NOTE ON THE ANALYSIS OF A RANDOMIZED BLOCK DESIGN

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Suppose that we are given b blocks each of which contains k more or less homogeneous experimental units or plots and k treatments or varieties to be compared by experiment. We shall say that the design is a randomized (complete) block design if we assign k treatments at random (for instance take k cards with the numbers 1,...,k on them, shuffle them well and lay them out in a row to determine the position of the first block. Repetition of this process will produce the assignment of the treatments in the second block and so forth.) "A mathematically rigorous treatment of this arrangement is at present not yet available. An approximate test of varietal effects is possible by treating the arrangement as a two-way classification design ignoring the variation of soil fertility within the rows." [1]

The purpose of the present note is to give a justification of the usual analysis of the randomized block design. Although the same argument can easily be extended to general incomplete balanced block design, for the sake of simplicity the simplest case, a randomized complete block, is treated.

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To begin with the explanation of some useful concepts of design, we shall be concerned with general incomplete block design with v treatments, and b blocks each of which contains k plots.

There are $n = kb$ experimental units or plots on the whole. We number them in some way but once for all; for instance, the i-th experimental unit in the j-th block bears the number $(j-1)+i$. The observation at i-th plot is denoted by x_i and the whole observations are represented by an n-dimensional vector $\underline{x}$ whose i-th component is x_i, and is called an <u>observation vector</u>.

We shall define <u>the incidence vectors of treatments</u> as follows:

(1) $$\underline{\zeta}_\alpha = \begin{bmatrix} \zeta_{\alpha 1} \\ \zeta_{\alpha 2} \\ \vdots \\ \zeta_{\alpha n} \end{bmatrix}, \quad \text{where} \quad \zeta_{\alpha f} = \begin{cases} 1, & \text{if the plot } f \text{ receives the treatment } \alpha \\ 0, & \text{otherwise} \end{cases}$$

and the matrix

(2) $$\Phi = \| \underline{\zeta}_1 \ \dots \ \underline{\zeta}_v \|$$

is called the <u>incidence matrix of treatments</u>. The linear subspace which is generated by $\underline{\zeta}_1 \dots \underline{\zeta}_v$ is called <u>treatment space</u>.

Likewise the <u>incidence vectors of blocks</u> are defined by

(3) $$\underline{\eta}_a = \begin{bmatrix} \eta_{a1} \\ \eta_{a2} \\ \vdots \\ \eta_{an} \end{bmatrix}, \quad \text{where} \quad \eta_{af} = \begin{cases} 1, & \text{if the plot } f \text{ belongs to the block } a \\ 0, & \text{otherwise} \end{cases}$$

and the <u>incidence matrix of blocks</u> is defined by

(4) $$\Psi = \| \underline{\eta}_1 \; \ldots \; \underline{\eta}_b \|$$

and the linear subspace which is generated by $\underline{\eta}_1 \ldots \underline{\eta}_b$ is called the block space.

The components of Φ projected into the block space with respect to the basis vectors Ψ is expressed as

(5) $$N = \Phi' \Psi$$

and is called the incidence matrix of the design.

(6) $$N = \| n_{\alpha a} \| \text{, where } n_{\alpha a} = \begin{cases} 1, & \text{if treatment } \alpha \text{ occurs in block } a \\ 0, & \text{otherwise} \end{cases}$$

Now from the very definitions it is seen that

(7) $$\Phi'\Phi = \begin{Vmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_v \end{Vmatrix} , \quad \Psi'\Psi = kI_b$$

where r_α stands for the replication of treatment α. Thus if in particular

$$r_1 = r_2 = \ldots = r_v = r$$

then

(8) $$T^* = \frac{1}{r} \Phi\Phi' , \quad B^* = \frac{1}{k} \Psi\Psi'$$

are idempotents and they are the projection operators.

Evidently the treatment space and the block space has the intersection which is generated by the vector $\underline{I}$ whose components are all unity. The projection operator on this intersection is

(9) $$G^* = \frac{1}{n} G$$

where G is the $n \times n$ matrix whose elements are all unity

In the special case which we are now going to discuss

(10) $$v = k \;, \quad r = b$$

and four matrices

(11) $$I \;, G \;, B = kB^* \;, T = bT^*$$

are so-called "relationship matrices" of the design [2].

It is known that the linear closure of the matric set $\{I, G, B, T\}$ is a linear associative algebra and it is commutative. Indeed, the multiplication table is as follows:

I	B	T	G
B	kB	G	kG
T	G	bT	bG
G	kG	bG	nG .

The decomposition of the unit element into orthogonal idempotents is given by

$$I = \frac{1}{n} G + (\frac{1}{k}B - \frac{1}{n}G) + (\frac{1}{b}T - \frac{1}{n}G) + (I - \frac{1}{k}B - \frac{1}{b}T + \frac{1}{n}G)$$

hence

(12) $$I - \frac{1}{n}G = (\frac{1}{k}B - \frac{1}{n}G) + (\frac{1}{b}T - \frac{1}{n}G) + (I - \frac{1}{k}B - \frac{1}{b}T + \frac{1}{n}G) \;.$$

The meaning of (12) is: $\frac{1}{k}B - \frac{1}{n}G$ is the projection operator of the contrast space into block space, $\frac{1}{b}T - \frac{1}{n}G$ is the projection operator into treatment space and $I - \frac{1}{k}B - \frac{1}{b}T + \frac{1}{n}G$ is the projection operator into error space, i.e.,

(13) $$(\frac{1}{k}B - \frac{1}{n}G)\underline{x} = \frac{1}{k}\,\Psi\Psi'\underline{x} - \frac{1}{n}\,\underline{I}\underline{I}'\underline{x} = \underline{\underline{\Psi}}\underline{B} - \underline{I}\,\bar{x} \;,$$

where

$$\underline{B} = \begin{bmatrix} B_1 \\ \vdots \\ B_b \end{bmatrix} \;, \quad B_i = \frac{1}{k} \sum_{f \,\varepsilon\, \text{block } i} x_f \quad \text{and} \quad \bar{x} = \frac{1}{n} \sum_{f=1}^{n} x_f \;,$$

and

(14) $$(\frac{1}{b}T - \frac{1}{n}G)\underline{x} = \frac{1}{b}\Phi\Phi'\underline{x} - \frac{1}{n}\underline{I}\underline{I}'\underline{x} = \Phi\underline{T} - \underline{I}\bar{x} ,$$

where

$$\underline{T} = \begin{bmatrix} T_1 \\ \vdots \\ T_k \end{bmatrix} , \quad T_\alpha = \frac{1}{k} \sum_{\zeta_{\alpha f}=1} x_f , \quad \alpha = 1,\ldots,v .$$

The essential difference between the randomized block design and two-way classification design is that the incidence vectors of treatment are random in the former case whereas they are fixed vectors in the latter case. If the plot effect can be ignored, the underlying model for a randomized block design is that the conditional distribution of the residual

(15) $$\underline{e} = \underline{x} - g\underline{I} - \Phi\cdot\underline{t} - \Psi\cdot\underline{b}$$

where g is the general mean and $\underline{t}' = (t_1, \ldots, t_k)$ and $\underline{b}' = (b_1, \ldots, b_b)$ are treatment effects and block effects respectively, given Φ is $N(\underline{0}, \sigma^2 I)$. There will be no loss of generality by assuming that

(16) $$\sum_{\alpha=1}^{k} t_\alpha = \sum_{a=1}^{b} b_a = 0 .$$

Obviously the probability of Φ is discrete and $1/(k!)^b$.

If we denote the $n \times k$ matrix whose elements are all unity by J, then

(17) $$G = J\Phi' = \Phi J'$$

and since

$$\frac{1}{b}T - \frac{1}{n}G = (\frac{1}{b}T - \frac{1}{n}G)(\frac{1}{b}T - \frac{1}{n}G) = (\frac{1}{b}\Phi - \frac{1}{n}J)\Phi'\Phi(\frac{1}{b}\Phi' - \frac{1}{n}J')$$

$$= b(\frac{1}{b}\Phi - \frac{1}{n}J)(\frac{1}{b}\Phi' - \frac{1}{n}J')$$

and

$$I-\frac{1}{b}T-\frac{1}{k}B+\frac{1}{n}G = (I-\frac{1}{b}T-\frac{1}{k}B+\frac{1}{n}G)(I-\frac{1}{b}T-\frac{1}{k}B+\frac{1}{n}G) ,$$

it follows that

(18) $$(\frac{1}{b}\Phi'-\frac{1}{n}J')\underline{x} = (\frac{1}{b}\Phi'-\frac{1}{n}J')\underline{e}+g(\frac{1}{b}\Phi'-\frac{1}{n}J')\underline{I}+(\frac{1}{b}\Phi'-\frac{1}{n}J')\Phi\cdot\underline{t}+(\frac{1}{b}\Phi'-\frac{1}{n}J')\Psi\underline{b}$$

$$= (\frac{1}{b}\Phi'-\frac{1}{n}J')\underline{e}+\underline{t}$$

and

$$(I-\frac{1}{b}T-\frac{1}{k}B+\frac{1}{n}G)\underline{x} = (I-\frac{1}{b}T-\frac{1}{k}B+\frac{1}{n}G)\underline{e}+g(I-\frac{1}{b}T-\frac{1}{k}B+\frac{1}{n}G)\underline{I}$$

(19) $$+(I-\frac{1}{b}T-\frac{1}{k}B+\frac{1}{n}G)\Phi\cdot\underline{t}+(I-\frac{1}{b}T-\frac{1}{k}B+\frac{1}{n}G)\Psi\cdot\underline{b}$$

$$= (I-\frac{1}{b}T-\frac{1}{k}B+\frac{1}{n}G)\underline{e} .$$

Thus

(20) $$\underline{x}'(\frac{1}{b}T-\frac{1}{n}G)\underline{x} = \underline{e}'(\frac{1}{b}T-\frac{1}{n}G)\underline{e}+2\underline{t}'b(\frac{1}{b}\Phi'-\frac{1}{n}J')\underline{e}+b\underline{t}'\underline{t}$$

and

(21) $$\underline{x}'(I-\frac{1}{b}T-\frac{1}{k}B+\frac{1}{n}G)\underline{x} = \underline{e}'(I-\frac{1}{b}T-\frac{1}{k}B+\frac{1}{n}G)\underline{e} .$$

Take an orthogonal matrix of the form

(22) $$P = \left\| \begin{matrix} \sqrt{b}\,\underline{t}'(\frac{1}{b}\Phi'-\frac{1}{n}J')\Big/\sqrt{\underline{t}'b(\frac{1}{b}\Phi'-\frac{1}{n}J')(\frac{1}{b}\Phi-\frac{1}{n}J)\underline{t}} \\ \underline{p}_2' \\ \vdots \\ \underline{p}_n' \end{matrix} \right\|$$

and let

$$\underline{e}^* = P\,\underline{e}$$

or

(23) $$\underline{e} = P'\underline{e}^*$$

then we have

$$P(\tfrac{1}{b}\Phi - \tfrac{1}{n}J) = \left\| \begin{matrix} \sqrt{b}\ \underline{t}'(\tfrac{1}{b}\Phi' - \tfrac{1}{n}J')(\tfrac{1}{b}\Phi - \tfrac{1}{n}J) \Big/ \sqrt{\underline{t}'b(\tfrac{1}{b}\Phi' - \tfrac{1}{n}J')(\tfrac{1}{b}\Phi - \tfrac{1}{n}J)\underline{t}} \\ \underline{p}_2'(\tfrac{1}{b}\Phi - \tfrac{1}{n}J) \\ \vdots \\ \underline{p}_n'(\tfrac{1}{b}\Phi - \tfrac{1}{n}J) \end{matrix} \right\|$$

and

$$P(I - \tfrac{1}{b}T - \tfrac{1}{k}B + \tfrac{1}{n}G) = \left\| \begin{matrix} \dfrac{\sqrt{b}\ t'(\tfrac{1}{b}\Phi' - \tfrac{1}{n}J')(I - \tfrac{1}{b}T - \tfrac{1}{k}B + \tfrac{1}{n}G)}{\sqrt{\underline{t}'b(\tfrac{1}{b}\Phi' - \tfrac{1}{n}J')(\tfrac{1}{b}\Phi - \tfrac{1}{n}J)\underline{t}}} \\ \underline{p}_2'(I - \tfrac{1}{b}T - \tfrac{1}{k}B + \tfrac{1}{n}G) \\ \vdots \\ \underline{p}_n'(I - \tfrac{1}{b}T - \tfrac{1}{k}B + \tfrac{1}{n}G) \end{matrix} \right\|$$

and hence

$$(24) \qquad Pb(\tfrac{1}{b}\Phi - \tfrac{1}{n}J)(\tfrac{1}{b}\Phi' - \tfrac{1}{n}J')P' = \left\| \begin{matrix} 1 & 0 & \cdots\cdots & 0 \\ 0 & & & \\ \vdots & & Q & \\ 0 & & & \end{matrix} \right\|$$

and since

$$\left(\frac{1}{b}\Phi' - \frac{1}{n}J'\right)\left(I - \frac{1}{b}T - \frac{1}{k}B + \frac{1}{n}G\right) = \left(\frac{1}{b}\Phi' - \frac{1}{n}J'\right)\left(I - \frac{1}{k}\Psi\Psi'\right) - \left(\frac{1}{b}\Phi' - \frac{1}{n}J'\right)\left(\frac{1}{b}\Phi - \frac{1}{n}J\right)\Phi'$$

$$= \frac{1}{b}\Phi' - \frac{1}{n}J' - \frac{1}{n}\Phi'\Psi\Psi' + \frac{1}{nk}J'\Psi\Psi' - \left(\frac{1}{b}I_k - \frac{1}{n}G_k\right)\Phi'$$

$$= \frac{1}{b}\Phi' - \frac{1}{n}J' - \frac{1}{n}J^*\Psi' + \frac{1}{n}J^*\Psi' - \frac{1}{b}\Phi' + \frac{1}{n}J' = 0 ,$$

where J^* stands for the $k \times b$ matrix whose elements are all unity

$$(25) \quad P\left(I - \frac{1}{b}T - \frac{1}{k}B + \frac{1}{n}G\right)P' = \left\|\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & R & \\ 0 & & & \end{array}\right\| .$$

Since $\frac{1}{b}T - \frac{1}{n}G$ and $I - \frac{1}{b}T - \frac{1}{k}B + \frac{1}{n}G$ are orthogonal idempotents with rank $k-1$ and $(b-1)(k-1)$ respectively, Q and R are orthogonal idempotents of rank $k-2$ and $(b-1)(k-1)$ respectively.

Finally since

$$(26) \quad \mathcal{T} \equiv \underline{x}'\left(\frac{1}{b}T - \frac{1}{n}G\right)\underline{x} = \left(e_1^* + \sqrt{bt't}\right)^2 + (e_2^* \ldots e_n^*)Q\begin{bmatrix} e_2^* \\ \vdots \\ e_n^* \end{bmatrix}$$

and

$$(27) \quad \mathcal{E} \equiv \underline{x}'\left(I - \frac{1}{b}T - \frac{1}{k}B + \frac{1}{n}G\right)\underline{x} = (e_2^* \ldots e_n^*)R\begin{bmatrix} e_2^* \\ \vdots \\ e_n^* \end{bmatrix} ,$$

if we consider the conditional joint distribution of $\mathcal{T}/\sigma^2$ and $\mathcal{E}/\sigma^2$, given Φ, then they are mutually independent, i.e., $\mathcal{T}/\sigma^2$ obeys the non-central chi-square distribution of d.f. $k-1$ with non-centrality parameter $bt't/\sigma^2$ and $\mathcal{E}/\sigma^2$ obeys the chi-square distribution of d.f.

(b-1)(b-1). Thus the conditional (given Φ) distribution of the statistic

$$(28) \qquad (b-1)\,\frac{\underline{x}'(\frac{1}{b}T - \frac{1}{n}G)\underline{x}}{\underline{x}'(I - \frac{1}{b}T - \frac{1}{k}B + \frac{1}{n}G)\underline{x}}$$

is the non-central F distribution of d.f. (k-1, (k-1)(b-1)) with non-centrality parameter $b\underline{t}'\underline{t}/\sigma^2$. Consequently the absolute distribution of the above statistic is the same.

Thus this seems to offer a way of justification of the traditional treatment of this problem provided that we can ignore the plot effect.

References

[1] Mann, H.B., Analysis and design of experiments. Dover Publications, Inc., 1949, Chapt. VII, p. 76.

[2] James, A.T., "The relationship algebra of an experimental design," Ann. of Math. Stat., vol. 28, 1957.