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# A NOTE ON CONFIDENCE BOUNDS CONNECTED WITH THE HYPOTHESIS OF EQUALITY OF TWO DISPERSION MATRICES

by

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1. Summary and introduction. In some previous work by the author and R. Gnanadesikan [1,2], among other results, confidence bounds with a confidence coefficient greater than or equal to $1 - \alpha$, on the characteristic roots $c(\Sigma_1 \Sigma_2^{-1})$'s of $\Sigma_1$ and $\Sigma_2$ from $N(\underline{\xi}_1, \Sigma_1)$ and $N(\underline{\xi}_2, \Sigma_2)$ were given in the form (numbered (2.6) in [1])

$$(1.1) \qquad \lambda_1 c_{max} (S_1 S_2^{-1}) \geq \text{all } c(\Sigma_1 \Sigma_2^{-1}) \geq \lambda_2 c_{min} (S_1 S_2^{-1}),$$

where (for $i = 1,2$) $S_i$ denotes the dispersion matrix of a random sample of size $n_i+1$ from $N(\underline{\xi}_i, \Sigma_i)$, $c_{max}$ and $c_{min}$ denote the largest and smallest characteristic roots of $S_1 S_2^{-1}$ and $\lambda_1$ and $\lambda_2$ are two constants depending upon $\alpha$, $n_1$, $n_2$, $p$ (the number of variates) and the (known) joint distribution of the smallest and largest roots of $S_1 S_2^{-1}$ under $H_o$: $\Sigma_1 = \Sigma_2$. Furthermore, with any one variate $i$ ($i = 1,2,\dots,p$) cut out and the (truncated) sample and population dispersion matrices being denoted by $S_1^{(i)}$, $S_2^{(i)}$, $\Sigma_1^{(i)}$, $\Sigma_2^{(i)}$, or with any two variates $i$ and $j$ ($i \neq j = 1,2,\dots,p$) cut out and the (truncated) sample and population dispersion matrices being denoted by $S_1^{(i,j)}$, $S_2^{(i,j)}$, $\Sigma_1^{(i,j)}$ and $\Sigma_2^{(i,j)}$, and so on, the following types of confidence bounds (numbered (2.7) and (2.8) in [1]) were given:

$$(1.2) \quad \lambda_1 c_{max} (S_1^{(i)} S_2^{(i)^{-1}}) \geq \text{all } c(\Sigma_1^{(i)} \Sigma_2^{(i)^{-1}}) \geq \lambda_2 c_{min} (S_1^{(i)} S_2^{(i)^{-1}})$$

and

$$(1.3) \qquad \lambda_1 c_{max} (S_1^{(i,j)} S_2^{(i,j)^{-1}}) \geq \text{all } c(\Sigma_1^{(i,j)} \Sigma_2^{(i,j)^{-1}}) \geq \lambda_2 c_{min} (S_1^{(i,j)} S_2^{(i,j)^{-1}}),$$

with obvious extensions to truncations of a higher order. However, in obtaining the truncated confidence bounds like (1.2) and (1.3) (numbered (2.7) and (2.8) in [1]) from (2.3) of [1] certain steps were omitted, thus rendering the derivation abrupt. It is the purpose of the present note to supply the missing steps and make the derivation complete.

2. Derivation of the confidence bounds on the roots connected with the truncated matrices. We go back to [1] and note that it is perfectly clear how the passage from (2.1) to (2.3) and from (2.3) on to (2.6), which is the confidence statement on the total matrices, are made. Now starting from (2.1) or rather from the step just back of (2.1), we shall deduce a result on truncated matrices which is exactly analogous to and would be implied by (2.1) on the total matrices. The passage from this analogous result to the analogous of (2.3) and from there on to (2.7) and (2.8) of [1] will obviously be exactly identical with the passage from (2.1) to (2.3) and from there on to (2.6). Back in (2.1) of [1], we observe that the step back of (2.1) of [1] is the distribution of $Y_1(p \times n_1)$ and $Y_2(p \times n_2)$ given by

$$(2.1) \qquad \text{Const. } \exp\left[ -\frac{1}{2} \operatorname{tr} (\Sigma_1^{-1} S_1 + \Sigma_2^{-1} S_2) \right] dY_1\, dY_2,$$

where $Y_1Y_1' = n_1S_1$ and $Y_2Y_2' = n_2S_2$, and where $S_1$, $S_2$, $\Sigma_1$ and $\Sigma_2$ have been already defined. Put $\Sigma_1 = \tau_1\tau_1'$ and $\Sigma_2 = \tau_2\tau_2'$, where $\tau_1$ and $\tau_2$ are lower triangular matrices and notice that

$$\mathrm{tr}\,(\Sigma_1^{-1} S_1 + \Sigma_2^{-1} S_2) = \mathrm{tr}\,[(\tau_1\tau_1')^{-1} S_1 + (\tau_2\tau_2')^{-1} S_2]$$

$$= \mathrm{tr}\,[\tau_1^{-1} S_1 \tau_1'^{-1} + \tau_2^{-1} S_2 \tau_2'^{-1}].$$

As before, we note that the distribution of $c(S_1S_2^{-1})$'s when $\Sigma_1 = \Sigma_2$ is exactly the same as of $c[\tau_1^{-1} S_1 \tau_1'^{-1})(\tau_2^{-1} S_2 \tau_2'^{-1})^{-1}]$'s when $\Sigma_1 \neq \Sigma_2$. Thus given $\alpha$, we can find $\lambda_1$ and $\lambda_2$ and make, with a probability $1 - \alpha$, the statement

$$\lambda_1 \geq \text{all } c\,[(\tau_2^{-1} S_2 \tau_2'^{-1})(\tau_1' S_1^{-1} \tau_1)] \geq \lambda_2 , \tag{2.2}$$

which is exactly equivalent to (2.1) of [1] and also to

$$\lambda_1 \geq \frac{\underline{a}'[\tau_2^{-1} S_2 \tau_2'^{-1}]\underline{a}}{\underline{a}'[\tau_1^{-1} S_1 \tau_1'^{-1}]\underline{a}} \geq \lambda_2, \tag{2.3}$$

for all nonnull $\underline{a}$. Let us now partition $\tau_1$ and $\tau_2$ into

$$\tau_1 = \begin{bmatrix} \tau_{11} & 0 \\ \tau_{12} & \tau_{13} \end{bmatrix} \begin{matrix} r \\ p-r \end{matrix} \qquad \tau_2 = \begin{bmatrix} \tau_{21} & 0 \\ \tau_{22} & \tau_{23} \end{bmatrix} \begin{matrix} r \\ p-r \end{matrix} , \tag{2.4}$$

$$\qquad\quad r \quad p-r \qquad\qquad\qquad\qquad r \quad p-r$$

where, of course, $\tau_{11}, \tau_{13}, \tau_{21}, \tau_{23}$ are lower triangular, and $\tau_{12}$ and $\tau_{22}$ are solid. A so put

$$S_1 = T_1T_1' , \quad S_2 = T_2T_2' , \tag{2.5}$$

where $T_1$ and $T_2$ are lower triangular matrices, and partition $T_1$ and $T_2$ into

$$(2.6) \qquad T_1 = \begin{array}{ccl} T_{11} & 0 & r \\ T_{12} & T_{13} & p-r \\ r & p-r & \end{array} , \qquad T_2 = \begin{array}{ccl} T_{21} & 0 & r \\ T_{22} & T_{23} & p-r \\ r & p-r & \end{array}$$

where, as in (2.4), $T_{11}$, $T_{13}$, $T_{21}$, $T_{23}$ are lower triangular, and $T_{12}$ and $T_{22}$ are solid. It is easy to check that $\tau_1^{-1}$ and $\tau_2^{-1}$ are also lower triangular matrices and can be expressed as

$$(2.7) \qquad \tau_1^{-1} = \left[ \begin{array}{cc} \tau_{11}^{-1} & 0 \\ -\tau_{13}^{-1}\, \tau_{12}\, \tau_{11}^{-1} & \tau_{13}^{-1} \end{array} \right] \begin{array}{l} r \\ p-r \end{array} \qquad \text{and}$$
$$\qquad\qquad\qquad\qquad\; r \qquad\qquad p-r$$

$$\tau_2^{-1} = \left[ \begin{array}{cc} \tau_{21}^{-1} & 0 \\ -\tau_{23}^{-1}\, \tau_{22}\, \tau_{21}^{-1} & \tau_{23}^{-1} \end{array} \right] \begin{array}{l} r \\ p-r \end{array} \qquad .$$
$$\qquad\qquad\qquad\qquad\; r \qquad\qquad p-r$$

Using (2.5), (2.6) and (2.7) we can now rewrite (2.3) as

$$(2.8) \qquad \lambda_1 \geq \frac{\underline{a}' \left[ \tau_2^{-1}\, T_2\, T_2'\, \tau_2'^{-1} \right] \underline{a}}{\underline{a}' \left[ \tau_1^{-1}\, T_1\, T_1'\, \tau_1'^{-1} \right] \underline{a}} \geq \lambda_2 ,$$

for all nonnull $\underline{a}$, where

$$(2.9) \qquad \tau_2^{-1}\, T_2 = \left[ \begin{array}{cc} \tau_{21}^{-1}\, T_{21} & 0 \\ -\tau_{23}^{-1}\, \tau_{22}\, \tau_{23}^{-1}\, T_{21} + \tau_{23}^{-1}\, T_{22} & \tau_{23}^{-1}\, T_{23} \end{array} \right] \begin{array}{l} r \\ p-r \end{array}$$
$$\qquad\qquad\qquad\qquad\qquad\quad r \qquad\qquad\qquad\qquad\qquad p-r$$

and

$$(2.10) \qquad \tau_1^{-1} T_1 = \begin{bmatrix} \tau_{11}^{-1} T_{11} & 0 \\ -\tau_{13}^{-1}\tau_{12}\tau_{13}^{-1} T_{11} + \tau_{13}^{-1} T_{12} & \tau_{13}^{-1} T_{13} \end{bmatrix} \begin{matrix} r \\ p-r \end{matrix} \quad .$$

$$\qquad\qquad\qquad r \qquad\qquad\qquad\qquad p-r$$

The matrices $T_2' \tau_2'^{-1}$ and $T_1' \tau_1'^{-1}$ are the transposes of the right sides of (2.9) and (2.10) nd can be easily written in the analogous partitioned forms. If we now specialize $\underline{a}$ so that the last p-r components are zero, and denote by $\underline{a}^*$ the r-dimensional vector with the first r components of the specialized $\underline{a}$, then it is easy to check that (2.8) $\Longrightarrow$

$$(2.11) \qquad \lambda_1 \geq \frac{\underline{a}^{*\prime} [\tau_{21}^{-1} T_{21} T_{21}' \tau_{21}'^{-1}] \underline{a}^*}{\underline{a}^{*\prime} [\tau_{11}^{-1} T_{11} T_{11}' \tau_{11}'^{-1}] \underline{a}^*} \geq \lambda_2 ,$$

for all nonnull (r-dimensional) $\underline{a}^*$. Furthermore, notice that

$$(2.12) \qquad S_1^{(r+1,\ldots,p)} = T_{11}T_{11}', \quad S_2^{(r+1,\ldots,p)} = T_{21}T_{21}' ,$$

$$\Sigma_1^{(r+1,\ldots,p)} = \tau_{11}\tau_{11}', \quad \Sigma_2^{(r+1,\ldots,p)} = \tau_{21}\tau_{21}' ,$$

where the symbols on the left side of all the equations stand for the truncated matrices obtained by cutting out the last p-r variates. Now since (2.3) $\Longleftrightarrow$ (2.2) $\Longleftrightarrow$ (2.1) of [1] $\Longleftrightarrow$ (2.3) of [1] $\Longrightarrow$ (2.6) of [1], therefore, (2.11) $\Longleftrightarrow$ (2.13) analogous to (2.2), but on the truncated matrices $\Longleftrightarrow$ (2.14) analogous to (2.1) of [1], but on the truncated matrices $\Longleftrightarrow$ (2.15) analogous to (2.3) of [1], but on the truncated matrices $\Longrightarrow$ (2.16) analogous to (2.6) of [1], but on the truncated matrices. Now, since (2.8) $\Longrightarrow$ (2.11) and (2.11) $\Longrightarrow$ (2.16), we have, with a confidence coefficient $\geq 1-\alpha$, the confidence statement.

$$(2.16) \qquad \lambda_1 \, c_{max} [S_1^{(r+1,\ldots,p)} \, S_2^{(r+1,\ldots,p)^{-1}}]$$
$$\geq \text{all } c[\Sigma_1^{(r+1,\ldots,p)} \, \Sigma_2^{(r+1,\ldots,p)^{-1}}]$$
$$\geq \lambda_2 \, c_{min} [S_1^{(r+1,\ldots,p)} \, S_2^{(r+1,\ldots,p)^{-1}}].$$

It is clear that, starting from (2.1), we could have cut out any p-r variates instead of the last (p-r) ones (with $r = 1,2,\ldots,p-1$), and thus we have, with a confidence coefficient $\geq 1-\alpha$, the set of $\binom{p}{2}$ confidence statements of the type (1.2) (with $i = 1,2,\ldots,p$), of the type (1.3) (with $i \neq j = 1,2,\ldots,p$), and so on.

## References

[1] Roy, S. N. and Gnandesikan, R., "Further contributions to multivariate confidence bounds," North Carolina Institute of Statistics Mimeograph Series No. 155.

[2] Roy, S. N. and Gnandesikan, R., "Further contributions to multivariate confidence bounds," Biometrica, Vol. 44 (1957), pp. 399-410.