MOMENTS OF ORDER STATISTICS FROM A NORMAL POPULATION

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1. Introduction and summary.

Statistics based on ordered observations have been called systematic statistics by Mosteller [12]. They are now being increasingly used in new statistical procedures [1, 2, 3, 4, 7, 8, 13, 14, 15, 16, 17, 20, 22]. The present paper deals with the problem of obtaining the moments of $X_{(k)}$, the k-th order statistic for a sample of size n from a normal population N(0, 1). This problem has been considered among others by Cole [5], Godwin [6], Hastings, Mosteller, Tukey and Winsor [9], Jones [11], Ruben [18] and Tippett [23].

It has been shown that $\mu'_t(n, k)$, the t-th moment of $X_{(k)}$, can be expressed in terms of lower moments of order $t - 2i$ $(i = 1, 2, \ldots, t/2 \text{ or } (t-1)/2)$ and the integral

$$(1.1) \qquad \int_{-\infty}^{+\infty} P_{t+1}(x)\, e^{-(t+1)x^2/2}\, dx$$

where $P_{t+1}(x)$ for $t > 0$, is defined by

$$(1.2) \qquad P_{t+1}(x) = k \binom{n}{k} \frac{d^t}{d\Phi^t} \left[\Phi^{k-1}(1-\Phi)^{n-k} \right],$$

it being understood that in (1.2), Φ is replaced after differentiation by $\Phi(x)$, the c. d. f. of N(0, 1). $P_t(x)$ is thus a polynomial of degree $(n - t)$ in $\Phi(x)$ if $n \le t$ and is zero if $n > t$. Exact values of all odd order moments can be derived when $n \le 5$, and the exact values of all even order moments can be derived when $n \le 6$. Godwin [6] and Jones [11] have given tables of exact moments $\mu'_t(n, k)$ for t = 1 and 2. The corresponding tables for t = 3 and 4 are provided in this paper. In general the numerical evaluation of the integral (1.1) can be expeditiously done by using the Gauss-Jacobi method of mechanical quadrature [21] based on the zeros and the weight factors of the Hermite-polynomials for which tables have been provided by Salzer, Zucker and Capuano [19]. It is believed that the formulae derived here are better suited for numerical computation than those given elsewhere.

2. The function $P_t(n, k, x)$.

Let $x_1, x_2, \ldots, x_n$ be n independent observations from a normal population $N(0, 1)$ with zero mean and unit variance, and let

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)} \tag{2.1}$$

be the n ranked observations among $x_1, x_2, \ldots, x_n$. Then the cumulative distribution function of $X_{(k)}$, the random variable corresponding to $x_{(k)}$, $(1 \leq k \leq n)$, is given by

$$\begin{aligned} P_0(n, k, x) &= \text{Prob}\left\{ X_{(k)} \leq x \right\} \\ &= \frac{C}{(2\pi)^{1/2}} \int_{-\infty}^{x} [\Phi(x)]^{k-1} [1 - \Phi(x)]^{n-k} e^{-x^2/2}\, dx \end{aligned} \tag{2.2}$$

where $\Phi(x)$ is defined as

$$\Phi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-x^2/2}\, dx \tag{2.3}$$

and C is the constant

$$C = \frac{n!}{(k-1)!\,(n-k)!} \quad . \tag{2.4}$$

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Let us now define the function $P_t(n, k, x)$, which we shall abbreviate to $P_t(x)$ for convenience, by the relation

$$P_{t+1}(x) = (2\pi)^{1/2} e^{x^2/2} \frac{dP_t}{dx} \tag{2.5}$$

where $P_0(x)$ is given by (2.2). Then

$$P_1(x) = C[\Phi(x)]^{k-1} [1 - \Phi(x)]^{n-k} . \tag{2.6}$$

It is clear that $P_t(x)$ is a polynomial of degree n-t in $\Phi(x)$ if $1 \leq t \leq n$, and is zero for $t > n$. In fact, we can write

$$P_{t+1}(x) = C \frac{d^t}{d\Phi^t} [\Phi^{k-1}(1 - \Phi)^{n-k}] \tag{2.7}$$

where Φ is replaced by $\Phi(x)$ after the differentiation. It follows that for given t, n, k, $\Phi_t(x)$ is a bounded function of x. The functions $P_2(x)$, $P_3(x)$, $P_4(x)$ and $P_5(x)$ are given below explicitly, where Φ is written for $\Phi(x)$.

$$P_2(x) = C\, \Phi^{k-2}(1 - \Phi)^{n-k-1}[(k-1) - (n-1)\Phi] \tag{2.8}$$

$$P_3(x) = C\, \Phi^{k-3}(1 - \Phi)^{n-k-2}[(k-1)(k-2) - 2(k-1)(n-2)\Phi + (n-1)(n-2)\Phi^2] \tag{2.9}$$

$$\begin{aligned} P_4(x) = C\Phi^{k-4}(1 - \Phi)^{n-k-3}[&(k-1)(k-2)(k-3) - 3(k-1)(k-2)(n-3)\Phi \\ &+ 3(k-1)(n-2)(n-3)\Phi^2 - (n-1)(n-2)(n-3)\Phi^3] \end{aligned} \tag{2.10}$$

$$\begin{aligned} P_5(x) = C\, \Phi^{k-5}(1 - \Phi)^{n-k-4}[&(k-1)(k-2)(k-3)(k-4) - 4(k-1)(k-2)(k-3)(k-4)\Phi \\ &+ 6(k-1)(k-2)(n-3)(n-4)\Phi^2 \\ &- 4(k-1)(n-2)(n-3)(n-4)\Phi^3 + (n-1)(n-2)(n-3)(n-4)\Phi^4] \end{aligned} \tag{2.11}$$

3. A system of differential equations satisfied by $P_0(x)$.

From (2.5)

$$(3.1) \qquad P_1(x) = (2\pi)^{1/2} e^{x^2/2} \frac{dP_0}{dx}$$

$$(3.2) \qquad P_2(x) = (2\pi)^{1/2} e^{x^2/2} \frac{d}{dx} \left[(2\pi)^{1/2} e^{x^2/2} \frac{dP_0}{dx} \right]$$

$$= (2\pi)\, e^{x^2} \left[\frac{d^2 P_0}{dx^2} + x \frac{dP_0}{dx} \right] .$$

In general let us assume

$$(3.3) \qquad P_t(x) = (2\pi)^{t/2} e^{tx^2/2} \sum_{r=0}^{t-1} g_{r,t}(x) \frac{d^{t-r} P_0}{dy^{t-r}}$$

where

$$(3.4) \qquad g_{0,t}(u) = 0$$

and $g_{r,t}(x)$ is a polynomial in x of the r-th degree. Differentiating (3.3) and using (2.5), we have

$$(3.5) \quad P_{t+1}(x) = (2\pi)^{(t+1)/2} e^{(t+1)x^2/2} \sum_{r=0}^{t-1} \left[g_{r,t}(x) \frac{d^{t-r+1} P_0}{dx^{t-r+1}} + \left\{ txg_{r,t}(x) + g'_{r,t}(x) \right\} \frac{d^{t-r} P_0}{dx^{t-r}} \right] .$$

This leads to the recurrence relation

$$(3.6) \qquad g_{r,t+1}(x) = g_{r,t}(x) + \left\{ tx + \frac{d}{dx} \right\} g_{r-1,t}(x)$$

where $g_{r,r}(x)$ should be interpreted as zero. This together with (3.4) determines all the polynomials $g_{r,t}(x)$. Starting from

$$(3.7) \qquad g_{0,1}(x) = 1$$

we can successively calculate

(3.8) $g_{0,2}(x) = 1, \quad g_{1,2}(x) = x$

(3.9) $g_{0,3}(x) = 1, \quad g_{1,3}(x) = 3x, \quad g_{2,3}(x) = 2x^2+1$

(3.10) $g_{0,4}(x) = 1, \quad g_{1,4}(x) = 6x, \quad g_{2,4}(x) = 11x^2+4, \quad g_{3,4}(x) = 6x^3+7x$

(3.11) $g_{0,5}(x) = 1, \quad g_{1,5}(x) = 10x, \quad g_{2,5}(x) = 35x^2+10, \quad g_{3,5}(x) = 50x^3+45x$

$$g_{4,5}(x) = 24x^4+46x^2+7 \; .$$

Hence we have the set of equations

$$(3.12) \quad \frac{dP_0}{dx} = \frac{1}{(2\pi)^{1/2}} e^{-x^2/2} P_1(x)$$

$$(3.13) \quad \frac{d^2P_0}{dx^2} + x \frac{dP_0}{dx} = \frac{1}{(2\pi)} e^{-x^2} P_2(x)$$

$$(3.14) \quad \frac{d^3P_0}{dx^3} + 3x \frac{d^2P_0}{dx^2} + (2x^2+1)\frac{dP_0}{dx} = \frac{1}{(2\pi)^{3/2}} e^{-3x^2/2} P_3(x)$$

$$(3.15) \quad \frac{d^4P_0}{dx^4} + 6x \frac{d^3P_0}{dx^3} + (11x^2+4)\frac{d^2P_0}{dx^2} + (6x^3+7x)\frac{dP_0}{dx} = \frac{1}{(2\pi)^2} e^{-2x^2} P_4(x)$$

$$(3.16) \quad \frac{d^5P_0}{dx^4} + 10x\frac{d^4P_0}{dx^3} + (35x^2+10)\frac{d^3P_0}{dx^3} + (50x^3+45x)\frac{d^2P_0}{dx^2} + (24x^4+46x^2+7)\frac{dP_0}{dx}$$

$$= \frac{1}{(2\pi)^{5/2}} e^{-5x^2/2} P_5(x) \; .$$

We can proceed in this manner up to any order but it should be noted that $P_n(x)$ is a constant and $P_t(x) = 0$ if $t > n$. The general equation is

$$(3.17) \quad g_{0,t}(x) \frac{d^tP_0}{dx^t} + g_{1,t}(x) \frac{d^{t-1}P_0}{dx^{t-1}} + \ldots + g_{t-1,t}(x)\frac{dP_0}{dx} = \frac{1}{(2\pi)^{t/2}} e^{-tx^2/2} P_t(x).$$

4. Moments of $X_{(k)}$.

We shall first prove the following Lemma:

Lemma. If α and r are non-negative integers, then

(4.1) $$\int_{-\infty}^{+\infty} x^{\alpha} \frac{d^{r+1}P_0}{dx^{r+1}} dx = (-1)^r \alpha(\alpha - 1) \dots (\alpha - r + 1)\mu'_{\alpha-r} \quad \text{or } 0$$

according as

$$r \leq \alpha \qquad \text{or} \qquad r > \alpha$$

where $\mu'_{\alpha-r}$ is the $(\alpha$-r)-th order moment of $X_{(k)}$ about the origin.

It should be noted that by definition

(4.2) $$\int_{-\infty}^{+\infty} x^{\alpha} \frac{dP_0}{dx} dx = \mu'_{\alpha} .$$

From (3.12) and (3.13)

(4.3) $$\frac{dP_0}{dx} = \frac{1}{(2\pi)^{1/2}} e^{-x^2/2} P_1(x)$$

(4.4) $$\frac{d^2P_0}{dx^2} = - \frac{x}{(2\pi)^{1/2}} e^{-x^2/2} P_1(x) + \frac{1}{2\pi} e^{-x^2} P_2(x)$$

and in general using the system of equations (3.12)-(3.17) we can write

(4.5) $$\frac{d^rP_0}{dx^r} = \sum_{i=1}^{r} h_i(x) e^{-ix^2/2} P_i(x)$$

where $h_i(x)$ is a polynomial in x. Now

(4.6) $$\int_{-\infty}^{+\infty} x^{\alpha} \frac{d^{r+1}P_0}{dx^{r+1}} dx = \left| x^{\alpha} \frac{d^rP_0}{dx^r} \right|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \alpha x^{\alpha-1} \frac{d^rP_0}{dx^r} dx .$$

Since $P_t(x)$ for any non-negative integer t is a bounded function of x, it follows from (4.5) that the first part on the right hand side of (4.6) vanishes.

Repeating this process we get if $r \leq \alpha$

$$\int_{-\infty}^{+\infty} x^{\alpha} \frac{d^{r+1}P_0}{dx^{r+1}}\,dx = (-1)^r \alpha(\alpha-1)\ldots(\alpha-r+1) \int_{-\infty}^{+\infty} x^{\alpha-r} \frac{dP_0}{dx} = (-1)^r \alpha(\alpha-1)\ldots(\alpha-r+1)\mu'_{\alpha-r}.$$

If $r > \alpha$, we get on repeating the process α times,

$$\int_{-\infty}^{+\infty} x^{\alpha} \frac{d^{r+1}F_0}{dx^{\alpha}}\,dx = (-1)^{\alpha}\alpha(\alpha-1)\ldots 3.2.1 \int_{-\infty}^{+\infty} \frac{d^{r+1-\alpha}P_0}{dx^{r+1-\alpha}}\,dx$$

$$= (-1)^{\alpha}\alpha(\alpha-1)\ \ldots\ 3.2.1 \left| \frac{d^{r-\alpha}P_0}{dx^{r-\alpha}} \right|_{-\infty}^{+\infty}$$

$$= 0\ .$$

This proves the Lemma.

On applying the Lemma and integrating the equations (3.13) ... (3.16) we get

(4.7) $$\mu'_1 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} P_2(x) e^{-x^2}\,dx$$

(4.8) $$-3 + (2\mu'_2 + 1) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} P_3(x) e^{-3x^2/2}\,dx$$

(4.9) $$-22\mu'_1 + (6\mu'_3 + 7\mu'_1) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} P_4(x) e^{-2x^2}\,dx$$

(4.10) $$70 - 150\mu'_2 - 45 + (24\mu'_4 + 46\mu'_2 + 7) = \frac{1}{(2\pi)^{5/2}} \int_{-\infty}^{+\infty} P_5(x) e^{-5x^2/2}\,dx\ .$$

We may write $\mu'_{\alpha}(n,k)$ instead of μ'_{α} to denote the fact that we have the α -th moment about the origin of the k-th order statistic out of a sample of n observations from N(0,1). We then have

(4.11) $$\mu_1'(n,k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} P_2(x) e^{-x^2} dx$$

(4.12) $$\mu_2'(n,k) = 1 + \frac{1}{2!(2\pi)^{3/2}} \int_{-\infty}^{+\infty} P_3(x) e^{-3x^2/2} dx$$

(4.13) $$\mu_3'(n,k) = \frac{5}{2} \mu_1'(n,k) + \frac{1}{3!(2\pi)^2} \int_{-\infty}^{+\infty} P_4(x) e^{-4x^2/2} dx$$

(4.14) $$\mu_4'(n,k) = -\frac{4}{3} + \frac{13}{3} \mu_2'(n,k) + \frac{1}{4!(2\pi)^{5/2}} \int_{-\infty}^{+\infty} P_5(x) e^{-5x^2/2} dx$$

In general applying the Lemma to (3.17) we can express $u_t'(n,k)$ in terms of lower moments of even (odd) order when t is even (odd) and the integral

(4.15) $$\int_{-\infty}^{+\infty} P_{t+1}(x) e^{-(t+1)x^2/2} dx$$

where the polynomials $P_2(x) \ldots P_5(x)$ are given by (2.8) ... (2.11). In the particular case when n = k, i.e., $x_{(n)}$ is the largest of $x_1, x_2, \ldots, x_n$, $P_t(x)$ assumes the very simple form

(4.16) $$P_t(x) = n(n-1)(n-2) \ldots (n-t+1) [\Phi(x)]^{n-t} .$$

Hence we get

(4.17) $$\mu_1'(n,n) = \frac{2\binom{n}{2}}{2\pi} \int_{-\infty}^{+\infty} [\Phi(x)]^{n-2} e^{-x^2} dx$$

(4.18) $$\mu_2'(n,n) = 1 + \frac{3\binom{n}{3}}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} [\Phi(x)]^{n-3} e^{-3x^2/2} dx$$

$$(4.19) \quad \mu_3'(n,n) = \frac{5}{2}\mu_1'(n,n) + \frac{4\binom{n}{4}}{(2\pi)^2} \int_{-\infty}^{+\infty} [\Phi(x)]^{n-4} e^{-4x^2/2} dx$$

$$(4.20) \quad \mu_4'(n,n) = -\frac{4}{3} + \frac{13}{3}\mu_2'(n,n) + \frac{5\binom{n}{5}}{(2\pi)^{5/2}} \int_{-\infty}^{+\infty} [\Phi(x)]^{n-5} e^{-5x^2/2} dx \quad .$$

It should be noted that in the formulae (4.17) through (4.20)

$$[\Phi(x)]^{n-t}$$

should be interpreted as zero if $t > n$.

Some integrals of the type occurring in (4.17) through (4.20) have been numerically evaluated by Hojo [10].

5. Exact values of some moments.

Let

$$(5.1) \quad I_n(a) = \int_{-\infty}^{+\infty} [\Phi(ax)]^n e^{-x^2} dx ,$$

then

$$(5.2) \quad I_0(a) = \pi^{1/2} .$$

Now

$$(5.3) \quad \int_{-\infty}^{+\infty} [\Phi(ax) - \tfrac{1}{2}]^{2m+1} e^{-x^2} dx = 0,$$

since the integrand is an odd function of x. Hence

$$(5.4) \quad I_{2m+1}(a) = \sum_{r=1}^{2m+1} (-1)^{r+1} \binom{2m+1}{r} I_{2m-r+1}(2) \Big/ 2^r \quad .$$

In particular

$$I_1(a) = \frac{1}{2} I_0(a) = \frac{1}{2} \pi^{1/2} , \tag{5.5}$$

and

$$I_3(a) = \frac{3}{2} I_2(a) - \frac{3}{4} I_1(a) + \frac{1}{8} I_0(a) = \frac{3}{2} I_2(a) - \frac{1}{4} I_0(a). \tag{5.6}$$

In general, $I_{2m+1}(a)$ can be expressed as a linear function of $I_{2m}(a)$, $I_{2m-2}(a)$, ..., $I_0(a)$.

Differentiating (5.1) with respect to a, (this is justified in virtue of the uniform convergence of the integrals with respect to a, $-\infty < a < \infty$, and the continuity of the integrands), we get for n = 2,

$$\frac{dI_2}{da} = \frac{1}{\pi^{1/2}} \frac{a}{(a^2+2)(a^2+1)^{1/2}} , \tag{5.7}$$

so that

$$I_2(a) = \frac{1}{\pi^{1/2}} \arctan(\sqrt{a^2 + 1}) . \tag{5.8}$$

Using (5.6)

$$I_3(a) = \frac{3}{2\pi^{1/2}} \arctan(\sqrt{a^2 + 1}) - \frac{\pi^{1/2}}{4} . \tag{5.9}$$

Since $P_{t+1}(x)$ is a polynomial in $\Phi(x)$ of degree n-t-1, by using (5.2), (5.5), (5.8) and (5.9), we can exactly evaluate (4.15) if $n \leq t + 4$. Hence we can exactly evaluate $\mu_t'(n,k)$ for all odd values of t , if $n \leq 5$ and all even values of t if $n \leq 6$. Godwin [6] and Jones [11] have given tables of exact moments μ_t' for t = 1 and 2. The corresponding tables for t = 3 and 4 are given below.

Table I.

$\mu_3'(n,k)$

n \ k	k = n	k = n - 1	k=n-2
2	2A		
3	3A	0	
4	$2B_1 + 2C$	$12A - 6B_1 - 6C$	
5	$-5A + 5B_1 + 5B_2$	$20A - 10B_1 - 10C$	0

Here

$$A = \frac{5}{4\pi^{1/2}}$$

$$B_1 = \frac{1}{2\sqrt{2}\,\pi^{3/2}}$$

$$B_2 = \frac{15}{2\pi^{3/2}}$$

$$C = \frac{15}{2\pi^{3/2}} \arctan(\sqrt{2})$$

Table II

$\mu_4'(n,k)$

n \ k			
2	3		
3	3+a	3-2a	
4	3+2a	3-2a	
5	3+b+c	3+10a-4b-4c	3-20a+6b+6c
6	3-5a+3b+3c	3+25a-9b-9c	3-20a+6b+6c

Here

$$a = \frac{13}{\sqrt{3}(2\pi)}$$

$$b = \frac{65}{\sqrt{3}\,\pi^2} \text{ arc } \tan(\sqrt{5}/3)$$

$$c = \frac{\sqrt{5}}{4\pi^2} \quad .$$

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