ON MINIMUM VARIANCE ESTIMATION OF LOCATION AND SCALE PARAMETERS

by

Sujit Kumar Mitra

This research was supported in part
by the United States Air Force, through
the Office of Scientific Research of
The Air Research and Development Command.

Institute of Statistics
Mimeograph Series No. 126
February, 1955

ON MINIMUM VARIANCE ESTIMATION OF LOCATION AND SCALE PARAMETERS

by

Sumit Kumar Mitra
Indian Statistical Institute and University of North Carolina

1. Summary. It is shown that uniformly minimum variance unbiased estimators (UMVUE) of location and scale parameters, when they exist, satisfy Pitman's configurational condition [4]. As an illustration the case of rectangular population with range depending on location and scale parameters has been considered in some detail.

2. Introduction. We often come across point estimation problems where the statistical distribution concerned does not admit UMVUE for the unknown parameters. In such cases we require an alternative set of criteria for chosing an estimator possessing locally desirable properties.

For purposes of estimating the unknown location parameter λ and the scale parameter σ in any population, Pitman [4] considers only such functions of the sample observations $\underline{x} = (x_1, x_2, \ldots, x_n)$ which respectively satisfy the configurational conditions.

$$(2.1) \qquad b(kx_1 + c, kx_2 + c, \ldots, kx_n + c) = kb(x_1, x_2, \ldots, x_n) + c$$

$$(2.2) \qquad s(kx_1 + c, kx_2 + c, \ldots, kx_n + c) = ks(x_1, x_2, \ldots, x_n)$$

for all real c and $k > 0$.

Among all functions which satisfy the above conditions he picks out as a suitable estimator the one which has the minimum second moment about the unknown parameter.

---

1. This research was supported in part by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command.

When σ is known the condition (2.1) is replaced by

$$b(x_1 + c, x_2 + c, \ldots, x_n + c) = b(x_1, x_2, \ldots, x_n) + c. \tag{2.3}$$

Similarly when λ is known, (2.2) is replaced by

$$s[k(x_1-\lambda) + \lambda, k(x_2-\lambda) + \lambda, \ldots, k(x_n-\lambda) + \lambda] = k\,s(x_1, x_2, \ldots, x_n). \tag{2.4}$$

The above configurational conditions although appearing to be arbitrary at first are in fact no less general than the condition of unbiasedness.

Girshick and Savage [2] have considered estimators with the translation property (2.3) and have shown that if λ is a translation (or location) parameter, and is the only unknown parameter in the distribution, and the loss function be of the form $L(b - \lambda)$, then, under certain further restrictions on L, the optimal estimator in this class of pure invariant estimators (which incidentally happens to be the best Pitman estimator) is unbiased and minimax.

Dealing with a positive random variable whose distribution is known except for some multiplicative constant σ, (the known location parameter λ may be assumed to be zero) Blackwell and Girshick [1] have shown that, if the loss on estimating σ by s depend only on the relative error σ/s, there always exists a minimax estimator for σ which is a pure invariant estimator having the multiplicative property (2.4). Nothing, however, is known about the admissibility of such estimators.

3. The UMVUE of Location and Scale Parameters. We shall prove

Theorem 3.1. The UMVUE for location and scale parameters (when they are the only unknown parameters) in any statistical distribution, if they exist, satisfy Pitman's configurational conditions.

Let us consider the case where both λ and σ are unknown and let the joint probability distribution of $\underline{x} = (x_1, x_2, \ldots, x_n)$ be given by

$$(3.1) \qquad \text{Prob}\left\{x_1 \leq X_1, x_2 \leq X_2, \ldots, x_n \leq X_n\right\} = F\left(\frac{X_1-\lambda}{\sigma}, \frac{X_2-\lambda}{\sigma}, \ldots, \frac{X_n-\lambda}{\sigma}\right)$$

We shall first prove

Lemma 3.1. If $\delta(\underline{x})$ be the UMVUE for any parametric function $e(\lambda,\sigma)$, $\delta(k\underline{x} + c\epsilon)$ is the UMVUE for $e(k\lambda + c, k\sigma)$ where k and c are real constants and $k > 0$ and ϵ is the n-vector (1, 1, ..., 1)

It is easy to verify that the joint probability distribution of $\underline{y} = (y_1, y_2, \ldots, y_n) = k\underline{x} + c\epsilon$ is given by.

$$(3.2) \qquad \text{Prob}\left\{y_1 \leq Y_1, y_2 \leq Y_2, \ldots, y_n \leq Y_n\right\} = F\left(\frac{Y_1-k\lambda-c}{k\sigma}, \frac{Y_2-k\lambda-c}{k\sigma}, \ldots, \frac{Y_n-k\lambda-c}{k\sigma}\right).$$

Hence if,

$$(3.3) \qquad E\left\{\delta(\underline{x})\right\} = e(\lambda,\sigma),$$

then

$$(3.4) \qquad E\left\{\delta(\underline{y})\right\} = E\left\{\delta(k\underline{x} + c\epsilon)\right\} = e(k\lambda + c, k\sigma).$$

Thus it can be easily seen that, there exists a one-one correspondence between unbiased estimators of $e(\lambda,\sigma)$ and $e(k\lambda + c, k\sigma)$ implying thereby that any unbiased estimator $\delta^*(\underline{x})$ of $e(k\lambda + c, k\sigma)$ can be uniquely expressed as,

$$(3.5) \qquad \delta^*(\underline{x}) = \delta(k\underline{x} + c\epsilon),$$

where $\delta(\underline{x})$ is an unbiased estimator for $e(\lambda, \sigma)$, and vice versa.

Now let $\delta(\underline{x})$ be the UMVUE for $e(\lambda,\sigma)$, $\delta_1(\underline{x})$ be any other unbiased estimator for $e(\lambda,\sigma)$, $\delta^*(\underline{x}) = \delta(k\underline{x} + c\epsilon)$ and $\delta_1^*(\underline{x}) = \delta_1(k\underline{x} + c\epsilon)$ be the corresponding unbiased estimators for $e(k\lambda + c, k\sigma)$. Let $v(\lambda,\sigma)$, $v_1(\lambda,\sigma)$, $v^*(\lambda,\sigma)$ and $v_1^*(\lambda,\sigma)$ be the variances of δ, δ_1, δ^*, δ_1^* respectively.

It follows from (3.2) that

$$v^*(\lambda,\sigma) = v(k\lambda + c, k\sigma) \tag{3.6}$$

$$v_1^*(\lambda,\sigma) = v_1(k\lambda + c, k\sigma) \tag{3.7}$$

Since $\delta(\underline{x})$ is the UMVUE for $e(\lambda,\sigma)$, we have

$$v(\lambda,\sigma) \leq v_1(\lambda,\sigma), \text{ for all } \lambda \text{ and } \sigma. \tag{3.8}$$

Therefore,

$$v^*(\lambda,\sigma) \leq v_1(\lambda,\sigma), \text{ for all } \lambda \text{ and } \sigma, \tag{3.9}$$

which proves lemma (3.1).

In particular, when $e(\lambda,\sigma) = \lambda$, we see that if $b(\underline{x})$ be a UMVUE for λ, $b(k\underline{x} + c\epsilon)$ is a UMVUE for $k\lambda + c$ and following an argument exactly similar to that in lemma 3.1, it can be shown that in this case $\frac{b(k\underline{x} + c\epsilon) - c}{k}$ is also a UMVUE for λ. From the uniqueness property of the UMVUE [3], [5] it follows, therefore, that

$$b(k\underline{x} + c\epsilon) = kb(\underline{x}) + c, \tag{3.10}$$

that is, $b(\underline{x})$ should satisfy condition (2.1).

The necessity of conditions (2.2), (2.3) and (2.4) in appropriate cases can be similarly established.

We have thus a method of formal derivation of statistics having uniformly mimimum variance among unbiased estimators of location and scale parameters.

4. Illustrations. Rectangular Population (Range depending on parameters).

(a) Both λ and σ are unknown.

Let the elementary probability density of the chance variable $\underline{x} = (x_1, x_2, \ldots, x_n)$ be of the form

$$(4.1) \qquad f(\underline{x};\ \lambda,\sigma) = \frac{1}{[(\beta-\alpha)\sigma]^n} \quad \text{if} \quad \lambda + \alpha\sigma \leq x_i \leq \lambda + \beta\sigma; \quad \text{for } i = 1, 2, \ldots, n.$$

where α and $\beta > \alpha$ are known constants, λ, σ are unknown parameters and the parameter space consists of the upper half of the Euclidean plane (to be denoted by R_2^*). In this case it is well known that

$$(4.2) \qquad \xi = \max\left\{x_1, x_2, \ldots, x_n\right\}$$

and

$$(4.3) \qquad \eta = \min\left\{x_1, x_2, \ldots, x_n\right\}$$

are minimal sufficient for λ and σ and that, for $(\lambda,\sigma) \in R_2^*$, the probability density of ξ, η is given by

$$(4.4) \qquad g(\xi,\eta;\ \lambda,\sigma) = \frac{n(n-1)}{[(\beta-\alpha)\sigma]^n}(\xi-\eta)^{n-2} \quad \text{if } \lambda + \alpha\sigma \leq \eta \leq \xi \leq \lambda + \beta\sigma \text{ and}$$

$$= 0 \quad \text{otherwise.}$$

It is thus clear that the UMVUE for λ and σ, if they exist, must be explicit functions of ξ and η alone [3].

The only unbiased function of ξ and η which satisfy (2.1) and (2.2) are those given by

$$(4.5) \qquad b(\xi,\eta) = \eta + b\,(\xi-\eta,0) = \eta + (\xi-\eta)\,b\,(1,0)$$

and

$$(4.6) \qquad s(\xi,\eta) = s(\xi-\eta,0) = (\xi-\eta)s(1,0),$$

where the constants b(1,0) and s(1,0) are properly chosen so that for all $\lambda,\ \sigma \in R_2^*$,

(4.7) $$E\{\eta + (\xi-\eta)\ b(1,0) \mid \lambda,\sigma\} = \lambda\ ,$$

and

(4.8) $$E\ \{(\xi-\eta)s(1,0) \mid \lambda,\sigma\} = \sigma\ .$$

In fact it will follow from the following lemma that they are respectively the only unbiased estimators of λ and σ based on ξ and η and as such are the UMVUE for λ and σ.

Lemma 4.1. The family $\mathcal{F}$ of distributions (4.4) for $(\lambda,\sigma)\in R_2^*$ is complete [3].

Proof. Let $t(\xi,\eta)$ be such that

(4.9) $$E\ \{t(\xi,\eta) \mid \lambda,\sigma\} \equiv 0, \quad \text{for all } (\lambda,\sigma)\in R_2^*,$$

that is,

(4.10) $$\int_{\lambda+\alpha\sigma}^{\lambda+\beta\sigma} \left\{ \int_{\eta}^{\lambda+\beta\sigma} t(\xi,\eta)\ g(\xi,\eta;\lambda,\sigma)d\xi \right\} d\eta \equiv 0 \quad \text{for all } (\lambda,\sigma)\in R_2^*,$$

or, if we put $\theta_2 = \lambda + \beta\sigma$,

(4.11) $$I_1(\theta_2,\sigma) = \int_{\theta_2 + (\alpha-\beta)\sigma}^{\theta_2} \left\{ \int_{\eta}^{\theta_2} t(\xi,\eta)(\xi-\eta)^{n-2}d\xi \right\} d\eta \equiv 0, \text{ for all } (\theta_2,\sigma)\in R_2^*.$$

Therefore,

(4.12) $$\frac{1}{(\beta-\alpha)}\ \frac{\partial I_1(\theta_2,\sigma)}{\partial\sigma} = \int_{\theta_2+(\alpha-\beta)\sigma}^{\theta_2} t\,(\xi,\theta_2+(\alpha-\beta)\sigma)(\xi-\theta_2-(\alpha-\beta)\sigma)^{n-2}d\xi \equiv 0$$

for all $(\theta_2,\sigma)\in R_2^*$

If we put $\theta_2 + (\alpha-\beta)\sigma = \theta_1$, we have

(4.13) $$I_2(\theta_1,\sigma) = \int_{\theta_1}^{\theta_1+(\beta-\alpha)\sigma} t(\xi,\theta_1)(\xi-\theta_1)^{n-2}d\xi \equiv 0, \text{ for all } (\theta_1, \sigma) \in R_2^*.$$

Therefore,

(4.14) $$\frac{1}{(\beta-\alpha)} \frac{\partial I_2(\theta_1,\sigma)}{\partial \sigma} = t(\theta_1 + (\beta-\alpha)\sigma,\theta_1) [(\beta-\alpha)\sigma]^{n-2} \equiv 0,$$

for all $(\theta_1, \sigma) \in R_2^*$.

Hence if $\sigma > 0$

(4.15) $$t(\theta_1 + (\beta-\alpha)\sigma, \theta_1) = t(\lambda + \beta\sigma, \lambda + \alpha\sigma) \equiv 0, \text{ for all real } \lambda ,$$

that is,

(4.16) $$t(\xi,\eta) = 0, \text{ for all } \xi > \eta.$$

We may note however that $t(\xi,\eta)$ may be assigned arbitrary finite values on the set $S\ (\xi,\eta):\xi=\eta$ which has zero probability measure, with respect to $\mathcal{F}$, which proves lemma 4.1.

(b) <u>σ known and equal to 1</u>.

In this case without any loss of generality we may assume $\alpha = 0$ and $\beta = c > 0$.

A function $b(\xi,\eta)$ satisfying (2.3) must be of the form

(4.17) $$b(\xi,\eta) = \eta + b(\xi - \eta) = \eta + b(R) \text{ where } R = \xi - \eta.$$

Hence to determine the best unbiased invariant estimator of λ we have to choose $b(R)$ in such a way that $\eta + b(R)$ is unbiased and has minimum variance among all other estimators of a similar type.

(4.18) Now let, $E\{\eta | R\} = \lambda + e(R)$, and,

(4.19) $$E\{(\eta - \lambda)^2 \mid R\} = v(R)$$

(4.20) We have $E\{\eta + b(R)\} = \lambda + E(e(R) + b(R))$ and

(4.21) $$E\{(\eta + b(R) - \lambda)^2 | R\} = v(R) + b^2(R) + 2b(R)e(R)$$

$$= [b(R) + e(R)]^2 + v(R) - e^2(R)$$

If $\eta + b(R)$ is to be unbiased, $b(R)$ has to satisfy the condition

(4.22) $$E\{e(R) + b(R)\} = 0.$$

Hence from (4.21) it follows that the proper choice of $b(R)$ will be

(4.23) $$b(R) = -e(R) = \frac{R-c}{2}.$$

Hence the best unbiased invariant estimator of λ is given by

(4.24) $$b^*(\xi,\eta) = \eta + \frac{R-c}{2} = \frac{\xi + \eta}{2} - \frac{c}{2}$$

Let us now consider the following estimator for λ:

(4.25) $$b(\xi,\eta) = \frac{n}{n-1}\xi - \frac{1}{n-1}\eta - 1, \text{ if } \xi > \lambda_0 + c;\ \xi - c \le \eta \le \xi,$$

$$= -\frac{1}{n-1}\xi + \frac{n}{n-1}\xi, \text{ if } \eta < \lambda_0;\ \eta \le \xi \le \eta + c \quad \text{and}$$

$$= \lambda_0 \qquad \text{if } \lambda_0 \le \eta \le \xi \le \lambda_0 + c,$$

where λ_0 is any point on the real axis.

It is easily seen that $b(\xi,\eta)$ is uniformly unbiased and moreover has zero variance at $\lambda = \lambda_0$. Thus $b^*(\xi,\eta)$ cannot be the UMVUE for λ; hence by Theorem 3.1, in this case, the UMVUE for λ does not exist.

In fact, Lehmann and Scheffe [3] have shown that for such a population the only estimable parametric functions possessing UMVUE are the real constants.

(c) <u>λ known and equal to zero.</u>

In this case a function $s(\xi,\eta)$ satisfying (2.4) is of the form

(4.26) $$s(\xi,\eta) = \xi s(1,\tfrac{\eta}{\xi}) = \xi h(t) \quad \text{where } t = \xi/\eta.$$

Hence to determine the best unbiased invariant estimator for σ one has to consider functions of the type $\xi h(t)$ and choose $h(t)$ such that $\xi h(t)$ is unbiased and has uniformly the smallest variance among all other unbiased functions of a similar type.

(4.27) If $$E(\xi|t) = \sigma e(t)$$ $$E(\xi^2|t) = \sigma^2 v(t)$$

it follows therefore

(4.28) $$E(e(t)h(t)) = 1, \text{ and,}$$

(4.29) $$\begin{aligned} E\{\xi h(t)\}^2 &= \sigma^2 E\{h^2(t)v(t)\} \\ &= \sigma^2 \Big[E\Big\{h(t)\sqrt{v(t)} - \frac{e(t)}{\lambda\sqrt{v(t)}}\Big\}^2 \\ &\quad + \frac{2}{\lambda} E(h(t)e(t)) - \frac{1}{\lambda^2} E\Big(\frac{e^2(t)}{v(t)}\Big)\Big] \\ &= \sigma^2 \Big[E\Big\{h(t)\sqrt{v(t)} - \frac{e(t)}{\lambda\sqrt{v(t)}}\Big\}^2 + \frac{1}{\lambda}\Big], \end{aligned}$$

where $\lambda = E\left(\frac{e^2(t)}{v(t)}\right)$

Hence the proper choice of h(t) will be

(4.30) $$h(t) = \frac{1}{\lambda}\ \frac{e(t)}{v(t)}$$ [6]

and the best unbiased invariant estimator will be given by

(4.31) $$s^*(\xi,\eta) = \frac{1}{\lambda}\ \frac{e(t)}{v(t)}\ \xi \ .$$

If α and β are of opposite signs, $s(\xi,\eta)$, when simplified, will reduce to

(4.32) $$s^*(\xi,\eta) = \frac{n+1}{n}\,\varsigma \ , \text{ where } \varsigma = \max\left\{\frac{\xi}{\beta}, \frac{\eta}{\alpha}\right\}.$$

It is also well known that in this case ς is minimal sufficient for σ and that the family of probability distributions of ς for $\sigma > 0$ is complete. Hence $\frac{n+1}{n}\varsigma$ is the unique UMVUE for σ.

Now let α and β be of the same sign and, for simplicity, let us consider the case $\alpha = 1$, $\beta = 2$. In this case $s^*(\xi,\eta)$ will reduce to

(4.33) $$s^*(\xi,\eta) = \frac{1}{\lambda}\ \frac{(n+2)}{(n+1)}\ \xi\ \frac{[2^{n+2}-t^{n+2}]}{[2^{n+1}-t^{n+1}]} \ ,$$

where

(4.34) $$\lambda = \frac{n(n+2)}{(n+1)^2}\ E\left[\frac{(2^{n+1}-t^{n+1})^2}{(2^n-t^n)(2^{n+2}-t^{n+2})}\right]$$

$$= \frac{n(n+2)}{(n+1)^2}\left[1 + \frac{4}{n^2}\{Z(3) - 1\} + 0\left(\frac{1}{n^3}\right)\right],$$

where Z is the Riemann's Zeta function

Hence

(4.35) $$\lambda = 1 - \frac{.192}{n^2} + 0\left(\frac{1}{n^3}\right) .$$

It may be noted that the variance of the best unbiased invariant estimation is given by

(4.36) $$\sigma^2(\frac{1}{\lambda} - 1) = \frac{.192}{n^2}\sigma^2 + 0(\frac{1}{n^3}) \sigma^2 .$$

Let us now consider the following estimator for σ

(4.37) $$\begin{aligned} s(\xi,\eta) &= \frac{n}{2(n-1)}\xi - \frac{1}{2(n-1)}\eta \quad \text{if } \xi > 2\sigma_0;\ \frac{\xi}{2} \leq \eta \leq \xi, \\ &= -\frac{\xi}{n-1} + \frac{n}{n-1}\eta \quad \text{if } \eta < \sigma_0;\ \eta \leq \xi \leq 2\eta \text{ and} \\ &= \sigma_0 \quad \text{if } \sigma_0 \leq \eta \leq \xi \leq 2\sigma_0, \end{aligned}$$

where σ_0 is any point on the positive part of the real axis. It is easy to verify that $s(\xi,\eta)$ is unbiased and has zero variance at $\sigma = \sigma_0$. Hence $s^*(\xi,\eta)$ cannot be the UMVUE for σ. We shall in fact prove a more general result similar to that of Lehmann and Scheffe stated in 4(b).

<u>Theorem 4.1</u>. For a rectangular population with range $\alpha\sigma$ to $\beta\sigma$ if α and β are of the same sign and if the parameter space for σ consists of the positive part of the real axis, the class of estimable parametric functions of σ, possessing UMVUE consists only of real constants.

<u>Proof</u>. As before, let us, for simplicity, consider the case $\alpha = 1$, $\beta = 2$ and let $f(\xi,\eta)$ be a function defined over the triangle $\sigma_0 \leq \eta \leq \xi \leq 2\sigma_0$ such that

(4.38) $$E\left\{f(\xi,\eta) \mid \sigma_0\right\} = 0, \text{ where } \sigma_0 \text{ is any positive number.}$$

We shall show that it is always possible to extend the definition of $f(\xi,\eta)$ over all ordered number pairs ξ,η in the strip $\eta \leq \xi \leq 2\eta$, such that

(4.39) $$E\left\{f(\xi,\eta) \mid \sigma\right\} = 0 \quad \text{for all } \sigma > 0.$$

A possible way of achieving this is as follows.

(A) If (ξ,η) is an ordered number pair and if

$$\sigma_0 2^k \leq \eta \leq \xi < \sigma_0 2^{k+1} ,$$

(k being any integer negative or positive) define

(4.40) $$f(\xi,\eta) = f(\xi/2^k, \eta/2^k) \quad 2^{-kn}.$$

(B) If (ξ,η) is an ordered number pair and if

$$\sigma_0 2^{k-1} < \eta < \sigma_0 2^k; \quad \sigma_0 2^k < \xi < 2\eta,$$

define

(4.41) $$f(\xi,\eta) = \tfrac{1}{2} f(\eta,\xi/2) \quad \left[\frac{(\eta-\xi/2)}{(\xi-\eta)} \right]^{n-2} .$$

It is easy to verify that with this extended definition of $f(\xi,\eta)$, (4.39) is satisfied. Now let $t(\xi,\eta)$ be a UMVUE of its expectation. It is well known [3], [5] that in this case if $f(\xi,\eta)$ be such that

(4.42) $$E \{ f(\xi,\eta) \mid \sigma \} \equiv 0, \text{ for all } \sigma > 0,$$

and

$$E \{ f^2(\xi,\eta) \mid \sigma_0 \} < \infty, \text{ for some } \sigma_0 > 0, \text{ then}$$

(4.43) $$E \{ t(\xi,\eta) \; f(\xi,\eta) \mid \sigma_0 \} = 0.$$

Now, for any $\sigma_0 > 0$, for any real c and $\delta > 0$, let $L^+(c,\delta,\sigma_0)$ be the lebesgue measure of the set

(4.44) $$s^+(c,\delta,\sigma_0) = \{ (\xi,\eta): \sigma_0 \leq \eta \leq \xi \leq 2\sigma_0; \; \xi-\eta \geq \delta \text{ and } t(\xi,\eta) - c \geq 0 \} ,$$

and $L^-(c,\delta,\sigma_0)$ be the lebesgue measure of

(4.45) $$s^-(c,\delta,\sigma_0) = \{(\xi,\eta): \sigma_0 \leq \eta \leq \xi \leq 2\sigma_0; \; \xi-\eta \geq \delta \text{ and } t(\xi,\eta) - c < 0\} .$$

Unless $t(\xi,\eta)$ is identically a real constant except for a set of zero lebesgue measure on $s_{\sigma_0} = \{(\xi,\eta): \sigma_0 \leq \eta \leq \xi \leq 2\sigma_0\}$ there always exists a $\delta > 0$ and a c such that both $L^+(c,\delta,\sigma_0)$ and $L^-(c,\delta,\sigma_0)$ are positive. Let us define

$$(4.46) \qquad f(\xi,\eta) = \frac{1}{L^+(c,\delta,\sigma_0)(\xi-\eta)^{n-2}}, \text{ for all } (\xi,\eta) \epsilon \ s^+(c,\delta,\sigma_0), \text{ and}$$

$$= -\frac{1}{L^-(c,\delta,\sigma_0)(\xi-\eta)^{n-2}}, \text{ for all } (\xi,\eta) \epsilon \ S^-(c,\delta,\sigma_0) \text{ and}$$

$$= 0, \text{ for all } (\xi,\eta) \ \epsilon \ s_{\sigma_0} - s^+(c,\delta,\sigma_0) - s^-(c,\delta,\sigma_0).$$

It is obvious that

$$(4.47) \qquad E\left\{f(\xi,\eta) \mid \sigma_0\right\} = 0 \text{ and } E\left\{f^2(\xi,\eta) \mid \sigma_0\right\} < \infty$$

and we may extned the definition of $f(\xi,\eta)$ to all ordered number pairs in the strip $\eta \leq \xi \leq 2\eta$ such that

$$(4.48) \qquad E\left\{f(\xi,\eta) \mid \sigma\right\} = 0, \text{ for all } \sigma > 0.$$

$$(4.49) \qquad E\left\{f(\xi,\eta)\ t(\xi,\eta) \mid \sigma_0\right\} = E\left[f(\xi,\eta)\left\{t(\xi,\eta) - c\right\} \mid \sigma_0\right\} > 0,$$

which contradicts (4.43).

Hence $t(\xi,\eta) \equiv c$, a.e., on s_σ, for all $\sigma > 0$, which proves theorem 4.1.

References

[1] Blackwell, D and Givishick, M. A, Theory of Games and Statistical Decisions John Wiley and Sons, Inc., New York (1954).

[2] Givishick, M.A. and Savage L. J., "Bayes and minimax estimates for quadratic loss functions," Proc. Sec. Berk. Symp. on Math. Stat. and Prob., University of California Press, Berkeley and Los Angeles (1950) 53-75.

[3] Lehmann, E. L. and Scheffe, H.,"Completeness, similar regions and unbiased estimation," Sankhya, Vol. 10, (1950) 305-339.

[4] Pitmann, E. J. G., "The estimation of the location and scale parameters of a continuous population of any form" Biometrica, Vol. 30 (1938) 391-421.

[5] Rao, C. R.,"Some theorems on minimum variance estimation," Sankhya, Vol. 12 (1952), 27-42.

[6] Rao, C. R.,"Minimum variance estimation in distributions admitting ancilliary statistics," Sankhya, Vol. 12 (1952), 53-56.