ON ESTIMATION OF THE SPECTRUM

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The object of this note is to extend some results of Grenander [1] relating to discrete real stationary normal process with absolutely continuous spectrum to the case in which the spectrum also contains a step function with a finite number of saltuses.

It is shown by Grenander [1] that the periodogram is an asymptotically unbiassed estimate of the spectral density $f(\lambda)$ and that its variance is $[f(\lambda)]^2$ or $2[f(\lambda)]^2$ according as $\lambda \neq 0$ or $\lambda = 0$. In the present note the same results are established at a point of continuity.

The consistency of a suitably weighted periodogram for estimating $f(\lambda)$ is established by Grenander [1]. In this note a weighted periodogram estimate similar to that of Grenander [1] (except that the weight function in this case is more restricted) is constructed which consistently estimates the spectral density at a point of continuity.

It appears that this extended result leads to a direct approach to the location of a single periodicity irrespective of the presence of others in the time series.

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We shall now proceed to establish our results.

Let x(n) be a discrete, real, stationary, normal process. It is known (Karhunen [1]) that the process can be decomposed into two mutually orthogonal stationary processes as $x(n) = x_1(n) + x_2(n)$ where $x_1(n)$ is a purely periodic process and $x_2(n)$ is a purely non-periodic process.

Let [x(-N), x(-N + 1), ... x(-1), x(0), x(1), ... x(N - 1), x(N)] be a realization of size 2N + 1 from the process x(n). Let us consider the statistic proposed by Grenander [1]

$$I_N(\lambda) = \frac{1}{2\pi(2N+1)} \left| \sum_{\nu=-N}^{N} x(\nu)e^{-i\nu\lambda} \right|^2 \tag{1}$$

This is the usual periodogram based on observations. We have

$$\begin{aligned} I_N(\lambda) = {} & \frac{1}{2\pi(2N+1)} \left| \sum_{\nu=-N}^{N} x_1(\nu)e^{-i\nu\lambda} \right|^2 + \frac{1}{2\pi(2N+1)} \left| \sum_{\nu=-N}^{N} x_2(\nu)e^{-i\nu\lambda} \right|^2 \\ & + \frac{1}{2\pi(2N+1)} \left(\sum_{\nu=-N}^{N} x_1(\nu)e^{-i\nu\lambda} \right) \overline{\left(\sum_{\nu=-N}^{N} x_2(\nu)e^{-i\nu\lambda} \right)} \\ & + \frac{1}{2\pi(2N+1)} \overline{\left(\sum_{\nu=-N}^{N} x_1(\nu)e^{-i\nu\lambda} \right)} \left(\sum_{\nu=-N}^{N} x_2(\nu)e^{-i\nu\lambda} \right) \end{aligned} \tag{2}$$

The two stationary parts $x_1(n)$ and $x_2(n)$ have the spectral representations

$$x_1(n) = \int_{-\pi}^{\pi} e^{in\lambda} dz_1(\lambda)$$

$$x_2(n) = \int_{-\pi}^{\pi} e^{in\lambda} dz_2(\lambda)$$

where $z_1(\lambda)$ and $z_2(\lambda)$ are orthogonal processes.

We shall have to use the following two lemmas in our further work.

Lemma 1:

If $z(s)$ is an orthogonal process with the associated measure $\sigma(s)$ on the subsets s of the elements (λ) of W, and if $g_1(\lambda)$ and $g_2(\lambda)$ are complex valued functions of the real variable λ such that each of them is quadratically integrable on W with respect to the σ-measure, then we have (Karhunen [1])

$$E\left[\int_W g_1(\lambda)dz(\lambda)\ \overline{\int_W g_2(\lambda)dz(\lambda)}\right] = \int_W g_1(\lambda)\overline{g_2(\lambda)}d\sigma(\lambda) \qquad (3)$$

$$\text{where } d\sigma(\lambda) = E\left\{dz(\lambda)\ \overline{dz(\lambda)}\right\}.$$

Lemma 2:

For any discrete, real, stationary, normal process with absolutely continuous spectrum, Grenander [1] has shown that

a) $$E\left[I_N(\lambda)\right] = \frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2(2N+1)(\frac{\ell-\lambda}{2})}{\sin^2\frac{\ell-\lambda}{2}} f(\ell)d\ell,$$

b) $$D^2\left[I_N(\lambda)\right] = \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2(2N+1)(\frac{\ell-\lambda}{2})}{\sin^2\frac{\ell-\lambda}{2}} f(\ell)d\ell\right]^2 + \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin(2N+1)(\frac{\ell-\lambda}{2})}{\sin\frac{\ell-\lambda}{2}}\ \frac{\sin(2N+1)\frac{\ell+\lambda}{2}}{\sin\frac{\ell+\lambda}{2}} f(\ell)d\ell\right]^2$$

where $D^2\ I_N(\lambda)$ denotes the variance of $I_N(\lambda)$; also that

c) $$\operatorname{cov}\left[I_N(\lambda),\ I_N(\mu)\right] = R_N(\lambda,\ \mu)$$

$$= \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin(2N+1)(\frac{\ell-\lambda}{2})}{\sin\frac{\ell-\lambda}{2}}\ \frac{\sin(2N+1)\frac{\ell-\mu}{2}}{\sin\frac{\ell-\mu}{2}} f(\ell)d\ell\right]^2 + \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin(2N+1)(\frac{\ell-\lambda}{2})}{\sin\frac{\ell-\lambda}{2}}\ \frac{\sin(2N+1)(\frac{\ell+\mu}{2})}{\sin\frac{\ell+\mu}{2}} f(\ell)d\ell\right]^2,$$

where $f(\lambda)$ is the spectral density.

Using lemmas (1) and (2), it is easily seen that for our processes, i.e. for discrete, real, stationary, normal processes whose spectrum includes besides the absolutely continuous part, a step part with a finite number of saltuses,

$$E\, I_N(\lambda) = \frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2(2N+1)(\frac{\ell-\lambda}{2})}{\sin^2 \frac{\ell-\lambda}{2}}\, d\sigma_1(\ell) + \frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2(2N+1)\frac{\ell-\lambda}{2}}{\sin^2 \frac{\ell-\lambda}{2}}\, d\sigma_2(\ell), \tag{4}$$

$$D^2\, I_N(\lambda) = \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2(2N+1)\frac{\ell-\lambda}{2}}{\sin^2 \frac{\ell-\lambda}{2}}\, d\left(\sigma_1(\ell) + \sigma_2(\ell)\right)\right]^2 + \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin(2N+1)\frac{\ell-\lambda}{2}}{\sin \frac{\ell-\lambda}{2}}\, \frac{\sin(2N+1)\frac{\ell+\lambda}{2}}{\sin \frac{\ell+\lambda}{2}}\, d\left(\sigma_1(\ell) + \sigma_2(\ell)\right)\right]^2 \tag{5}$$

$$\operatorname{cov}\left[I_N(\lambda),\, I_N(\mu)\right] = R_N(\lambda, \mu) = R_N^{(1)}(\lambda, \mu) + R_N^{(2)}(\lambda, \mu),$$

where

$$R_N^{(1)}(\lambda, \mu) = \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin(2N+1)\frac{\ell-\lambda}{2}}{\sin \frac{\ell-\lambda}{2}}\, \frac{\sin(2N+1)\frac{\ell-\mu}{2}}{\sin \frac{\ell-\mu}{2}}\, d\left(\sigma_1(\ell) + \sigma_2(\ell)\right)\right]^2,$$

$$R_N^{(2)}(\lambda, \mu) = \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin(2N+1)\frac{\ell-\lambda}{2}}{\sin \frac{\ell-\lambda}{2}}\, \frac{\sin(2N+1)\frac{\ell+\mu}{2}}{\sin \frac{\ell+\mu}{2}}\, d\left(\sigma_1(\ell) + \sigma_2(\ell)\right)\right]^2. \tag{6}$$

From the nature of the two parts $x_1(n)$ and $x_2(n)$ of the process $x(n)$, their spectra $\sigma_1(\lambda)$ and $\sigma_2(\lambda)$ are respectively a pure step function and an absolutely continuous bounded measure function. Also it is evident that the spectrum $\sigma(\lambda)$ of the process $x(N)$ is the sum of $\sigma_1(\lambda)$ and $\sigma_2(\lambda)$, the spectra of the two parts.

We shall now prove

Theorem 1:

For any real, discrete, stationary, normal process whose spectrum consists of the absolutely continuous part and a step function with a finite number of saltuses, $I_N(\lambda)$ is an asymptotically unbiassed estimate of $f(\lambda)$ at a point of continuity of $\sigma(\lambda)$.

Proof:

Let $S_1, S_2, \ldots, S_p$ be the steps of $\sigma_1(\lambda)$ corresponding to the values $\lambda_1, \lambda_2, \ldots, \lambda_p$ of λ in $(-\pi, \pi)$. We have from (4)

$$E\left[I_N(\lambda)\right] = \frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2(2N+1)\frac{\ell-\lambda}{2}}{\sin^2\frac{\ell-\lambda}{2}}\, d\sigma_1(\ell) + \frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2(2N+1)\frac{\ell-\lambda}{2}}{\sin^2\frac{\ell-\lambda}{2}}\, f(\ell)d\ell, \qquad (7)$$

where $d\sigma_2(\ell) = f(\ell)d\ell$.

The first term on the right-hand side (R.H.S.) of (7) can be written as

$$\frac{1}{2\pi(2N+1)} \sum_{k=1}^{p} S_k \frac{\sin^2(2N+1)(\frac{\lambda_k-\lambda}{2})}{\sin^2\frac{\lambda_k-\lambda}{2}} .$$

If λ is a point of continuity of the spectrum $\sigma(\lambda)$ it does not coincide with any one of λ_k $k=1,2,\ldots,p$, and hence all the p terms in the above expression are finite. As $N \to \infty$ the above expression tends to zero. By Fejer's theorem the second term on the R.H.S. of (7) tends to $f(\lambda)$ as $N \to \infty$. We have thus established that

$$\lim_{N \to \infty} E\left[I_N(\lambda)\right] = f(\lambda) \text{ at a point of continuity.}$$

<u>Theorem 2</u>:

For any discrete, real, stationary, normal process whose spectrum consists of the absolutely continuous part and a step function with a finite number of saltuses, the variance $D^2\left[I_N(\lambda)\right]$ of Grenander's statistic is equal to $\left[f(\lambda)\right]^2$ or $2\left[f(\lambda)\right]^2$ according as $\lambda \neq 0$ or $\lambda = 0$ at a point of continuity of the spectrum.

<u>Proof</u>:

We have from (5)

$$D^2\, I_N(\lambda) = \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin^2(2N+1)\frac{\ell-\lambda}{2}}{\sin^2\frac{\ell-\lambda}{2}}\, d\left(\sigma_1(\ell) + \sigma_2(\ell)\right)\right]^2$$

$$+ \left[\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin(2N+1)\frac{\ell-\lambda}{2}}{\sin\frac{\ell-\lambda}{2}}\, \frac{\sin(2N+1)\frac{\ell+\lambda}{2}}{\sin\frac{\ell+\lambda}{2}}\, d\left(\sigma_1(\ell) + \sigma_2(\ell)\right)\right]^2 \quad (8)$$

By an argument similar to the one used in the proof of the previous theorem, the first term on the R.H.S. of (8) tends to $\left[f(\lambda)\right]^2$ at a point of continuity of $\sigma(\lambda)$. In the second term the contribution of the term containing $\sigma_1(\ell)$ tends to zero as $N \to \infty$ so that we have only to investigate the nature of

$$\frac{1}{2\pi(2N+1)} \int_{-\pi}^{\pi} \frac{\sin(2N+1)\frac{\ell-\lambda}{2}}{\sin\frac{\ell-\lambda}{2}}\, \frac{\sin(2N+1)\frac{\ell+\lambda}{2}}{\sin\frac{\ell+\lambda}{2}}\, f(\ell)\,d\ell \quad ..(9)$$

Case I: $\lambda = 0$.

Then in view of Fejer's theorem it is easily seen that (9) tends to $\left[f(\lambda)\right]^2_{\lambda=0}$ as $N \longrightarrow \infty$.

Case II: $\lambda \neq 0$.

We divide the range of integration $(-\pi, \pi)$ into six parts as follows. Let $\lambda > 0$ and $(-\pi, -\lambda-\epsilon)$, $(-\lambda-\epsilon, -\lambda+\epsilon)$, $(-\lambda+\epsilon, 0)$, $(0, \lambda-\epsilon')$, $(\lambda-\epsilon', \lambda+\epsilon')$, $(\lambda+\epsilon', \pi)$ be the six parts into which the range is divided where ϵ, ϵ' are small, arbitrary, positive constants and denote the corresponding integrals by I_1, I_2, I_3, I_4, I_5 and I_6. Applying the first mean value theorem, it is easily seen that I_1, I_3, I_4 and I_6 tend to zero as $N \longrightarrow \infty$. Consider

$$I_5 = \frac{1}{2\pi(2N+1)} \int_{\lambda-\epsilon'}^{\lambda+\epsilon'} \frac{\sin(2N+1)\frac{\ell-\lambda}{2}}{\sin\frac{\ell-\lambda}{2}} \; \frac{\sin(2N+1)\frac{\ell+\lambda}{2}}{\sin\frac{\ell+\lambda}{2}} f(\ell)d\ell .$$

Putting $\ell-\lambda = t$, we have

$$I_5 = \frac{1}{2\pi(2N+1)} \int_{-\epsilon'}^{\epsilon'} \frac{\sin(2N+1)\frac{t}{2}}{\sin\frac{t}{2}} \; \frac{\sin(2N+1)\frac{t+2\lambda}{2}}{\sin\frac{t+2\lambda}{2}} f(t+\lambda)dt$$

$$= \frac{1}{2\pi(2N+1)} \int_{0}^{\epsilon'} \frac{\sin(2N+1)\frac{t}{2}}{\sin\frac{t}{2}} \; \frac{\sin(2N+1)(\frac{2\lambda-t}{2})}{\sin\frac{2\lambda-t}{2}} f(\lambda-t)dt$$

$$+ \frac{1}{2\pi(2N+1)} \int_{0}^{\epsilon'} \frac{\sin(2N+1)\frac{t}{2}}{\sin\frac{t}{2}} \; \frac{\sin(2N+1)\frac{t+2\lambda}{2}}{\sin\frac{t+2\lambda}{2}} f(\lambda+t)dt \qquad (10)$$

$$\therefore \quad I_5 \leq \frac{k}{2\pi(2N+1)} \int_{0}^{\epsilon} \frac{\left|\sin(2N+1)\frac{t}{2}\right|}{\sin\frac{t}{2}} dt$$

$$< \frac{k}{2\pi(2N+1)} \int_{0}^{\pi} \frac{\left|\sin(2N+1)\frac{t}{2}\right|}{\sin\frac{t}{2}} dt ,$$

which can be written (Zygmund [1]) as

$$I_5 < \frac{k}{2\pi(2N+1)} \; O(\log N) \; .$$

Hence $\lim_{N \to \infty} I_5 = 0$. Similarly $\lim_{N \to \infty} I_2 = 0$.

$\therefore$ the expression (9) in the case $\lambda \neq 0$ tends to zero as $N \to \infty$. We have thus established that, at a point of continuity $\lambda = \lambda_0 \neq 0$ $\lim_{N\to\infty} D^2[I_N(\lambda)] = [f(\lambda_0)]^2$; while at $\lambda = 0$, $\lim_{N\to\infty} D^2[I_N(\lambda)] = 2[f(\lambda)]^2_{\lambda=0}$. Thus except in the trivial case $f(\lambda) = 0$, $I_N(\lambda)$ is not a consistent estimate of the spectral density at a point of continuity of $\sigma(\lambda)$.

We will now try to construct a weighted estimator which estimates consistently the spectral density at a point of continuity.

Consider

$$I_N(\lambda) = \frac{1}{2\pi(2N+1)} \sum_{\nu=-N}^{N} x(\nu)e^{-i\nu\lambda} \overline{\sum_{\nu=-N}^{N} x(\nu)e^{-i\nu\lambda}}$$

$x(\nu)$ being real it is easy to verify that $I_N(\lambda) = I_N(-\lambda)$, i.e. $I_N(\lambda)$ is an even function of λ in $(-\pi, \pi)$.

Let $w(\lambda)$ be an even function of λ such that in $(0, \pi)$, $w(\lambda)$ vanishes outside $(\lambda_0 \pm h)$ and h is so chosen that the h-neighborhood of λ_0 does not contain any saltus of $\sigma(\lambda)$. (12)

Consider

$$f_N^*(\lambda_0) = \int_{-\pi}^{\pi} I_N(\ell)w(\ell)d\ell$$

$$= 2 \int_{\lambda_0-h}^{\lambda_0+h} I_N(\ell)w(\ell)d\ell \; . \qquad (13)$$

Taking expectations on both sides of (13) we have

$$E\left[f_N^*(\lambda_o)\right] = 2\int_{\lambda_o-h}^{\lambda_o+h} E\left[I_N(\lambda)\right]w(\lambda)d\lambda.$$

Taking limits as $N \longrightarrow \infty$ we have at a point of continuity

$$\lim_{N\longrightarrow\infty} E\left[f_N^*(\lambda_o)\right] = 2\int_{\lambda_o-h}^{\lambda_o+h} f(\ell)w(\ell)d\ell \ . \tag{14}$$

Putting the condition for $f_N^*(\lambda)$ to estimate asymptotically unbiassedly $f(\lambda)$ at a point of continuity λ of $\sigma(\lambda)$, we have

$$2\int_{\lambda_o-h}^{\lambda_o+h} f(\ell)w(\ell)d\ell = f(\lambda_o) \ . \tag{15}$$

If $f(\lambda)$ does not vary too much in the neighborhood of λ_o the approximate condition for asymptotic unbiassedness, is

$$\int_{\lambda_o-h}^{\lambda_o+h} w(\lambda)d\lambda = \frac{1}{2} \ . \tag{16}$$

<u>Theorem 3</u>:

Let $w(\lambda)$ be a continuous weight function satisfying (12) and (16). Let the spectral density $f(\lambda)$ be continuous. Then at a point of continuity λ_o of $\sigma(\lambda)$ the variance of the weighted estimator $f_N^*(\lambda_o)$ goes to zero as $N \longrightarrow \infty$.

<u>Proof</u>:

We have from Grenander [1]

$$4\pi^2(2N+1)^2D^2f_N^*(\lambda_o) = \sum_{n,m,k,\ell=-N}^{N} r(n+m)r(k+\ell)W(n+k)W(m+\ell)$$
$$+ \sum_{n,m,k,\ell=-N}^{N} r(n+m)r(k+\ell)W(m+\ell)\overline{W(n+k)} \ , \tag{17}$$

where

$$r(n) = \int_{-\pi}^{\pi} e^{in\lambda} d\sigma(\lambda) = \int_{-\pi}^{\pi} \cos n\lambda \, d\sigma(\lambda),$$

$$W(n) = \int_{-\pi}^{\pi} e^{in\lambda} w(\lambda) d\lambda = \int_{-\pi}^{\pi} \cos n\lambda \, w(\lambda) d\lambda \tag{18}$$

Since $w(\lambda)$ is an even function we have

$$4\pi^2 (2N+1)^2 D^2 \left(f_N^*(\lambda_o) \right) = 2 \sum_{n,m,k,\ell = -N}^{N} r(n+m) r(k+\ell) W(n+k) W(m+\ell) \tag{19}$$

Again following Grenander [1] we have

$$2\pi^2 (2N+1) D^2 \left[f_N^*(\lambda_o) \right] < \sum_{\alpha,\beta,\gamma} r(\alpha) r(\beta) W(\gamma) W(\alpha+\beta-\gamma)$$

$$= \sum_{\nu=-2N}^{2N} \sum_{n=-2N}^{2N} r(n) W(n+\nu) \times \sum_{n=-2N}^{2N} r(n) W(n-\nu) \quad . \tag{20}$$

<u>Case 1</u>:

$$d\sigma(\lambda) = f(\lambda) d\lambda$$

where $f(\lambda)$ is an even function being the spectral density of a real process. We have

$$f(\lambda) = \lim_{N \to \infty} \sum_{-N}^{N} r(n) e^{-in\lambda} = \sum_{-\infty}^{\infty} r(n) \cos n\lambda \, ,$$

$$w(\lambda) = \lim_{N \to \infty} \sum_{-N}^{N} W(n) e^{-in\lambda} = \sum_{-\infty}^{\infty} W(n) \cos n\lambda \, ,$$

$$f(\lambda) w(\lambda) = \lim_{N \to \infty} \sum_{-N}^{N} d(n) e^{-in\lambda} \, ,$$

where $$d(\nu) = \sum_{-\infty}^{\infty} r(n) W(n+\nu) = \sum_{-\infty}^{\infty} W(n) r(n+\nu) \, .$$

Let us write $\quad d^{2N}(\nu) = \sum_{-2N}^{2N} r(n)W(n+\nu)$.

We have from (20)

$$2\pi^2(2N+1)D^2\left[f_N^*(\lambda_o)\right] < \sum_{-2N}^{2N} \left\{d^{2N}(\nu)\right\}^2 . \tag{21}$$

Taking the limit as $N \longrightarrow \infty$ we have,

since $\quad \sum_{-\infty}^{\infty} d^2(\nu) < \infty$,

that $\quad \lim_{N\longrightarrow\infty} D^2\left[f_N^*(\lambda_o)\right] = 0$.

Case 2:

$$d\sigma(\lambda) = f(\lambda)d\lambda + d\sigma_1(\lambda) ,$$

where $\sigma_1(\lambda)$ is a step function with a finite number of saltuses $S_1, S_2, \ldots, S_p$ at $\lambda_1, \lambda_2, \ldots, \lambda_p$ respectively.

We have from (18)

$$r(n) = r_1(n) + \sum_{i=1}^{p} S_i \cos n\lambda_i$$

We have from (20) in another form

$$\begin{aligned}
&2\pi^2(2N+1)D^2\left(f_N^*(\lambda_o)\right) \\
&< \sum_{\nu=-2N}^{2N}\left[\sum_{n=-2N}^{2N} W(n)r(n+\nu)\right]\left[\sum_{n=-2N}^{2N} W(n)r(n-\nu)\right] \\
&= \sum_{\nu=-2N}^{2N}\left\{\left[\sum_{n=-2N}^{2N} W(n)r_1(n+\nu) + \sum_{i=1}^{p} S_i \sum_{n=-2N}^{2N} W(n)\cos \overline{n+\nu}\,\lambda_i\right]\right. \\
&\qquad \left. \times \left[\sum_{n=-2N}^{2N} W(n)r_1(n-\nu) + \sum_{i=1}^{p} S_i \sum_{n=-2N}^{2N} W(n)\cos \overline{n-\nu}\,\lambda_i\right]\right\} \\
&= \sum_{\nu=-2N}^{2N}\left\{ (d^{2N}(\nu))^2 + d^{2N}(\nu)\sum_{i=1}^{p} S_i \sum_{n=-2N}^{2N} W(n)\left[\cos n\lambda_i \cos \nu\lambda_i \right.\right. \\
&\qquad\qquad \left. + \sin n\lambda_i \sin \nu\lambda_i\right] \\
&\qquad + d^{2N}(\nu)\sum_{i=1}^{p} S_i \sum_{n=-2N}^{2N} W(n)\left[\cos n\lambda_i \cos \nu\lambda_i \right. \\
&\qquad\qquad \left. - \sin n\lambda_i \sin \nu\lambda_i\right]
\end{aligned}$$

$$+ \sum_{i,j=1}^{p} s_i s_j \sum_{n=-2N}^{2N} W(n) \Big[\cos n\lambda_i \cos \nu\lambda_i + \sin n\lambda_i \sin \nu\lambda_i \Big] \times \sum_{n=-2N}^{2N} W(n) \Big[\cos n\lambda_j \cos \nu\lambda_j - \sin n\lambda_j \sin \nu\lambda_j \Big] \Big\}. \quad (22)$$

But we have, in view of the conditions imposed on the weight function, that

$$\left.\begin{aligned} &\sum_{-\infty}^{\infty} W(n) \cos n\lambda_i = W(\lambda_i) = 0 \\ &\sum_{-\infty}^{\infty} W(n) \sin n\lambda_i = 0 \\ \text{and} \quad &\sum_{-\infty}^{\infty} d^2(\nu) < \infty \quad . \end{aligned}\right\} \quad (23)$$

Taking limits on both sides of (22) as $N \to \infty$ and taking into account (23) we have, at a point of continuity of $\sigma(\lambda)$, that

$$\lim_{N \to \infty} D^2 f_N^*(\lambda_o) = 0,$$

which proves the theorem.

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References

1. Ulf Grenander: "On Empirical Spectral Analysis of Stochastic Processes" Arkiv for Mathematik, Vol. 1 (1951) pp. 503-531.

2. Kari Karhunen: "Uber Lineare Methoden In Der Wahrscheinlichkeitsrechung" Ann. Acad. Sc. Fenn. Soc. AI, Maths-Phys No. 37, 1947.

3. Antoni Zygmund: "Trigonometric Series" page 172.