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RELATION BETWEEN CERTAIN INCOMPLETE BLOCK DESIGNS

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1. Summary. Relations between balanced incomplete block designs derived from the symmetrical balanced incomplete block designs with $\lambda = 1$ and 2 and certain partially balanced incomplete block designs are investigated in this paper.

2. Introduction. Let v, b, r, k, λ be the parameters of a balanced incomplete block design [1] (b.i.b.d.). Then we have

(2.1) $$vr = bk$$

(2.2) $$\lambda(v-1) = r(k-1)$$

For a symmetrical b.i.b.d. it is known that any two blocks have exactly λ treatments in common. Hence from the design with parameters

(2.3) $$v = b,\ r = k,\ \lambda$$

we get by the method of block section [1] another b.i.b.d. with parameters.

(2.4) $$v' = v-k,\ b' = v-1,\ k' = k-\lambda,\ r' = k,\ \lambda' = \lambda.$$

Thus the existence of a b.i.b.d. with parameters (2.3) implies the existence of another b.i.b.d. with parameters (2.4). The converse is not true in general [2]. However, when $\lambda = 1$, (2.4) corresponds to a finite Euclidean Plane and it is well known that such a plane can be

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embedded into a Finite Projective Plane which is the symmetrical design with parameters (2.3) for $\lambda=1$. A similar result has been proved by Hall and Connor [3] for the case $\lambda = 2$. We thus have the following result.

Let C, C_1, D, D_1 be b.i.b. designs with parameters indicated below.

(2.5) C: $v = b = s^2 + s + 1,\ r = k = s + 1,\ \lambda = 1$

(2.6) C_1: $v = s^2,\ b = s^2 + s,\ r = s + 1,\ k = s,\ \lambda = 1$

(2.7) D: $v = b = \frac{n^2 + n + 2}{2},\ r = k = n + 1,\ \lambda = 2$

(2.8) D_1: $v = \frac{n(n-1)}{2},\ b = \frac{n(n+1)}{2},\ r = n + 1,\ k = n - 1,\ \lambda = 2$

Then either C and C_1 (D and D_1) are both existent or both nonexistent and any solution of one yields a solution of the other.

From [4,5] it follows that the dual of C, is the group divisible designs C_1*.

(2.9) C_1*" $v = s^2 + s,\ b = s^2,\ r = s,\ k = s + 1,\ \lambda_1 = 1,\ \lambda_2 = 0$

$$p = s + 1,\ q = s$$

where

p = number of groups of treatments

q = size of each group

λ_1= number of times any two treatments from different groups occur together.

λ_2 = number of times any two treatments from any group occur together.

Again from [4] it follows that the dual D_1* of D_1 is a partially balanced incomplete block design (b.i.b.d.) with two associate classes with parameters

(2.10) D_1*: $v = \frac{n(n+1)}{2},\ b = \frac{n(n-1)}{2},\ r = n-1,\ k = n + 1,$

$$n_1 = 2n - 2,\ n_2 = \frac{(n-1\ (n-2)}{2},\ \lambda_1 = 1,\ \lambda_2 = 2$$

$$p^1_{11} = n - 1,\quad p^2_{11} = 4.$$

Conversely using the results of Roy and Laha [6] it can be seen that the designs with parameters given by (2.9) and (2.10) are linked block designs and hence the dual of a design with parameters (2.9) is the b.i.b.d. with parameters (2.6). Similarly the dual of a design with parameters (2-10) is the b.i.b.d. with parameters (2.8). Hence the designs C_1 and C_1* (D_1 and D_1*) are either both existent or both nonexistent.

A p.b.i.b.d. with two associate classes is said to be triangular [7] if the number of treatments is $\frac{n(n-1)}{2}$ and the association scheme is an array of n rows and n columns with the following properties.

(a) The positions in the principal diagonal are blank.

(b) The $\frac{n(n-1)}{2}$ positions above the prinicpal diagonal are filled by the numbers 1, 2, . . ., $\frac{n(n-1)}{2}$ corresponding to the treatments.

(c) The positions below the principal diagonal are filled so that the array is symmetrical about the principal diagonal.

(d) For any treatment i, the first associates are exactly those treatments which lie in the same row and the same column as treatment i.

The following relations obviously hold.

(1) The number of first associates of any treatment is $n_1 = 2n - 4$.

(2) With respect to any two treatments θ_1 and θ_2 which are first associates, the number of treatments which are first associates of both θ_1 and θ_2 is

$$p^1_{11} = n - 2$$

(3) With respect to any two treatments θ_3 and θ_4 which are second associates, the number of treatments which are first associates of both θ_3 and θ_r is

$$p^2_{11} = 4.$$

Conversely connor [8] has shown that for a p.i.b.d. with two associate classes if $v = \frac{n(n-1)}{2}$, $n \geq 9$, (1), (2) and (3) above imply that the association scheme is triangular, denoted by (T_n). In a paper [9] submitted to the Annals of Mathematical Statistics, the author has shown that the same result is true for n = 5,6. The cases n = 7 and 8 are as yet undecided.

3. Results for b.i.b.d. with $\lambda = 1$. In this section we derive some results for designs obtained from the symmetrical b.i.b.d. with $\lambda = 1$. They provide the motivation for the results obtained in sections 4 and 5.

We have already seen that a design C_1 with paramenters (2.6) is dual of a design C_1* with parameters (2.9) and conversely. Now since C_1* is a group divisible design, the blocks of C_1 can be divided into (s + 1) sets of s each such that any two blocks of the same set have no treatment in common, whereas any two blocks from different sets have exactly one treatment in common. Hence if we omit one set of blocks we get a group divisible design C_{11} given by parameters.

(3.1) C_{11}: $u = b = s^2, \ r = k = s, \ \lambda_1 = 1, \ \lambda_2 = 0$

Further the omitted set of blocks gives rise to another disconnected group divisible design C_{12} with parameters

(3.2): $u = s^2, \ b = s, \ r = 1, \ k = s, \ \lambda_1 = 0, \ \lambda_2 = 1$

In C_{11} and C_{12}, the s groups of s treatments are given by the s blocks of C_{12}. We may indicate this result symbolically by $C_1 = C_{11} + C_{12}$.

Now consider the dual C_{11}* of C_{11}. By virtue of the relation $C_1 = C_{11} + C_{12}$, C_{11}* can be obtained from C_1^* the dual of C_1, by removing from the blocks of C_1*, the treatments corresponding to the blocks of C_{12}. We note that each block of C_1* contains just one of the s treatments corresponding to the s blocks of C_{12} and that any two of these s treatments are second associates in C_1^*. Now C_1^* is a group divisible design with (s+1) groups of s treatments each, where any two treatments from the same group are second associates. Hence, obviously the s treatments corresponding to the blocks of C_{12} must all belong to the same group. Omitting these treatments from C_1^*, we are left with C_{11}^*. The association scheme of C_{11}^* is obviously group divisible with the remaining s groups of size s each. Thus C_{11}^* is again a group divisible design with exactly the same parameters as C_{11}.

4. Results connected with b.i.b.d. with $\lambda = 2$. In this section we prove four theorems for designs connected with b.i.b.d. with $\lambda = 2$.

Theorem (1). If for any value of n, a design D_1^* with parameters (2.10) exists, then its association scheme is of the type (T_{n+1}).

Proof. We note that the design D_1 given by parameters (2.8) does not exist for n = 6 and 7 [10]. Hence D_1^* does not exist for these values of n.

For $n \geq 8$, the theorem follows from Connor's result [8] and for n = 4 and 5, it follows from the result of the author [9] referred to previously.

Theorem (2). For any value of n, the existence of the design D_1 with parameters (2.8), implies the existence of two design D_{11} and D_{12} given respectively by the follow parameters.

(4.1) D_{11}:
$$v = b = \frac{n(n-1)}{2}, \quad r = k = n - 1,$$
$$n_1 = 2n - 4, \; n_2 = (n-2)\;(n-3)/2, \; \lambda_1 = 1, \; \lambda_2 = 2.$$
$$p_{11}^1 = n - 2, \; p_{11}^2 = 4.$$

(4.2) D_{12}:
$$v = n\,(n-1)/2, \; b = n, \; k = n-1, \; r = 2$$
$$n_1 = 2n - 4, \; n_2 = (n-2)\;(n-3)/2, \; \lambda_1 = 1, \; \lambda_2 = 0$$
$$p_{11}^1 = n - 2, \; p_{11}^2 = 4.$$

Such that
$$D_1 = D_{11} + D_{12}.$$

Proof. Instead of D_1 we consider its dual D_1^* given by (2.10). Let D_{11}^* and D_{12}^* denote the duals of D_{11} and D_{12} respectively. It is sufficient to prove that the treatments of D_1^* can be divided into two distinct sets T_1 and T_2 say that by retaining treatments of the set T_1 in the blocks of D_1^* we get the design D_{11}^*, whereas by omitting these treatments (and hence retaining treatments of the set T_2) in the blocks of D_1^* we get the design D_{12}^*. We note that D_{12}^* is nothing but the unreduced b.i.b.d. with

$v = n$, $k = 2$, $\lambda = 1$. From Theorem 1, we know that the association scheme of D_1^* is triangular (T_{n+1}) which may be written without loss of generality as follows.

$$
(4.3)\qquad
\begin{bmatrix}
x & 1 & 2 & 3 & . & . & n \\
1 & x & n+1 & n+2 & . & . & 2n-1 \\
2 & n+1 & x & . & . & . & . \\
3 & n+2 & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & x & \frac{n^2+n}{2} \\
n & 2n-1 & . & . & . & \frac{n^2+n}{2} & x
\end{bmatrix}
$$

We show that any block of D_1^* contains exactly two treatments from any row of (4.3). Consider (n+1) blocks s_i of size (n+1), each containing a new treatment α and all the treatments of row i of (4.3), i = 1, 2, ..., (n+1). Adding these (n+1) blocks s_i to the blocks of D_1^* we obviously get a design with $v = b = \frac{n^2+n+2}{2}$, $r = k = n + 1$. The treatment α occurs with any treatments of D_1^* in exactly two of these blocks. Any two treatments of D_1^* say ℓ and m occuring in the same row or column of (4.3) occured once in a block of D_1^*. They also occur once in a block s_i. Again if ℓ and m do not lie in the same row or same column of (4.3), they occur together in two blocks of D_1^*, but they do not occur together at all in the new blocks s_i. Thus any two treatments ℓ and m of D_1^* occur together twice in the new design giving $\lambda = 2$. Hence the design so obtained is nothing but the b.i.b.d. with parameters (2.7) and hence any two blocks of this design have two treatments in common. Let θ_1 and θ_2 be two different treatments of D_1^* belonging to same row of (4.3). Then θ_1 and θ_2 occur together in exactly one block of D_1^*.

Further this block of D_1* cannot contain any other treatment θ_3 of the same row of (4.3). For otherwise the design (2.7) obtained above will have two blocks containing three treatments in common which is impossible. Consider in particular the treatments 1, 2, ..., n occuring in the first row of (4.3). Then each of the $\frac{n(n-1)}{2}$ pairs of these treatments uniquely determines a block of D_1*. Noticing that each pair determines only one block and that the number of blocks in D_1* is exactly $n(n-1)/2$, this correspondence can be set up in 1 - 1 manner. This proves the statement made at the beginning of this paragraph. If we now retain only the treatments of the set T_2 = (1, 2, ..., n) in the blocks of D_1*, we get the design D_{12}* which is the unreduced b.i.b.d. for v = n and k = 2. We now write the n_{c_2} combination of the n treatments of T_2 in the following array of n rows and n columns.

$$
(4.4)\quad \begin{bmatrix}
x & (1,2) & (1,3) & (1,4) & . & . & (1,n) \\
(2,1) & x & (2,3) & (2,4) & . & . & (2,n) \\
(3,1) & (3,2) & x & (3,4) & . & . & (3,n) \\
(4,1) & (4,2) & (4,3) & x & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & (n-1,n) \\
(n,1) & (n,2) & (n,3) & . & . & (n,n-1) & x
\end{bmatrix}
$$

As mentioned before each combination determines a unique block of D_1* to which the combination belongs. By removing these treatments from the corresponding blocks we are left with a design with $u = b = n(n-1)/2$, $r = k = n - 1$, the $n(n-1)/2$ treatments being the treatments of the set $T_1 = (n+1, n+2, ..., \frac{n^2+n}{2})$. Now in D_1* any two blocks have two treatments in common. Hence in this design for treatments of the set T_1, any two blocks corresponding to two combinations which are either in the same row or in the same column have exactly one treatment in common, since the corresponding combinations have one treatment in common and which are now omitted. Similarly, since any two comibnations not in the same row and not in the same column have no treatment in common, the corresponding blocks for the above design for the treatment of T_1 have two treatments in common. This implies that the blocks of the above design can be written in a triangular array like (4.4) such that any two blocks which are either in the same row or the same column have exactly one treatment in common, whereas any other pair of blocks have two treatments in common. We may say that the block structure for this design is triangular (T_n). It is now obvious that this design is nothing but $D_{11}*$, the dual of D_{11}.

Now considering the dual of D_1^*, D_{11}^*, D_{12}^* we see that the n(n+1)/2 blocks of D_1 can be divided into sets of n(n-1)/2 and n respectively such that the first set of blocks gives the design D_{11} and the second set gives the design D_{12}. We thus have $D_1 = D_{11} + D_{12}$. This completes the proof of the theorem.

We now prove a partial converse of the above theorem

Theorem 3. Existence of the design D_{11} with parameters (4.1), having triangular association scheme (T_n) for any value of n implies the existence of the corresponding design D_1 with parameters (2-8).

Proof Without loss of generality we can represerve the association schemes as follows.

$$(4.5)\qquad \begin{bmatrix} x & 1 & 2 & . & . & (n-1) \\ 1 & x & n & . & . & (2n-3) \\ 2 & n & x & . & . & . \\ . & . & . & . & x & \frac{n^2-n}{2} \\ n-1 & (2n-3) & . & . & \frac{n^2-n}{2} & x \end{bmatrix}$$

Consider the n blocks of size (n-1) given by the rows of (4.5). They constitute the design D_{12} given by (4.2). Adding the blocks of D_{12} to the blocks of D_{11}, we get a design with $v = n(n-1)/2$, $b = n(n+1)/2$, $r = n+1$, $k = n-1$. Further any two treatments of D_{11} occuring in the same row or the same column of (4.5) occur together once more in these new blocks. Also two treatments which are neither in the same now nor in the same column of (4.5) already occur together twice in blocks of D_{11} but they do not occur together at all in these new block. Hence in the new design $\lambda = 2$, for all hours of treatments. Thus the new design is exactly the design D_1. Futher we have $D_1 = D_{11} + D_{12}$. This completes the proof.

The design D_{11} is known to be impossible [11] for $n = 6$ so is the design D_1 for the same value of n [10]. The design D_1 for $n = 7$ is also known to be impossible [10], but the impossibility of D_{11} for this value of n has not been proved. If, however, D_{11} exists for $n = 7$, the association scheme cannot be triangular, as from the above theorem this would imply the existence of D_1 for $n = 7$. From [8,9] we know that the association scheme for D_{11} is triangular (T_n) for $n \geq 9$ and $n = 5$. We thus have the following corollary.

Corollary. If $n = 5$ or $n \geq 9$, the existence of D_{11} with parameters (4.1) implies the existence of D_1 with parameters (2.8).

We now state and prove another theorem for the design D_{11}.

Theorem 4. If $n = 5$ or $n \geq 9$, the dual of the design D_{11} given by (4.1) is another p.b.i.b.d. with the same parameters as (4.1).

Proof. From Theorem 3 and its corollary we have $D_{11} + D_{12} = D_1$. Consider the dual D_1^* of D_1. Then D_{11}^* is obtained by omitting from D_1^*, the treatments corresponding to the blocks of D_{12}. We note that in each block of D_1^*, there are two treatments corresponding to the blocks of D_{12} and (n-1) treatments corresponding to the blocks of D_{11}. Further since the dual of D_{12} is D_{12}^* which is a b.i.b.d. with parameters $v = n$, $b = n(n-1)/2$, $k = 2$, $r = n-1$, $\lambda = 1$, any two of the n treatments $\alpha_1\ \alpha_2\ \ldots\ \alpha_n$ of D_{12}^* are first associates in D_1^*. Now the association scheme of D_1^* is triangular (T_{n+1}), from [8,9] since $n + 1 = 6$ or $n + 1 \geq 10$. We now show that $\alpha_1, \alpha_2, \ldots, \alpha_n$ must all be in the same row of the association scheme (T_{n+1}). For consider any two treatments α_1 and α_2, say. Since they are first associates, they must lie either in the same row or the same column of the association scheme. Suppose they lie in in the same row, then since the association scheme is unchanged if we interchange any two rows and the same two columns of the array of (T_{n+1}), we can, without loss of generality, assume that α_1 and α_2 are in the first row in positions (1,2) and (1,3) respectively. If $\alpha_3, \alpha_4, \ldots, \alpha_n$ all lie in the first row we are through. Otherwise there is at least one treatment, say, α_3 which does not lie in this row. Since the first associates of α_1 must lie in the first and the second rows, α_3 lies in the second row in position $(2,j)$ say. Similarly since α_3 is first associate of α_2, it must be in the third row in position $(3,j_1)$, say. Since these two positions of α_3 must be symmetrical with respect to the main diagonal we have $j = 3$ and $j_1 = 2$, i.e., α_3 occurs in the array of (T_{n+1}) in positions (2,3) and (3,2). Now consider α_4. Since it is first associate of α_1, it must occur either in the first row or in the second row in column position $\ell \geq 4$. Suppose it occurs in position $(1,\ell)$, $\ell \geq 4$ then α_4 does not occur in the same row or same column as α_3 and hence is second associate of α_3 which is a contradiction. Suppose α_4 occurs in position $(2,\ell)$, $\ell \geq 4$, then it is not in the same row or the same column as α_2 and hence is second associate of α_2 which is again a contridiction. Thus if α_1 and α_2 lie in a row, all other treatments $\alpha_3, \ldots, \alpha_n$ must lie in the same row. We get a similar result if α_1 and α_2 lie in the same column. This proves the result that all the treatments $\alpha_1, \alpha_2 \ldots, \alpha_n$

must lie in the same row and hence the same column. We now omit the treatments $\alpha_1\ \alpha_2\ \ldots,\ \alpha_n$ from the blocks of D_1^*. This leaves the designs D_{11}^*. Further the association scheme of D_{11}^* is obviously obtained by omitting the row and column in which the treatments $\alpha_1, \alpha_2, \ldots, \alpha_n$ lie. The scheme is therefore, triangular (T_n). It is now easy to see that the parameters of D_{11}^* are exactly those of D_{11}. This completes the proof the theorem.

5. <u>A constructive proof for embedding the design D_1 into the design D</u>. As mentioned earlier Hall and Connor [3] proved that the existence of the design D_1 given by (2.8) implies the existence of the design D given by (2.7). Their proof is not constructive. We give in this section a constructive proof of their result.

Suppose the design D_1 exists, then D_1^* also exists and by Theorem 1, the association scheme of D_1^* is triangular (T_{n+1}). Hence, the blocks B_1, $B_2, \ldots, B_{\frac{n^2+n}{2}}$ of D_1 can be exhibited in the scheme (4.3) when i stands for B_i, such that any two blocks of D_1 in the same row or in the same column of (4.3) have exactly one treatment in common, whereas any two blocks neither in the same row nor in the same column of (4.3) have two treatments in common. Take a new block B' consisting of new treatments $u_1, u_2, \ldots, u_{n+1}$. Let the $n(n+1)/2$ combinations of these treatments be listed as shown below.

$$(5.1)\qquad \begin{bmatrix} x & (u_1,u_2) & (u_1,u_3) & . & . & . & (u_1,u_{n+1}) \\ (u_2,u_1) & x & (u_2,u_3) & . & . & . & (u_2,u_{n+1}) \\ (u_3,u_1) & (u_3,u_2) & x & . & . & . & (u_3,u_{n+1}) \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & x & (u_n,u_{n+1}) \\ (u_{n+1},u_1) & (u_{n+1},u_2) & (u_{n+1},u_3) & . & . & (u_{n+1},u_n) & x \end{bmatrix}$$

We now assign the above combinations to the respective blocks B_i in the corresponding positions of (4.3) to get new blocks B_i'. Thus B_1 consists of treatments of B_1 besides the treatments u_1 and u_2. It is now easy to verify that B' and B_i', $i=1,2\ldots,\frac{n^2+n}{2}$ give a design with $v = b = \frac{n^2+n+2}{2}$, $r = k = n+1$. Further if i and j belong to the same row or same column of (4.3), then B_i and B_j have one treatment in common and the corresponding combinations added to these blocks to give B_i' and B_j' have again one treatment in common. Thus B_i' and B_j' have two treatments in common. If on the other hand i and j neither belong to the same row nor to the same column of (4.3), then B_i and B_j have two treatments in common; but now the corresponding combinations to be added to them have no treatment in common. Thus again B_i' and B_j' have two treatments in common. Again B' has obviously two treatments in common with any B_i'. Thus we get a symmetric design in which every two blocks have two treatments in common. This implies that $\lambda = 2$. The design thus obtained is exactly the design D.

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