = 1 - P(x_{s'} > \xi_q) - P(x_{r'} < \xi_p) \leqq P(x_{s'} - x_{r'} \leqq \xi_q - \xi_p) \leqq P(x_{s'} \leqq \xi_q) + P(x_{r'} > \xi_p) = U' ;$$

$$(6) \qquad L + L' - 1 \leqq P(x_{s'} - x_{r'} \leqq \xi_q - \xi_p \leqq x_s - x_r) \leqq U + U' - 1.$$

Proof. Let $P(A, B)$ denote the probability of simultaneous occurrence of the events A and B. Clearly $P(x_s - x_r \geqq \xi_q - \xi_p) \geqq P(x_s \geqq \xi_q , x_r \leqq \xi_p) \geqq P(x_s \geqq \xi_q) + P(x_r \leqq \xi_p) - 1$. Therefore we have $L \leqq P(x_s - x_r \geqq \xi_q - \xi_p)$. Likewise we obtain the other inequalities.

Remark. For samples drawn from a binomial distribution, say with pdf (probability density function) $f(x) = p^{1-x}(1-p)^x$, $x = 0, 1$, if we define $\xi_p = 0$ and $\xi_q = 1$, then the lower bounds in Lemma 1 are actually attained.

Lemma 2. For all integers k and k'; $0 \leqq k, k' \leqq n - 1$, and

$$(7) \qquad c = q - p ,$$

choose

$$(8) \qquad r = [(n - k)p/(1 - c)] + 1 , \quad s = r + k ;$$

$$(9) \qquad r' = [(n - k')p/(1 - c)] + 1 , \quad s' = r' + k' ,$$

where $[a]$ denotes the integral part of a. Then the corresponding RHS (right hand sides) of (14) and (15), lower bounds for L and L', are respectively non-decreasing and non-increasing functions of k and k'. Further, let c_1 and c_2 be given such that $0 < c_1 < c < c_2 < 1$. If $k = [nc_2]$ and k'

$= [nc_1]$, then

(10) $$\lim_{n \to \infty} L = \lim_{n \to \infty} L' = 1.$$

On the other hand, if $k = [nc_1]$ and $k' = [nc_2]$, and if, in addition, the parent cdf $F(x)$ is continuous at $x = \xi_p$ and $x = \xi_q$, then

(11) $$\lim_{n \to \infty} U = \lim_{n \to \infty} U' = 0.$$

Proof. It can be seen that $1 \leqq r, r' \leqq n$ and $1 \leqq s, s' \leqq n$, e.g., $s \leqq n$ because $s < (n - k) p / (1 - c) + k + 1 < n + 1$. Hence L and L', as well as the RHS of (14) and (15), are well defined functions of k and k'. Now

(12) $$P(x_s < \xi_q) = 1 - B_n(s - 1, F(\xi_q - 0)) \leqq 1 - B_n(s - 1, q).$$

where

(13) $$B_n(r, p) = \sum_{i=0}^{r} \binom{n}{i} p^i (1 - p)^{n-i}$$

is, for fixed n and r, a decreasing function of p , $0 < p < 1$. Hence

(14) $$L \geqq B_n(s - 1, q) - B_n(r - 1, p);$$

(15) $$L' \geqq -B_n(s' - 1, q) + B_n(r' - 1, p).$$

Now r is a non-increasing function of k. But if k is increased by 1 , r is decreased at most by 1 . Hence s is a non-decreasing function of k , consequently so is the RHS of (14). In a similar way, we show that the RHS of (15) is a non-increasing function of k'.

It is well known, [2, p. 200] and [3, p. 193] that, as n tends to ∞,

(16) $$B_n(r, p) - \Phi(x) \longrightarrow 0,$$

uniformly in r , $0 \leqq r \leqq n$, where $x = (r - np)[np(1 - p)]^{-1/2}$ and

(17) $$\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} e^{-t^2/2} dt.$$

As n tends to ∞ , it can be shown that if $k = [nc_2]$, where $c_2 > c$ and < 1 is fixed, and r and s are defined by (8),

(18) $\quad n^{-1/2}(r-1-np) \longrightarrow -\infty, \quad n^{-1/2}(s-1-nq) \longrightarrow \infty,$

for $(r-1-np)/n = (c-c_2)p/(1-c) + O(n^{-1})$ and $(s-1-nq)/n = b + O(n^{-1})$ where $b = (1-c_2)p/(1-c) + c_2 - q$ $= (c_2-c)(1-q)/(1-c) > 0$. Combining (14), (16), and (18), we obtain $\lim_{n \to \infty} L = 1$. Likewise we prove the rest of (10) and (11).

Lemma 3. Corresponding to given n and α, $0 < \alpha < 1$, let

(19) $\quad k\ (k')$ = the least (greatest) integer between 0 and $n-1$ such that, when r, s, r', and s' are defined accordingly by (8) and (9), the RHS of (14) and (15) $\geq 1-\alpha$.

(From Lemma 2, such k and k' exist for any α, $0 < \alpha < 1$, if n is sufficiently large). For fixed p_i and q_i, $i = 1, 2$, where $p_1 < p < p_2$ and $q_1 < q < q_2$, define

(20) $$r_i = [np_i] + 1 \ , \quad s_i = [nq_i] + 1 \ .$$

Assume that $F(x)$ is continuous at $x = \xi_p$ and $x = \xi_q$. Then, for sufficiently large n,

(21) $$r_1 \leq r,\ r' \leq r_2 \ , \qquad s_1 \leq s,\ s' \leq s_2 \ ,$$

where r, s, r', and s' are defined by (8) and (9) with k and k' given by (19).

Proof. By (10), if k is the integer defined by (19), then $k \leq [nc_2]$ for any fixed $c_2 > c$, provided that n is sufficiently large. Choose c_2 sufficiently close to c, then $r \geq n(1-c_2)p/(1-c)$ $\geq np_1 + 1 \geq r_1$ and $s \leq n[(1-c_2)p/(1-c) + c_2] \leq nq_2 \leq s_2$. Similarly following (11), we have $r \leq r_2$ and $s \geq s_1$.

The following lemma is a known fact [2, p. 369]. We state it without a proof.

Lemma 4. Let a continuous population be given with cdf $F(x)$ and pdf $f(x)$. Suppose that for $0 < p < q < 1$, ξ_p and ξ_q are uniquely defined and $f'(x)$, the derivative, exists and is continuous in some neighborhoods of $x = \xi_p$ and $x = \xi_q$. If $\mu = [np] + 1$ and $\nu = [nq] + 1$, (we assume that np and nq are not integers), and x_μ and x_ν are the corresponding order statistics in a sample of size n, then as $n \to \infty$, $x_\nu - x_\mu$ has an asymptotically normal distribution with mean $\xi_q - \xi_p$ and variance

(22) $$\sigma^2(p, q) = \frac{1}{n} \left[\frac{p(1-p)}{f^2(\xi_p)} + \frac{q(1-q)}{f^2(\xi_q)} - \frac{2p(1-q)}{f(\xi_p)\,f(\xi_q)} \right].$$

As a consequence of the previous lemmas, we have

Theorem. Let a continuous population be given whose cdf and pdf satisfy the continuity conditions stated in Lemma 4. For given n and α, let k and k' be the integers defined by (19), and r, s, r', and s' be defined by (8) and (9). If x_r etc. are respectively the r-th etc. order statistics in a sample of size n, then both $x_s - x_r$ and $x_{s'} - x_{r'}$ are consistent estimates of $\xi_q - \xi_p$ in the sense that $\lim_{n \to \infty} (x_s - x_r) = \lim_{n \to \infty} (x_{s'} - x_{r'}) = \xi_q - \xi_p$ in probability.

Proof. Following Lemmas 3 and 4, for given $\delta, \varepsilon > 0$, if p_1 and q_2, and δ' are properly chosen and n is sufficiently large, then $P(x_s - x_r > \xi_q + \xi_p + \delta) \leqq P(x_{s_2} - x_{r_1} > \xi_q - \xi_p + \delta) \leqq P(x_{s_2} - x_{r_1} > \xi_{q_2} - \xi_{p_1} + \delta') \leqq \varepsilon$. In a similar way we easily complete the proof.

4. Inference.

In section 3 we proved among other things the existence of k and k', defined by (19), for sufficiently large n. To actually find such k and k', there is no difficulty either. As is well known, binomial cdfs can be evaluated by means of [5], through the relationship

(23) $$B_n(r-1, p) = 1 - I_p(r, n-r+1),$$

where $I_p(r, s) = \int_0^p x^{r-1}(1-x)^{s-1}dx \Big/ \int_0^\infty x^{r-1}(1-x)^{s-1}dx$

is the incomplete beta function. Therefore for a given α and n, we can easily find k and k', defined by (19) and the corresponding r, s, r', and s' defined by (8) and (9). Of course, for a small α, in order that such k and k' exist, n has to be large.

A. Confidence intervals.

When r, s, r', and s' arechosen in the way previously described, $x_s - x_r$ and $x_{s'} - x_{r'}$ are respectively confidence upper and lower bounds for $\xi_q - \xi_p$ with the same confidence coefficient $1 - \alpha$, and $(x_{s'} - x_{r'} , x_s - x_r)$ is a confidence interval with confidence coefficient $1 - 2\alpha$. If, in addition, the continuity conditions, stated in Lemma 4, are satisfied by the parent cdf and pdf, then both $x_s - x_r$ and $x_{s'} - x_{r'}$ are consistent estimates of $\xi_q - \xi_p$.

B. Tests of hypotheses.

Let

(24) $$H_o : \xi_q - \xi_p = d ,$$

Then the tests, using as critical regions:

$$x_s - x_r < d \; ; \qquad x_{s'} - x_{r'} > d \; ;$$
$$x_s - x_r < d \text{ or } x_{s'} - x_{r'} > d ,$$

are respectively: 1. of significance levels α, α, 2α ; and 2. consistent with respect to the alternatives $\xi_q - \xi_p < d$; $\xi_q - \xi_p > d$; and $\xi_q - \xi_p \neq d$, provided that the continuity conditions in Lemma 4 are satisfied. A test, for testing a given hypothesis H_o , is said to be consistent with respect to a certain alternative, if its power, when the alternative is true, tends to 1 as sample size tends to infinity.

C. A special case: $q = 1 - p$.

If, in particular, $q = 1 - p$, then, following (8) and (9), $r = [(n - k) / 2] + 1$, $s = r + k$, etc.. If we use only those k and k' for which $(n - k)/2$ and $(n - k')/2$ are not integers, then

(25) $$s = n - r + 1 , \text{ and } s' = n - r' + 1 .$$

From (14) and (15), it follows that $L \geqq 1 - 2I_q(n - r + 1 , r)$ and $L' \geqq 2I_q(n - r' + 1 , r') - 1$. For a given n , we use [5] to find the largest r and the smallest integer r' for which

(26) $I_q(n-r+1, r) \leqq \alpha/2$, $I_q(n'-r'+1, r') \geqq 1-\alpha/2$.

Then the corresponding L and $L' \geqq 1-\alpha$. Table 3 in section 5 is obtained in this way. For example: if $n = 30$, $\alpha = .05$, and $q = .75$, then 3 is the largest r satisfying the first inequality in (26). From (25), $s = 28$. Thus $P(x_{28} - x_3 \geqq \xi_{.75} - \xi_{.25}) \geqq .95$. Likewise, $P(x_{17} - x_{14} \leqq \xi_{.75} - \xi_{.25} \leqq x_{28} - x_3) \geqq .95$.

For large n , a fairly good approximation for $B_n(r,p)$ is $\Phi((r + 1/2 - np)/[np(1-p)]^{1/2})$ where $\Phi(x)$ is given by (17) . In this case, the largest integer r and smallest integer r' for which (26) holds are the largest integer r and smallest r' satisfying

(27) $r \leqq 1/2 + np - a_\alpha [np(1-p)]^{1/2}$, $r' \geqq 1/2 + np + a_\alpha [np(1-p)]^{1/2}$,

where $\Phi(-a_\alpha) = \alpha/2$. Table 2 in section 5 is computed in this way. We see that, with the exception of a few cases, $r' = 2np + 1 - r$.

5. The standard deviation of a normal distribution.

Suppose that the parent distribution is normal with mean ξ and variance σ^2 . Then, for each p , $0 < p < 1$, and $q = 1 - p$,

(28) $$\sigma = (\xi_q - \xi_p)/2a ,$$

where $\Phi(-a) = p$ and $\Phi(x)$ is given by (17). Obviously, as was pointed out in section 2 , confidence intervals for, and tests of hypotheses about σ , can be readily obtained from those concerning $\xi_q - \xi_p$. A new problem arises , however, as to how to choose p to maximize the efficiencies of the corresponding w_r and $w_{r'}$ in (3), since they, for given n and α , are determined by p . We shall now consider this problem.

A familiar statistic used to estimate and test σ is

(29) $$S' = \left(\frac{n-1}{2}\right)^{1/2} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n/2)} S ,$$

where $S^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/(n-1)$ and $\bar{x} = \sum_{i=1}^{n} x_i/n$. The relative efficiency of w_r (and similarly $w_{r'}$) with respect to S' is defined to be

(30) $$e = 100 \text{ var } S' / \text{var } [w_r \sigma / E(w_r)] .$$

The variance of S' is easy to compute and for large n is about $\sigma^2/2n$ [2, p. 484]. The variances of w_r and $w_{r'}$, for arbitrary n , r , and r' are, on the contrary, difficult to obtain, (Recently general formulas have been derived by Ruben [6, p. 213]), consequently so is the value of e . When the sample size is large, and $(r-1)/n$ and r/n are fairly close to each other, we suggest the following method of approximation for obtaining $E(w_r)$, var w_r , etc.. In section 3, Lemma 4 , it was stated that if $0 < p' < q' < 1$, and $\mu' = [np'] + 1$ and $\nu' = [nq'] + 1$, then the distribution of $x_{\nu'} - x_{\mu'}$ is asymptotically normal. For large samples, the mean and variance of the asymptotic distribution may be used as approximations to the true mean and variance of $x_{\nu'} - x_{\mu'}$. Now if p' and q' are such that $(r-1)/n < p' < r/n$ and $q' = 1 - p'$, then $x_r = x_{\mu'}$ and $x_{n-r+1} = x_{\nu'}$. Therefore an approximation to the denominator of (30) is the variance of the asymptotic distribution of $(x_{\nu'} - x_{\mu'})/2a'$, where $\Phi(-a') = p'$. If var S' is replaced by $\sigma^2/2n$, then an approximation to e is ,following Lemma 4 ,

(31) $$a'^2 \phi^2(a') / p'(1-2p') , \text{ where } \phi(x) = (2\pi)^{-1/2} e^{-x^2/2} ,$$

which, in fact, is the asymptotic efficiency of $(x_{\nu'} - x_{\mu'})/2a'$. Previous numerical investigations, e.g., [4] , revealed that (31) has a maximum value .65 at $p' = .07$ approximately. Therefore, for large n , the efficiency of w_r is maximized if p is so chosen that r/n is about $.07n$. For given n and α , this value of p can be found approximately by equating to $.07n$ the RHS of the first inequality in (27) and then solving a quadratic equation in p . In a similar way we can find the optimal p with respect to $w_{r'}$. Obviously , values of p which maximize the efficiencies of the corresponding w_r's are larger than .07 , and those maximize the efficiencies of the $w_{r'}$'s , smaller. However , as sample size increases , both r/n and r'/n tend to p (Lemma 3).

Therefore eventually the efficiencies of w_r and $w_{r'}$ are maximized if $p = .07$.

In the following, Table 1 gives, for different values of p, the asymptotic efficiencies (31) of $(x_\nu - x_\mu)/ 2a$, where $\mu = [np] + 1$, $\nu = [nq] + 1$, $q = 1 - p$, and $\Phi(-a) = p$. As was mentioned before, the maximum value .65 occurs at $p = .07$. In the neighborhood of $p = .07$, the variation is small of the corresponding efficiency. In Table 2 are given, for different choices of n and α, the values of p, and corresponding to each p, the subscripts r and r' of w_r and $w_{r'}$. They are not found, however, by the method of solving a quadratic equation mentioned in the previous paragraph. Instead, four different values of p are chosen, then for each p, the corresponding r and r' are found by (27). For most combinations of n and α given in that table, at least one of the four p's will provide a w_r or $w_{r'}$, with an efficiency fairly close to .65. For example, if $n = 500$, then $P((x_{464} - x_{37}) / 2.56 \geqq \sigma) \geqq .95$, ($p = .10$), and the efficiency of the statistic is about .65.

For samples of moderate sizes, Cadwell [1, p. 609] obtained recently the efficiencies e (30), for $n = 10, 20, \ldots, 60$, and $r = 1, 2, 3$. Using his results, we see (Tables 3 and 4) that if $p = .25$, then the corresponding w_r's have efficiencies around .70 for $n = 20, 30$.

Table 1. Aymptotic efficiency of $(x_\nu - x_\mu)/ 2a$

p	.01	.02	.03	.04	.05	.06	.07
AE	.39	.51	.58	.62	.64	.65	.65
p	.08	.09	.10	.11	.12	.13	.14
AE	.65	.64	.63	.62	.61	.59	.57

$\mu = [np] + 1$, $\nu = [nq] + 1$, $q = 1 - p$, $\Phi(-a) = p$.

Table 2 The Largest r and Smallest r'

p = .04, .07, .10, and .13

2a = 3.50, 2.95, 2.56, and 2.25

n \ α	.25		.10		.05		.025		.01		.005	
1000	33	48	30	51	28	53	26	55	24	57	23	58
	61	80	57	84	54	87	52	89	49	92	47	94
	89	112	84	117	81	120	79	122	76	125	73	128
	118	143	113	148	109	152	106	155	103	158	100	161
500	15	26	13	28	11	30	10	31	9	32	8	33
	28	43	26	45	24	47	22	49	20	51	19	52
	42	59	39	62	37	64	35	66	33	68	31	70
	56	75	53	78	50	81	48	83	46	85	44	87
200	5	12	3	14	3	14	2	15	1	16	0	17
	10	19	8	21	7	22	6	23	5	24	4	25
	15	26	13	28	12	29	10	31	9	32	8	33
	21	32	18	35	17	36	15	38	14	39	13	40
100	2	7	1	8	0	9	0	9	0	10	0	11
	4	11	3	12	2	13	1	14	0	15	0	15
	7	14	5	16	4	17	3	18	2	19	2	19
	9	18	7	20	6	21	5	22	4	23	3	24

n = Sample size   α = Level of significance

Table 3. The Largest r and Smallest r'

$p = .25 \qquad 2a = 1.35$

n \ α	.25		.10		.05		.025		.01		.005	
10	1	5	-	-	-	-	-	-	-	-	-	-
20	3	8	2	9	2	10	1	-	1	-	-	-
30	5	11	4	13	3	13	3	14	2	15	2	-
40	7	14	6	16	5	17	4	17	4	18	3	19
50	9	17	8	19	7	20	6	21	5	22	5	23

Table 4. Efficiencies of the Statistics

$(x_{n-r+1} - x_r) / 2a$ in Table 3

n \ α	.25	.10	.05	.025	.01	.005
10	.85	*	*	*	*	*
20	.66	.73	.73	.70	.70	*
30	*	*	.70	.70	.70	.70
40	*	*	*	*	*	.69

6. The parameter of a binomial distribution.

We shall consider a practical example.

To classify lots of, say, factory products, according to the proportion θ of defectives contained therein, the following way might be of practical interest. If $\theta > q$, a prescribed quantity, then the lot will be rejected for containing too many defectives. If $p \leqq \theta \leqq q$, where $p < q$ is another given quantity, then the lot will be accepted as of average quality. But if $\theta < p$, then the lot will be considered as superior (and probably will be sold at a higher price). While the problem, to decide between three alternatives, i.e., $\theta < p$; $p \leqq \theta \leqq q$; and $\theta > q$, is actually a 3-decision one, we may, perhaps as a preliminary approach, start with testing the hypothesis

(34) $$H_o: \quad p \leqq \theta \leqq q.$$

Corresponding to a random observation taken from a given lot, let $x = 0$ if it is found defective, and $x = 1$ if good. Then $f(x) = \theta^{1-x}(1-\theta)^x$; $x = 0, 1$, where $f(x)$ denotes the probability of obtaining an observation x.
For this distribution, whenever H_o is true, we may define, following (1), $\xi_p = 0$ and $\xi_q = 1$. Let x_r and x_s be the r-th and s-th order statistics in a sample of size n, then $P(x_s - x_r \geqq \xi_q - \xi_p \mid H_o) = P(x_s = 1, x_r = 0 \mid H_o) = 1 - P(x_s = 0 \text{ or } x_r = 1 \mid H_o)$, where $P(E \mid H_o)$ denotes the probability of E given that H_o is true. Following the general method given in section 3, a pair of integers r and s can be determined, (8) and (19), for given n and α such that

(35) $$P(x_s = 0 \text{ or } x_r = 1 \mid H_o) \leqq \alpha .$$

Therefore the test using as critical region

(36) $$x_s = 0 \text{ or } x_r = 1 ,$$

where r and s are given by (35), is of level of significance α for testing H_o of (34).

We showed (section 3, Lemma 3) that for arbitrary but fixed $p_1 < p$ and $q_2 > q$, if $r_1 = [np_1] + 1$ and $s_2 = [nq_2] + 1$, then, when n is sufficiently large, $r \geqq r_1$ and $s \leqq s_2$, where r and s are given in (35). Using (16), we easily show that the test defined by (36) is consistent with respect to the alternative $\theta < p$ or $\theta > q$.

The hypothesis

$$H_1 : \theta = p \tag{37}$$

may be considered as a limiting case of (34) as $q \to p$. To test H_1 at a prescribed level of significance α, we use critical region (36) with $s = r + k$, and $r = [(n - k)p] + 1$, where k is the least integer for which

$$B_n(s - 1, p) - B_n(r - 1, p) \geqq 1 - \alpha. \tag{38}$$

If, in particular, $p = 1/2$, then, from (25) and (26), $s = n - r + 1$ and r is the largest integer for which $B_n(r - 1, 1/2) \leqq \alpha/2$. It should be noted that the tests just suggested are not new [7, p. 14]. However, for testing H_1 when $p \neq 1/2$, according to the known method, one chooses a pair of r and s which satisfy (38) and uses the corresponding region (36) as critical region ; and if there are more than one pair of r and s satisfying (38), then selection should be made " in accordance with practical consideration " [7, p.15]. Our method removes this somewhat ambiguous way of selection, since r, s and k are uniquely determined for given n, p, and α. Further, the test thus obtained is consistent with respect to the alternative $\theta \neq p$.

See [7], for applications to interval estimations and testing of hypotheses of quantiles of an unknown distribution.

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