A CLASS OF TWO REPLICATE INCOMPLETE BLOCK DESIGNS

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0. Introduction and Summary.

Two replicate incomplete block designs for comparative trials are useful when experimental units are costly and/or when experimental error is small. Not many are known [1] [3] [6] [7] [8]. In this paper a new class of two replicate designs called Simple Partially Linked Block designs is introduced. It is shown that with any of these designs, the variance of the estimate of the difference in effects of two treatments can be at most of seven different types. The general procedure of intra and inter-block analysis is developed and illustrated with a numerical example. A list of these designs involving ten or fewer plots per block is given together with the values of parameters required in the analysis and the values of the efficiency-factor. It turns out that most of these designs are highly efficient with an efficiency-factor of the order of 75 o/o. It is indicated how other two replicate designs can be derived from these designs.

1. Two general methods of analysis of experiments in randomized incomplete blocks.

Consider an experiment in randomized incomplete blocks in which v treatments are tried in b blocks, each of k plots, each plot getting just one treatment and each treatment being applied on at most one plot in a block and altogether on r plots. Let λ_{ju} denote the number of blocks in which the j-th and the u-th treatments occur together and μ_{it} the number of treatments common between the i-th and the t-th blocks. $(j,u = 1, 2, \ldots, v; \; i,t = 1, 2, \ldots, b)$.

Let the "effect" of the j-th treatment be denoted by Θ_j and that of the i-th block by β_i. The usual assumption is that y_{ji} the "yield" of the plot in the i-th

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block getting the j-th treatment is given by

$$y_{ji} = \beta_i + \theta_j + e_{ji}$$

where e_{ji}'s are mutually uncorrelated random variables, each with expectation zero and variance σ^2. For intra-block estimation β_i's are regarded as constants whereas for combined inter- and intra-block estimation β_i's are regarded as random variables mutually uncorrelated and also uncorrelated with the e_{ji}'s, each with the same expectation β and variance σ_1^2. We shall write $\delta = \sigma^2/\sigma_1^2$.

Let B_i denote the total "yield" of all plots in the i-th block and T_j that for all plots getting the j-th treatment. We shall use the symbol $\{B\}_j$ to denote the total for all blocks in which the j-th treatment occurs and similarly the symbol $\{T\}_i$ to denote the total for all treatments which occur in the i-th block. Thus

$$\{B\}_j = \sum_{i=1}^{b} n_{ji} B_i$$

$$\{T\}_i = \sum_{j=1}^{v} n_{ji} T_j$$

where $n_{ji} = 1$ if the j-th treatment occurs in the i-th block and $n_{ji} = 0$ otherwise. Let

$$Q_j = T_j - \frac{1}{k}\{B\}_j$$

$$\bar{Q}_j = T_j - \frac{1}{k+\delta}\{B\}_j$$

$$P_i = B_i - \frac{1}{r}\{T\}_i \quad .$$

It is well known [9] that the equations for intra-block estimation of treatment effects are

$$(1.1) \qquad c_{j1}\theta_1 + c_{j2}\theta_2 + \dots + c_{jv}\theta_v = Q_j \qquad j = 1, 2, \dots, v$$

where

$$c_{ju} = \begin{cases} r(1 - \frac{1}{k}) & \text{if} \quad j = u \\ -\frac{\lambda_{ju}}{k} & \text{if} \quad j \neq u \end{cases} .$$

The design is said to be "connected" if the matrix $C = ((c_{ju}))$ is of rank (v-1). In whatever follows, we shall be concerned with connected designs only. We shall denote by $(t_1, t_2, \ldots, t_v)$ any particular solution of these equations (1.1).

For combined inter- and intra-block estimation, the equations are

$$\bar{c}_{j1}\theta_1 + \bar{c}_{j2}\theta_2 + \ldots + \bar{c}_{jv}\theta_v = \bar{Q}_j \qquad (j = 1, 2, \ldots, v) \tag{1.2}$$

where

$$\bar{c}_{ju} = \begin{cases} r(1 - \frac{1}{k+\delta}) & \text{if} \quad j = u \\ -\frac{\lambda_{ju}}{k+\delta} & \text{if} \quad j \neq u \end{cases} .$$

We shall denote a solution of these equations (1.2) by $(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_v)$.

We note that the equations (1.2) for combined estimation are obtained easily from the equations (1.1) for intra-block estimation merely by replacing k by k+δ both in the expressions for c_{ju}'s and Q_j's. This is somewhat simpler than what is suggested in [9]. We shall call this the Q-method of estimation.

An alternative method [10] is to estimate the treatment-effects from the equations

$$\theta_j = \frac{1}{r}[T_j - \{\beta\}_j] \tag{1.3}$$

where β_j denotes the total of the effects of all the blocks in which the j-th treatment occurs:

$$\{\beta\}_j = \sum_{i=1}^{b} n_{ji}\beta_i .$$

For intra-block estimation the block-effects are estimated from the equations

(1.4) $$d_{i1}\beta_1 + d_{i2}\beta_2 + \dots + d_{ib}\beta_b = P_i \qquad (i = 1, 2, \dots, b)$$

where

$$d_{it} = \begin{cases} k(1 - \frac{1}{r}) & \text{if} \quad i = t \\ -\frac{\mu_{it}}{r} & \text{if} \quad i \neq t \end{cases} .$$

We shall denote a solution of these equations (1.4) by $(b_1, b_2, \dots, b_b)$. For combined inter- and intra-block estimation, the block-effects are estimated from the equations:

(1.5) $$\bar{d}_{i1}\beta_1 + \bar{d}_{i2}\beta_2 + \dots + \bar{d}_{ib}\beta_b = P_i \qquad (i = 1, 2, \dots, b)$$

where

$$\bar{d}_{it} = \begin{cases} k(1 - \frac{1}{r}) + \delta & \text{if} \quad i = t \\ -\frac{\mu_{it}}{r} & \text{if} \quad i \neq t \end{cases} .$$

We note that the equations (1.4) and (1.5) are identical except that the diagonal terms d_{ii} and $\bar{d}_{ii}$ differ by δ. A solution of the equations (1.4) will be denoted by $(\bar{b}_1, \bar{b}_2, \dots, \bar{b}_b)$. The method here described is a somewhat simplified version of what is suggested in [10]. We shall call this the P-method of estimation.

To estimate the parameters σ^2 and δ, the analysis of variance can be carried out as follows:

ANALYSIS OF VARIANCE

Variation due to	Sum of squares	Degrees of freedom	Sum of squares	Variation due to
Blocks (Unadjusted)	S_B^*	b-1	S_B	Blocks(Adjusted)
Treatments(Adjusted)	S_T	v-1	S^*_T	Treatments(Unadjusted)
Error	S_E	n-b-v+1	S_E	Error
Total	T	n-1	T	Total

Here, $n = bk = vr$ denotes the total number of plots, G is the total and G_2 the total of the squares of all the "yield's" and

$$S_B^* = \frac{1}{k}\sum_{i=1}^{b} B_i^2 - \frac{G^2}{n} \qquad S_B = \sum_{i=1}^{b} b_i P_i = S_B^* + S_T - S_T^*$$

$$S_T = \sum_{j=1}^{v} t_j Q_j = S_B + S_T^* - S_B^* \qquad S_T^* = \frac{1}{r}\sum_{j=1}^{v} T_j^2 - \frac{G^2}{n}$$

$$S_E = T - S_B^* - S_T = T - S_B - S_T^*$$

$$T = G_2 - \frac{G^2}{n}$$

Then unbiassed estimates of σ^2 and σ_1^2 are obtained from the fact that the expectations of S_E and S_B are given by:

$$(S_E) = (n - b - v + 1)\sigma^2$$

$$(S_B) = (b-1)\sigma^2 + (n-v)\sigma_1^2 \quad .$$

As an estimate of δ one can take

$$d = \frac{(b-1)S_E}{(n-b-v+1)S_B - (n-v)S_E} \tag{1.6}$$

in the sense that the ratio of the expectations of the numerator and the denominator of d is equal to δ. Generally d is a consistent estimate of δ.

Consider the variance of the intra-block estimate of $\Theta_j - \Theta_u$ the difference between the effects of the j-th and the u-th treatments,

$$Var(t_j - t_u) = v_{ju}\,\sigma^2 , \quad \text{say.}$$

The average variance of all such differences is given by

$$\bar{V} = \frac{1}{\binom{v}{2}} \sum_{j=1}^{v} \sum_{u=j+1}^{v} v_{ju} \sigma^2$$

whereas in a randomized block experiment any such difference would be estimated with a variance

$$\frac{2}{r} \sigma^2$$

if the error variance were the same. This bears to the former the ratio

(1.7) $$E = \frac{v(v-1)}{r} \Big/ \sum_{j=1}^{v} \sum_{u=j+1}^{v} v_{ju}$$

which is called the "efficiency-factor" of the design. Designs with high efficiency factors are generally preferred. It has been shown [12] that

$$0 < E \leq \frac{1 - 1/k}{1 - 1/v} \quad .$$

2. Dualization of designs.

New designs have sometimes been [8] [11] [13] constructed by dualization, that is by interchanging the role of the blocks and treatments of a given design. Consider a design D^* involving v^* treatments in b^* blocks, each of k^* plots, such that each treatment occurs at most once on each plot and altogether on r^* plots. It is easy to see that the dual design D will involve $v = b^*$ treatments in $b = v^*$ blocks each of $k = r^*$ plots and each treatment will occur in $r = k^*$ plots. It is also obvious that if the design D^* is easily analyzable by the Q-method, the dual design D can be readily analyzed by the P-method. It has also been shown [12] that the efficiency factor E of the design D is given by

(2.1) $$E = \frac{(b^*-1)E^*}{(b^*-v^*)E^* + (v^*-1)}$$

where E^* is the efficiency-factor of the design D^*. Consequently

$$E \gtreqless E^* \quad \text{according as} \quad b^* \gtreqless v^* \ .$$

Therefore, if we start with a design D^* with a reasonably high efficiency-factor in which $b^* > v^*$, by dualizing it, we always get a design whose efficiency-factor is still higher.

3. Partially balanced association schemes.

Given $n_1 + n_2 + \dots + n_m + 1$ objects, a relation satisfying the following conditions is said to be a partially balanced association scheme with m associate classes: [2], [4]

(i) Any two objects are either 1-st, or 2-nd, ..., or m-th associates.

(ii) The relation of association is symmetrical, that is, if the object α is the i-th associate of the object β, then β is the i-th associate of α $(i = 1, 2, \dots, m)$

(iii) Each object has n_1 first associates, n_2 second associates, ..., n_m m-th associates.

(iv) If any two objects α and β are i-th associates, then the number of objects which are the j-th associates of α and the k-th associates of β is p^i_{jk}, independent of the pair of i-th associates $(i,j,k = 1,2,\dots,m)$

The parameters p^i_{jk} are not all independent. They satisfy, for instance, the following relations:

$$p^i_{jk} = p^i_{kj}$$

$$n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij}$$

$$p^i_{j1} + p^i_{j2} + \dots + p^i_{jm} = \begin{cases} n_j - 1 & \text{if } i = j \\ n_j & \text{if } i \neq j \end{cases} .$$

Partially balanced association schemes with two associate classes are classified in [4] and listed in [5]. Though not exhaustive, these cover all the known cases. The five types discussed in [4] are (1) Group Divisible (GD), (2) Triangular (T), (3) Latin square type (LS), (4) Cyclic (C) and (5) Simple (S1).

In a Group Divisible type of association scheme, there are mn objects which fall into m groups of n objects each. Any two objects in the same group are first associates and two objects from different groups are second associates. Thus

$$n_1 = n - 1, \qquad n_2 = n(m - 1)$$

$$((p^1_{ij})) = \begin{bmatrix} n-2 & 0 \\ 0 & n(m-1) \end{bmatrix}, \qquad ((p^2_{ij})) = \begin{bmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{bmatrix}.$$

In a Triangular type of association scheme, there are $p(p-1)/2$ objects arranged in a square array of p rows and p columns as follows: (i) The positions in the principal diagonal (running from the upper left-hand to the lower right-hand corner) are left blank. (ii) The $p(p-1)/2$ positions above the principal diagonal are filled by the numbers 1, 2, ..., $p(p-1)/2$ corresponding to the objects. (iii) The $p(p-1)/2$ positions below the principal diagonal are filled so that the array is symmetrical about the principal diagonal. Two objects are first associates if they lie in the same row (or column) in this array, otherwise they are second associates. For this scheme

$$n_1 = 2p - 4 \qquad n_2 = (p-2)(p-3)/2$$

$$((p^1_{ij})) = \begin{bmatrix} p-2 & p-3 \\ p-3 & (p-3)(p-4)/2 \end{bmatrix}, \qquad ((p^2_{ij})) = \begin{bmatrix} 4 & 2p-8 \\ 2p-8 & (p-4)(p-5)/2 \end{bmatrix}.$$

In the Latin square type of association scheme with t constraints, there are s^2 objects arranged in a square scheme. For the case $t = 2$, two objects are first associates if they occur in the same row or in the same column; otherwise they are second associates. For the general case, a set of $(t-2)$ mutually orthogonal Latin squares are taken and two objects are first associates if they occur in the same row or column or correspond to the same letter of one of the Latin squares. In this scheme

$$n_1 = t(s-1), \qquad n_2 = (s-1)(s-t+1)$$

$$((p^1_{ij})) = \begin{bmatrix} t^2 - 3t + s & (t-1)(s-t+1) \\ (t-1)(s-t+1) & (s-t)(s-t+1) \end{bmatrix}$$

$$((p^2_{ij})) = \begin{bmatrix} t(t-1) & t(s-t) \\ t(s-t) & (s-t)^2 + t - 2 \end{bmatrix} .$$

4. Simple partially linked block designs.

An allocation of v treatments in b blocks each of k plots will be called a simple partially linked block (SPLB) design if the following conditions are satisfied:

(i) Each treatment occurs at most on one plot in a block and altogether on two plots.

(ii) Any two blocks have at most one treatment in common.

(iii) Two blocks are first (second) associates if they have one (no) treatment in common and this association scheme is partially balanced with parameters n_1, n_2 and p^i_{jk}.

Thus,

$$v = \frac{1}{2}n_1(n_1+n_2+1)$$

$$b = n_1 + n_2 + 1$$

$$k = n_1$$

and, of course,

$$r = 2.$$

SPLB designs can be easily constructed as follows. Given any partially balanced association scheme $\mathcal{S}$ with two associate classes, a design D^* with $v^* = n_1 + n_2 + 1$ treatments and $k^* = 2$ plots per block can be easily constructed by considering the objects as treatments and forming one block with each pair of treatments that are first associates. In this design D^*, obviously there are $b^* = \frac{1}{2}n_1(n_1+n_2+1)$ blocks, each treatment occurs on $r^* = n_1$ plots and any pairs of treatments occur together on one or no block according as they are first or second associates. The dual D of this design D^* is obviously a SPLB design. By interchanging the nomenclature of "first" and "second" associates, a second SPLB design can similarly be constructed from the same association scheme $\mathcal{S}$.

A SPLB design can easily be analyzed by the P-method described in section 1. Let us use the notation S_1 to denote summation over first associates. The equations (1.4) for intra-block estimation reduce to

$$n_1\beta_i - S_1(\beta_i) = 2P_i \quad .$$

If $p_{11}^2 \neq 0$, the general solution of these equations, except for an arbitrary constant, is given by

$$b_i = \ell(2P_i) + \ell_1 S_1(2P_i) \qquad (4.1)$$

where

$$\ell = a/A$$

$$\ell_1 = 1/A$$

and

$$a = n_1 + p_{11}^2 - p_{11}^1$$
$$A = b\ p_{11}^2 \quad .$$

The intra-block estimates of treatment effects are then given by

$$(4.2) \qquad t_j = \tfrac{1}{2} \left[T_j - \{ b \}_j \right]$$

where b_j denotes the sum of the b_i's for the two blocks in which the j-th treatment occurs.

Take two treatments: say, the j-th and the u-th. Two cases may arise: (X) the two treatments occur together in a block or (Y) they do not.

In case (X) there are three blocks in which at least one of the two treatments occurs. In one of these blocks both the treatments occur. Consider the other two blocks. We shall say that the j-th and the u-th treatments form a pair of the type X_1 if these two blocks are first associates and of type X_2 if these two blocks are second associates.

In case (Y) there are four blocks, in two of which the j-th treatment occurs and in the other two the u-th treatment occurs. With these four blocks, it is possible to form four different pairs of blocks such that in each pair there is one block containing the j-th treatment and one block containing the u-th treatment. If ν is the number of first associate pairs amongst these four pairs of blocks, we shall say that the j-th and the u-th treatments form a pair of the type Y_ν (ν = 0, 1, 2, 3, 4).

We have thus classified all possible pairs of treatments into seven distinct types: X_1, X_2 and Y_0, Y_1, Y_2, Y_3, Y_4.

Consider now the variance of the intra-block estimate of $\theta_j - \theta_u$, the difference between the effects of the j-th and the u-th treatments. After a little computation, it is seen that

$$\mathrm{Var}(t_j - t_u) = v_{ju}\sigma^2$$

where the value of v_{ju} depends on the type of pair formed by the j-th and the u-th treatments and is tabulated below:

Type of pair of treatments	Value of v_{ju}
X_1	$1 + \lambda - \lambda_1$
X_2	$1 + \lambda$
Y_o	$1 + 2\lambda + 2\lambda_1$
Y_1	$1 + 2\lambda + \lambda_1$
Y_2	$1 + 2\lambda$
Y_3	$1 + 2\lambda - \lambda_1$
Y_4	$1 + 2\lambda - 2\lambda_1$

We thus see that in all seven different precisions are possible.

To compute the efficiency-factor E of the SPLB design, we observe that the efficiency-factor E^* of the dual design is given by

$$E^* = \frac{n_1 + n_2}{2n_1[\lambda(n_1+n_2) - \lambda_1 n_1]}$$

and therefore from (2.1),

$$E = \frac{(v-1)A}{(v-b)A + 2n_1\{a(b-1) - n_1\}} \quad . \tag{4.3}$$

For combined inter- and intra-block estimation, we have from (1.5)

$$(n_1 + 2\delta)\beta_i - S_1(\beta_i) = 2P_i \quad .$$

The general solution of this (except for an arbitrary constant) is given by

$$(4.4) \qquad b_i = \bar{\gamma}(2P_i) + \bar{\gamma}_1 S_1(2P_i)$$

where

$$\bar{\gamma} = \bar{a}/\bar{A}$$

$$\bar{\gamma}_1 = 1/\bar{A}$$

and

$$\bar{a} = a + 2\delta$$

$$\bar{A} = A + 2\delta(a+n_1) + 4\delta^2 .$$

The combined estimates of the treatment effects are then given by

$$(4.5) \qquad \bar{t}_j = \frac{1}{2}\left[T_j - \{\bar{b}\}_j\right] ,$$

where $\{\bar{b}\}_j$ denotes the sum of the $\bar{b}_i$'s for the two blocks in which the j-th treatment occurs.

The analysis of variance can be carries out as in section 1. The various components are computed in the following order: first the total sum of squares T, then the unadjusted block sum of squares S_B^* and the unadjusted treatment sum of squares S_T^*, next the adjusted block sum of squares $S_B = \sum_{i=1}^{b} b_i P_i$ and finally the adjusted sum of squares $S_T = S_B + S_T^* - S_B^*$ and the error sum of squares $S_E = T - S_B^* - S_T = T - S_B - S_T^*$.

5. <u>A list of simple partially linked block design with ten or fewer plots per block.</u>

A list of SPLB designs with $k \leq 10$ derivable from known partially balanced association schemes is presented here. The list is arranged in increasing order of v and under the same v, in increasing order of k. The values of the parameters v,b,k = n_1,a,A,E and the type of the association scheme are shown. The word "interchange" indicates that in the definition of the association schemes given in section 3, the nomenclature of first and second associates have to be interchanged. Of the designs listed, the lattice designs are, of course, well known and a few others with a GD type of association scheme are given in [8]. The other designs are new

List of SPLB designs with $k \leq 10$

Serial Number	v	$k=n_1$	b	a	A	E	Association scheme
	4	2	4	4	8	0.600	GD m = 2, n = 2 interchange
	9	3	6	6	18	0.667	GD m = 2, n = 3 interchange
	12	4	6	6	24	0.750	T p = 4
	15	3	10	4	10	0.565	T p = 5 interchange
	16	4	8	8	32	0.714	GD m = 2, n = 4 interchange
	18	4	9	5	18	0.680	LS s = 3, i = 2
	24	6	8	8	48	0.807	GD m = 4, n = 2 interchange
	25	5	10	10	50	0.750	GD m = 2, n = 5
	27	6	9	9	54	0.796	GD m = 3, n = 3 interchange
	30	6	10	7	40	0.782	T p = 5
	36	6	12	12	72	0.778	GD m = 2, n = 6 interchange
	39	6	13	7	39	0.760	C
	40	8	10	10	80	0.841	GD m = 5, n = 2 interchange
	45	6	15	8	45	0.755	T p = 6 interchange
	48	6	16	6	32	0.740	LS s = 4, i = 2
	48	8	12	12	96	0.829	GD n = 4, m = 3 interchange
	49	7	14	14	98	0.800	GD m = 2, n = 7
	54	9	12	12	108	0.848	GD n = 3, m = 4 interchange
	57	6	19	7	38	0.738	Sl.
	60	8	15	8	60	0.812	T p = 6
	60	10	12	12	120	0.863	GD m = 6, n = 2 interchange
	64	8	16	16	128	0.818	GD m = 2, n = 8 interchange
	68	8	17	9	68	0.807	C
	72	9	16	11	96	0.833	LS s = 4, i = 3
	75	10	15	15	150	0.854	GD m = 3, n = 5 interchange
	81	9	16	18	162	0.833	GD m = 2, n = 9 interchange
	100	8	25	7	50	0.784	LS s = 5, i = 2
	100	10	20	20	200	0.846	GD m = 2, n = 10
	105	10	21	9	84	0.836	T p = 7
	105	10	21	13	126	0.841	T p = 7 interchange
	130	10	26	11	104	0.832	Sl.
	135	10	27	14	135	0.835	Sl.
	180	10	36	8	72	0.813	LS s = 6, i = 2

*Lattice

6. Numerical illustration.

To illustrate the numerical procedure, let us consider the following artificial data (Table 6.1) giving the plan and the yields of a randomized experiment with a SPLB design involving 15 treatments in 10 blocks each of 3 plots. The figures in brackettes indicate the serial numbers for the treatments and the figures below them are the corresponding yields.

TABLE 6.1 FIELD PLAN AND YIELDS

Blocks				Blocks			
1	(13)	(6)	(3)	6	(14)	(11)	(4)
	4.5	5.8	4.2		4.7	5.1	3.8
2	(10)	(13)	(7)	7	(10)	(11)	(12)
	9.9	5.3	6.7		6.3	5.7	5.8
3	(2)	(3)	(1)	8	(15)	(8)	(5)
	4.6	4.4	2.3		4.9	8.0	7.5
4	(7)	(8)	(9)	9	(12)	(15)	(1)
	3.9	7.1	0.8		7.3	4.2	2.4
5	(6)	(4)	(5)	10	(2)	(9)	(14)
	2.2	3.5	5.3		8.6	3.0	5.4

This was obtained by taking the Triangular association scheme with p=5 represented by the following square array:

x	1	2	3	4
1	x	5	6	7
2	5	x	8	9
3	6	8	x	10
4	7	9	10	x

in which each number stands for an object and two objects are first associates if they do not occur together in the same row nor in the same column. Thus

$$n_1 = 3 \qquad\qquad n_2 = 6$$

$$((p^1_{ij})) = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix} \qquad ((p^2_{ij})) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Forming blocks with each pair of first associates, we get the design D*

Blocks	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Treatments	1,8	1,9	1,10	2,6	2,7	2,10	3,5	3,7	3,9	4,5	4,6	4,8	5,10	6,9	7,8

with parameters $v^* = 10$, $b^* = 15$, $k^* = 2$, $r^* = 3$. Dualizing we get the SPLB design:

Blocks	1	2	3	4	5	6	7	8	9	10
Treatments	1,2,3	4,5,6	7,8,9	10,11,12	7,10,13	4,11,14	5,8,15	1,12,15	2,9,14	3,6,13

with parameters $v = 15$, $b = 10$, $k = 3$, and of course, $r = 2$.

For this design, we have

$$a = n_1 + p^2_{11} - p^1_{11} = 3 + 1 - 0 = 4$$

$$A = bp^2_{11} = 10.1 = 10$$

$$\ell = a/A = 0.4$$

$$\ell_1 = 1/A = 0.1$$

and

$$E = \frac{(v-1)\,A}{(v-b)A + 2n_1\left\{a(b-1)-n_1\right\}} = \frac{(15-1)\ .\ 10}{(15-10)\ . + 2.3\left\{4\ .\ (10-1) - 3\right\}}$$

$$= \frac{140}{50 + 6.(33)} = \frac{140}{248} = 0.565$$

In the actual lay-out the blocks and the plots within a block have been re-arranged at r andom.

TABLE 6.2 DETAILS OF COMPUTATION

Blocks i	B_i	$\{T\}_i$	$2P_i$	First associates of i	$S_1(2P_i)$	b_i	$\bar{b}_i$
1	14.5	26.4	2.6	2, 3, 5	-2.8	0.76	0.5032
2	21.9	36.6	7.2	1, 4, 7	-7.8	2.10	1.3916
3	11.3	26.5	-3.9	1, 9, 10	10.4	-0.52	-0.4848
4	11.8	29.5	-5.9	2, 8, 10	17.9	-0.57	-0.6391
5	11.0	28.1	-6.1	1, 6, 8	5.4	-1.90	-1.2317
6	13.6	28.2	-1.0	5, 7, 10	-3.7	-0.77	-0.4017
7	17.8	40.1	-4.5	2, 6, 9	7.1	-1.09	-0.7728
8	20.4	37.0	3.8	4, 5, 9	-11.1	0.41	0.4303
9	13.9	26.9	0.9	3, 7, 8	-4.6	-0.10	0.0160
10	17.0	27.1	6.9	3, 4, 6	-10.8	1.68	1.1888
Total	153.2	306.4	0*		0*	0*	-0.0002*

Treatments	T_j	Blocks in which treatment j occurs	$\{b\}_j$	t_j	$\{\bar{b}\}_j$	$\bar{t}_j$
1	4.7	3, 9	-0.62	2.66	-0.4688	2.584
2	13.2	3, 10	1.16	6.02	0.7040	6.248
3	8.6	1, 3	0.24	4.18	0.0184	4.291
4	7.3	5, 6	-2.67	4.98	-1.6334	4.467
5	12.8	5, 8	-1.49	7.14	-0.8014	6.801
6	8.0	1, 5	-1.14	4.57	-0.7285	4.364
7	10.6	2, 4	1.53	4.54	0.7525	4.924
8	15.1	4, 8	-0.16	7.63	-0.2088	7.654
9	3.8	4, 10	1.11	1.34	0.5497	1.625
10	16.2	2, 7	1.01	7.60	0.6188	7.791
11	10.8	6, 7	-1.86	6.33	-1.1745	5.987
12	13.1	7, 9	-1.19	7.14	-0.7568	6.928
13	9.8	1, 2	2.86	3.47	1.8948	3.953
14	10.1	6, 10	0.91	4.60	0.7871	4.656
15	9.1	8, 9	0.31	4.40	0.4463	4.327
Total	153.2		0*	76.60**	-0.0006*	76.600**

* Check: sum is zero.
** Check: sum is G/2

To carry out the analysis of variance, we compute:

$G = 153.2$ $\quad n = 30$ $\quad G^2/n = 782.341$

$G_2 = 901.20$ $\quad T = G_2 - G^2/n = 118.859$

$\Sigma B_i^2 = 2477.96$ $\quad S_B^* = \frac{1}{3}\Sigma B_i^2 - G^2/n = 43.646$

$\Sigma T_j^2 = 1737.78$ $\quad S_T^* = \frac{1}{2}\Sigma T_j^2 - G^2/n = 86.549$

$$S_B = \Sigma b_i P_i = \tfrac{1}{2}(52.812) = 26.406$$

$$S_T = S_B + S_T^* - S_B^* = 69.309$$

$$S_E = T - S_B^* - S_T = 5.904$$

TABLE 6.3 ANALYSIS OF VARIANCE

Variation due to	Sum of squares	Degrees of freedom	Sum of squares	Variation due to
Blocks (Unadjusted)	43.646	9	26.406	Blocks (Adjusted)
Treatments (Adjusted)	69.309	14	86.549	Treatments (Unadjusted)
Error	5.904	6	5.904	Error
Total	118.859	29	118.859	Total

To test if treatment differences are significant, we compute the variance-ratio

$$F = \frac{69.309/14}{5.904/6} = 5.031$$

which with 14 and 6 degrees of freedom is significant at the 5 o/o level.

To test any particular treatment difference, say that between treatments 1 and 2, we proceed as follows: The best intra-block estimate of the difference is

$$t_1 - t_2 = -3.36 .$$

Now, treatments 1 and 2 occur together in block 3 and the other blocks in which they occur are: block 9 (in which treatment 1 occurs) and block 10 (in which treatment 2 occurs). But the pair of blocks 9 and 10 are second associates because they do not have a treatment in common. Hence, the treatments 1 and 2 form a pair of type X_2. The variance of $(t_1 - t_2)$ is thus

$$(1 + \gamma)\,\sigma^2 = 1.4\,\sigma^2$$

and this is estimated by

$$1.4 \times 5.904/6 = 1.3776$$

and the standard error is

$$\sqrt{1.3776} = 1.17371.$$

We then have the Student's ratio

$$t = \frac{-3.36}{1.17371} = -2.863$$

which with 6 degrees of freedom is significant at the 5 o/o level but not at the 1 o/o level against both-sided alternatives.

For combined estimation, we have:

$$d = \frac{9 \times 5.904}{6 \times 26.406 - 15 \times 5.904} = 0.76043$$

$$\bar{a} = a + 2d = 5.52086$$

$$\bar{A} = A + 2d(a + n_1) + 4d^2$$

$$= 10 + 1.52086(4+3) + 4 \times 0.57825 = 22.95902$$

$$\bar{\gamma} = \bar{a} / \bar{A} = 0.24047$$

$$\bar{\gamma}_1 = 1 / \bar{A} = 0.04356 \ .$$

The best combined estimate of the difference between treatments 1 and 2 is then found to be

$$\bar{t}_1 - \bar{t}_2 = -3.664$$

7. Construction of other two replicate designs:

In like manner, two replicate designs can be constructed from any partially balanced association scheme with $m > 2$ classes, by first constructing a partially balanced incomplete block design with $k=2$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_p = 1$, $\lambda_{p+1} = \ldots = \lambda_m = 0$ and then dualizing it. By replacing each object by a group of $t (t \geq 2)$ objects in a partially balanced association scheme with m associate classes, one gets again a partially balanced associate scheme with $(m+1)$ associate classes. This result can be used in constructing other two replicate designs. Another way would be to replace each treatment in a SPLB design by a group of $t (t \geq 2)$ treatments. This, however, will not be pursued in this paper.

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