A NOTE ON A CLASS OF PROBLEMS IN 'NORMAL' MULTIVARIATE ANALYSIS OF VARIANCE

by

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Summary: Let the columns of X (pxn) be independent non-singular p-dimensional normal variates with a common variance-covariance matrix and expectations given by

$$\mathcal{E} X' = A \xi$$

where A (nxm) is a matrix of known constants and ξ (mxp) is a matrix of unknown parameters. This will be called the model. Under this model consider the hypothesis

$$\mathcal{H} : \quad \xi = B \eta ,$$

where B (mxk) is a given matrix of constants and η (kxp) is a matrix of unknown parameters.

It is shown that the hypothesis $\mathcal{H}$ is "completely testable" if and only if

$$\text{rank } A + \text{rank } B - \text{rank } AB = m .$$

Further, if rank A $\leqq$ n-p, it is always possible to construct a testable hypothesis $\mathcal{H}^*$ which is implied by $\mathcal{H}$; the test-criterion proposed for $\mathcal{H}^*$ is based on the latent roots of the matrix $S_2 S_1^{-1}$ where S_1 and $(S_1 + S_2)$ are the "error-matrices of sums of squares and products" under the model and under $\mathcal{H}$, respectively. It is further shown that the rank of the matrix S_2 is min [p, rank A - rank(AB)].

Let X (pxn) be a matrix of random variables, the columns of which are independent p-dimensional normal variates with the same positive-definite variance-covariance matrix Σ (pxp) and with expectations given by

$$(1) \qquad \mathcal{E} X' = A \xi ,$$

where A (nxm) is a matrix of known constants and ξ (mxp) is a matrix of unknown parameters.

Let the rank of the matrix A be r. We shall assume that $r \leqq \min(m, n-p)$. Without loss of generality, the first r columns of A may be taken to be linearly independent and so to form a basis of A. Then [2] we can partition and factorize A in the form:

$$(2) \qquad A = [A_1(nxr) \vdots A_2(n x \overline{m-r})] = L'(nxr) [T_1'(rxr) \vdots T_2'(r x \overline{m-r})]$$

where A_1, L and T_1 are matrices each of rank r, T_1 being triangular and L semi-orthogonal, that is

$$(3) \qquad LL' = I (rxr) .$$

It is well known [2] that the error-matrix of sums of squares and products is given by

$$(4) \qquad S_1 = X E X' ,$$

where

$$(5) \qquad E = I - A_1(A_1'A_1^{-1})A_1' = I - L'L ,$$

and that this E is an nxn matrix of rank n-r. By our assumption that $r \leqq \min(m, n-p)$, the matrix S_1 is, a.e., non-singular.

Consider, now, the hypothesis that the parameters ξ can be expressed in terms of a smaller number of parameters η (kxp) in the form:

$$\mathcal{H} \quad : \quad \xi = B\,\eta ,$$

where B (mxk) is a given matrix. Under $\mathcal{H}$ the expectations are given by

(6) $$\mathcal{E}X' = AB\,\eta .$$

Let rank AB = s. Obviously $s \leqq \min(r,k)$. Here again, without any loss of generality, we can regard the first s columns of AB to be linearly independent. The rank of the matrix

$$[T_1' \,\vdots\, T_2']B$$

must be s [2] and it can be factorized the same way as (2). Thus:

(7) $$[T_1' \,\vdots\, T_2']B = M'(r\times s)\,[U_1(s\times s) \,\vdots\, U_2(s\times \overline{k-s})] ,$$

where matrices M and U_1 are each of rank s, U_1 is triangular and M semi-orthogonal, that is,

(8) $$MM' = I(s\times s) .$$

We thus have

(9) $$AB = (ML)'\,[U_1 \,\vdots\, U_2]$$

where LM (nxs) is seen to be semi-orthogonal. Using (5) it immediately follows that the error-matrix of sums of squares and products under the hypothesis is given by

(10) $$X \underset{\mathcal{H}}{E} X' ,$$

where

(11) $$E_{\mathcal{H}} = I - L'M'ML' ,$$

and that this $E_{\mathcal{H}}$ is an nxn matrix of rank n-s.

Let us choose a matrix N ($\overline{r-s}$ x r) which is an orthogonal completion of M; that is,

(12) $$NN' = I(\overline{r-s} \times \overline{r-s}) \quad \text{and} \quad NM' = 0.$$

The difference of the error-matrices (10) and (5) is the hypothesis-matrix of sums of squares and products, and is given by

(13) $$S_2 = X H X' ,$$

where

(14) $$H = E_{\mathcal{H}} - E = L'L - L'M'ML = L'N'NL \quad .$$

Using (3) it is easily checked that

(15) $$E H = 0 .$$

Thus the matrices E and H are orthogonal and S_2 is, a.e., of rank = min(p, r-s).

It will now be shown that the matrix S_2 is the appropriate hypothesis-matrix of sums of squares and products for testing a hypothesis $\mathcal{H}^*$ which is testable [2] and will be introduced presently. It will be shown that, in general, the hypotheses $\mathcal{H}$ and $\mathcal{H}^*$ are not identical; though $\mathcal{H}$ implies $\mathcal{H}^*$, the converse is not generally true.

Let the rank of the matrix B be t. Then we can find a matrix C ($\overline{m-t}$ x m) of rank (m-t) such that

(16) $$CB = 0 .$$

Since the row-vectors of C generate the vectorspace completely orthogonal to that generated by the column-vectors of B, it follows that, if C^* is any other matrix such that

(17) $$C^*B = 0 ,$$

we can factorize C^* in the form

(18) $$C^* = DC .$$

Define the matrix C^* ($\overline{r-s} \times m$) by

(19) $$C^* = N [T_1' \vdots T_2']$$

with T_1, T_2 defined by (2) and N defined by (12). Notice that this C^* is of rank r-s. Then

$$C^*B = N [T_1' \vdots T_2'] B = NM' [U_1 \vdots U_2] = 0 ,$$

because of (12). Thus, for the matrix C^*, the relation (17) holds and consequently a matrix D exists which satisfies (18).

It is easily seen that on elimination of η by pre-multiplication by C the hypothesis $\mathcal{H}$ may be expressed in the equivalent form:

(20) $$\mathcal{H} \quad : \quad C \xi = 0 .$$

Pre-multiplication by D gives

(21) $$\mathcal{H}^* \quad : \quad C^* \xi = 0 .$$

Note that D is a matrix of the form (r-s) x (m-t) of rank (r-s).

Obviously $\mathcal{H}$ implies $\mathcal{H}$ * but the converse is not true unless D is a non-singular matrix of form (m-t) x (m-t). A necessary and sufficient condition for this is that

$$r-s = m-t$$

or, in words, that

(22) $$\text{rank}(A) + \text{rank}(B) - \text{rank}(AB) = m .$$

Now partition C* in the form

$$C^* = [C_1 \vdots C_2] ,$$

where

(23) $$C_1 = NT_1' \text{ and } C_2 = NT_2' ,$$

so that

(24) $$C_2 = C_1 T_1'^{-1} T_2' .$$

Note that (23) is precisely the condition that the hypothesis $\mathcal{H}^*$ is testable [2].

The hypothesis-matrix of sums of squares and products for testing $\mathcal{H}^*$ (which is testable) computed directly from the formula given in [2] turns out to be

(25) $$S^* = X H^* X'$$

where

$$H^* = A_1(A_1'A_1^{-1}) C_1' [C_1(A_1'A_1^{-1}) C_1']^{-1} C_1(A_1'A_1^{-1})A_1'$$

$$= L'T_1^{-1}C_1' [C_1(T_1T_1')^{-1}C_1']^{-1} C_1 T_1'^{-1} L \qquad \text{(using (2))}$$

$$= L'N'(NN')^{-1} NL' \qquad \text{(using (23))}$$

$$= L'N'NL \qquad \text{(using (12))}$$

$$= H \qquad \text{(from (14))} .$$

Thus, we have

(26) $$S^* = S_2 .$$

An important special case is where we have $n > m > k$ and rank $A = m$ and rank $B = k$. In this case, rank $AB = k$. Consequently the condition (22) is satisfied and the hypotheses $\mathcal{H}$ and $\mathcal{H}^*$ are identical.

The statistical criterion for testing the hypothesis $\mathcal{H}^*$ would be based on the latent roots of the matrix $S_2S_1^{-1}$, the particular critical region proposed here [1,2] being given by

(27) $$c_{max} [S_2S_1^{-1}] \geq \lambda_\alpha(t^*, r-s, n-r)$$

where $c_{max} [S_2S_1^{-1}]$ denotes the largest characteristic root of the matrix $[S_2S_1^{-1}]$ all of whose roots are non-negative and, a.e., t^* roots are positive, $t^* = \min(p, r-s)$, and $\lambda_\alpha(t^*, r-s, n-r)$ is a constant, depending upon t^*, $r-s$, $n-r$ and the size of the critical region α, which can be obtained, since the distribution is known and the percentage points are being tabulated.

If $p=1$ we have the univariate problem, in which case (27) is replaced by a β-critical region or, after a little transformation, by an F-critical region.

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