INCOMPLETE BLOCK DESIGNS IN WHICH THE NUMBER OF REPLICATES IS NOT THE SAME FOR ALL TREATMENTS

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1. Introduction.

When designing experiments for comparison of the effects of treatments or varieties by means of an incomplete block arrangement, one often meets the difficulty that the usual requirement that every treatment occurs the same number of times, say r, cannot be satisfied. One has, e.g., only a very small quantity of seed of some (new) varieties available, while seed of other varieties is abundant. It is also possible that for some reason one has a minor interest in some of the varieties in comparison with the remaining varieties, and so does not wish to spend as many costs (replicates) on these varieties as on the more interesting ones.

Getting round this difficulty by including only as many replicates for <u>all</u> treatments as because of the limitations can be taken for part of the treatments is not adequate, because it sacrifices information which could be obtained with respect to the treatments for

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which such limitations do not exist, as well with respect to the comparison between the two kinds of treatments. For the last reason it is also not desirable to perform experiments for such groups of treatments separately.

We therefore consider the possibilities of construction of incomplete block designs in which the treatments have not all the same number of replicates, but which still have a certain degree of symmetry, usually indicated with the term "balance". Although a more general attack of this problem is of interest we confine ourselves here to the case in which the treatments have either r_1 or r_2 replicates ($r_1 < r_2$), and hence will be called rare and frequent treatments respectively.

2. Definition of parameters of incomplete block designs with rare and frequent treatments, and their relations.

An incomplete block design with rare and frequent treatments is a two-way classification of experimental units with the following properties: One classification is in b classes, numbered $1,\ldots,j,\ldots,b$, each of size k, and called blocks; the other is according to classes called treatments; v_1 of these treatments occur r_1 times and v_2 occur r_2 times. Requiring that $r_1 < r_2$ we call each treatment of the first group rare, and each treatment of the second group frequent. The total number of treatments, numbered $1,\ldots,i,\ldots$, with the rare treatments first, is $v_1 + v_2$. We see immediately that

$$r_1v_1 + r_2v_2 = kb = n$$

where n is the total number of units. We further require that any treatment occurs at most once in a block.

The design will completely be determined by the incidence matrix N with typical element n_{ij}, which is unity if treatment i occurs in block j, and is zero otherwise. So every column contains exactly k ones, and every row either r_1 or r_2 ones.

In order to obtain some symmetry we require further:

a) Any pair of rare treatments occurs together in λ_{11} blocks.

b) Any pair of frequent treatments occurs together in λ_{22} blocks.

c) Any pair of treatments, the one of which is a rare treatment and the other a frequent treatment, occurs together in $\lambda_{12} = \lambda_{21}$ blocks.

We see immediately that $\lambda_{11} \leq r_1$ and $\lambda_{22} \leq r_2$, with equality only if every of the rare or frequent treatments, respectively, occurs in the same set of blocks. Further $\lambda_{12} \leq r_1$, where equality means that, when a rare treatment occurs in a block, it is accompanied by all the frequent treatments. Hence $\lambda_{12} = r_1$ implies that every block contains either all frequent treatments together with some rare treatments, or some frequent treatments only. Because, however, all blocks must have the same size, blocks of the latter kind cannot occur. Thus from $\lambda_{12} = r_1$ follows $r_2 = \lambda_{22}$.

Because $\lambda_{12} = 0$ would imply a disconnected design consisting of blocks with either rare or frequent treatments only, we require: $\lambda_{12} \geq 1$.

Then it also follows that not at the same time $r_1 = \lambda_{11}$ and $r_2 = \lambda_{22}$; the case that the set of blocks in each of which the rare treatments occur is different from the set of blocks in each of which the frequent treatments occur has been excluded, while the presence of blocks containing both the rare and the frequent treatments would lead to unequal block sizes, as $r_1 < r_2$.

From a consideration of the units in the blocks containing a particular rare treatment follows: $r_1 k = r_1 + \lambda_{11}(v_1-1) + \lambda_{12}v_2$ or:

$$r_1(k-1) = \lambda_{11}(v_1-1) + \lambda_{12}v_2 \tag{1}$$

Similarly, $$r_2(k-1) = \lambda_{12}v_1 + \lambda_{22}(v_2-1) \quad .$$

In accordance with the methods of studying designs, as the balanced or partially balanced designs, we consider the non-negative matrix NN', which in this case is equal to the matrix of order v_1+v_2: $\begin{bmatrix} A & \vdots & B \\ \cdots & \vdots & \cdots \\ B' & \vdots & C \end{bmatrix}$ where A is a symmetric matrix of order v_1 with elements r_1 in the diagonal and λ_{11} elsewhere, C is a symmetric matrix of order v_2 with elements r_2 in the diagonal and λ_{22} elsewhere, and B is a matrix with v_1 rows and v_2 columns and all elements equal λ_{12}.

In order to find $|NN'|$ we subtract in $\begin{bmatrix} A \\ \cdots \\ B' \end{bmatrix}$ the last column from the other columns, and proceed similarly in $\begin{bmatrix} B \\ \cdots \\ C \end{bmatrix}$. Next the sum of the first (v_1-1) rows of the upper part of the partitioned matrix are added to the v_1-th row, and the matrix in the bottom is handled in a similar way.

We find

$$|NN'| = (r_1-\lambda_{11})^{v_1-1} \cdot (r_2-\lambda_{22})^{v_2-1} \cdot \begin{vmatrix} r_1+\lambda_{11}(v_1-1) & \lambda_{12}v_1 \\ \lambda_{12}v_2 & r_2+\lambda_{22}(v_2-1) \end{vmatrix}$$

We reduce the determinant on the right hand side, say $|D|$, using the relations (1) to: $|D| = \begin{vmatrix} kr_1 & kr_2 \\ \lambda_{12}v_2 & kr_2-\lambda_{12}v_1 \end{vmatrix} = k^2(r_1r_2-\lambda_{12}b)$.

Because the coefficient of $|D|$ in $|NN'|$ is non-negative, $|D| \geq 0$ or

$$r_1r_2 \geq \lambda_{12}b \quad . \qquad (2)$$

Considering the determinant $|D|$ we can express this condition as follows: In the set of blocks in which a fixed rare treatment occurs the ratio between the numbers of elements with rare treatments and those with frequent treatments is at least as large as the similar ratio in the set of blocks in which a fixed frequent treatment occurs. In the case of equality the proportions $\left\{(kr_1-\lambda_{12}v_2)v_1\right\} / \lambda_{12}v_2v_1$ and $\lambda_{12}v_1v_2 \ / \left\{(kr_2-\lambda_{12}v_1)v_2\right\}$ are equal, and thus can be replaced by $kr_1v_1 \ / \ kr_2v_2$, i.e., the proportion between the numbers of elements with rare treatments and those with frequent treatments in the design as a whole.

As rank (NN') = rank (N), an analogue of Fisher's inequality for balanced incomplete blocks holds: $b \geq v_1 + v_2$ unless $|NN'| = 0$; in other words, if $r_1 > \lambda_{11}$, $r_2 > \lambda_{22}$, and $r_1r_2 > \lambda_{12}b$, then $b \geq v_1 + v_2$.

In order to investigate the restrictions on the parameters in case NN' is singular, we consider the zero characteristic roots of NN'

and their multiplicity. The characteristic equation $|NN'-\lambda I| = 0$ can after a reduction similar to that of $|NN'|$ be written as:

$$(r_1-\lambda_{11}-\lambda)^{v_1-1}(r_2-\lambda_{22}-\lambda)^{v_2-1}\left[\lambda^2+\lambda\{\lambda_{12}(v_1+v_2)-k(r_1+r_2)\}+k^2(r_1r_2-\lambda_{12}b)\right]=0.$$

First we assert that the quadratic form in square brackets can have only one zero, namely if $r_1r_2 = \lambda_{12}b$. If there could be two zeros, this would imply that the corresponding matrix D could have rank zero, which because, e.g., $\lambda_{12}v_1$ is non-zero, is impossible.

Secondly we observe that, if $r_1 = \lambda_{11}$ (and hence $r_2 > \lambda_{22}$), in the set of blocks in each of which the rare treatments occur one must have λ_{12} replicates of the frequent treatments. As $\lambda_{12} \leq r_1 < r_2$, there must be other blocks containing $r_2-\lambda_{12}$ replicates of the frequent treatments only. But this means that in the set of blocks containing some rare treatment the ratio between the numbers of elements with rare treatments and those with frequent treatments is larger than in the set of blocks containing a frequent treatment. Hence $r_1r_2 > \lambda_{12}b$.

Thirdly we observe that, if $r_2 = \lambda_{22}$ (and hence $r_1 > \lambda_{11}$), in the set of blocks in each of which the frequent treatments occur one must have λ_{12} replicates of the rare treatments. If now in addition $r_1 = \lambda_{12}$, then in every block there are v_2 frequent treatments and r_1v_1/r_2 rare treatments, so that $r_1r_2 = \lambda_{12}b$. If, however, $r_1 > \lambda_{12}$ then there are also blocks with rare treatments only, and consequently $r_1r_2 > \lambda_{12}b$.

Finally it follows that if $r_1r_2 = \lambda_{12}b$ (and hence $r_1 > \lambda_{11}$), then $r_2 = \lambda_{22}$ only if $r_1 = \lambda_{12}$. But, as we saw, $r_1 = \lambda_{12}$ implies always $r_2 = \lambda_{22}$.

Summarizing we distinguish the following cases:

A: $r_1 = \lambda_{12}$; this implies $r_2 = \lambda_{22}$, $r_1 r_2 = \lambda_{12} b$ and $r_1 > \lambda_{11}$. The characteristic equation has v_2 roots zero, so that $b \geq v_1$.

B: $r_2 = \lambda_{22}$ and $r_1 > \lambda_{12}$; this implies $r_1 r_2 > \lambda_{12} b$ and $r_1 > \lambda_{11}$. The characteristic equation has v_2-1 roots zero, so that $b \geq v_1 + 1$.

C: $r_1 r_2 = \lambda_{12} b$ and $r_1 > \lambda_{12}$; then $r_2 > \lambda_{22}$ and $r_1 > \lambda_{11}$. The characteristic equation has one root zero, so that $b \geq v_1+v_2-1$.

D: $r_1 = \lambda_{11}$; then $r_2 > \lambda_{22}$, $r_1 r_2 > \lambda_{12} b$ and $r_1 > \lambda_{12}$. The characteristic equation has v_1-1 roots zero, so that $b \geq v_2+1$.

E: $r_1 > \lambda_{11}$, $r_2 > \lambda_{22}$, $r_1 r_2 > \lambda_{12} b$; then $r_1 > \lambda_{12}$. The characteristic equation has no zero roots and $b \geq v_1 + v_2$.

3. Some types of construction.

For each of the five cases distinguished above we constructed examples which will be considered now. We shall refer several times to the catalogue of plans of balanced incomplete block designs given in "Experimental Designs" by W. G. Cochran and G. M. Cox, pp. 327 sqq, and denote particular designs by the numbers given there.

Type A: From the above discussion it follows that designs of type A contain all the frequent treatments in every block. Further the number of rare treatments is in every block the same, namely $r_1 v_1/v_2$; every pair of these treatments has to occur together in λ_{11} of the $b = r_2$ blocks. Hence the rare treatments must be arranged according to a balanced incomplete block design, with the number of treatments equal to v_1, the number of replicates r_1, the number of blocks r_2, the block size r_1v_1/v_2 and λ-parameter λ_{11}, while each of the blocks is extended with a full replication of the now certainly frequent ($r_2 = b \geq r_1$) treatments. The inequality $b \geq v_1$ now also follows from Fisher's inequality for balanced incomplete blocks.

Example 1. Consider the treatments in design 11.7 as rare, and add two frequent treatments to every block. The parameters will be:

$$\lambda_{11} = 1 \quad r_1 = 3 \quad v_1 = 7 \quad b = 7 \quad k = 5$$
$$\lambda_{12} = 3$$
$$\lambda_{22} = 7 \quad r_2 = 7 \quad v_2 = 2$$

As the balanced incomplete block design for rare treatments we cannot choose the particular case of a complete block design, because this would lead to $r_1 = r_2$. Another extreme choice is a balanced design in which the treatments occur only once. Then we obtain the blocks of the design by adding to every replicate ($r_2 \geq 2$) of the frequent treatments one of the $v_1 = r_2$ rare treatments.

Example 2. The parameters of the designs of the particular kind as we just described will be:

$$\lambda_{11}=0 \qquad r_1=1 \qquad v_1=r_2 \qquad b=r_2=v_1 \qquad k=v_2+1$$
$$\lambda_{12}=1$$
$$\lambda_{22}=r_2=v_1=b \qquad r_2=v_1=b \qquad v_2=k-1$$

Type B: In this case every block contains either all frequent treatments or none. The simplest arrangement will then be such that the blocks without frequent treatments are complete replicates of the rare treatments, while the rare treatments in the blocks in which the frequent treatments occur form a balanced incomplete block design.

Of course the number of complete replicates of the rare treatments must be so small that $r_1 < r_2$.

Example 3. Consider the seven treatments of 11.7 as rare, add four frequent treatments to each of its blocks, and finally add two blocks each containing the seven rare treatments. The parameters are:

$$\lambda_{11}=3 \quad r_1=5 \quad v_1=7 \quad b=9 \quad k=7$$
$$\lambda_{12}=3$$
$$\lambda_{22}=7 \quad r_2=7 \quad v_2=4$$

A particular case is that in which the incomplete block design consists of v_1 blocks in each of which only one of the v_1 rare treatments occurs. We obtain the desired design by taking $r_2=v_1=k$ blocks, each containing the $v_2 = k-1$ frequent treatments, adding one of the v_1 rare treatments to each of these blocks, and finally adding r_1-1 blocks with the rare treatments to the design; r_1 must satisfy $1 \leq r_1-1 \leq r_2-2$, so that $r_1 \geq 2$ and $k = r_2 = v_1 \geq 3$.

Example 4. The parameters of the designs described above are:

$$\lambda_{11}=r_1-1 \quad r_1 \quad v_1=r_2=k=v_2+1 \quad b=r_1+r_2-1 \quad k=v_1=r_2=v_2+1$$
$$\lambda_{12}=1$$
$$\lambda_{22}=r_2=v_1=k \quad r_2=v_1=k \quad v_2=k-1$$

A variant on the last type of construction will be the case where also the blocks without frequent treatments form a balanced incomplete block design; for small numbers of replicates there are, however, not many pairs of different balanced incomplete block designs with the same number of treatments such that their total number of replicates is smaller than the number of blocks in that with the smallest block size.

Example 5. Consider the five treatments of 11.2 as rare, add to each of its ten blocks the two frequent treatments, and finally add the five possible combinations of the five rare treatments in groups of size four as five additional blocks. The parameters:

$$\lambda_{11}=4 \quad r_1=8 \quad v_1=5 \quad b=15 \quad k=4$$
$$\lambda_{12}=4$$
$$\lambda_{22}=10 \quad r_2=10 \quad v_2=2$$

Type C: In this case we found some types of construction of designs in which the ratio of the numbers of elements with rare and frequent treatments is equal not only in every set of blocks containing the replicates of any fixed rare or frequent treatment, but even in every individual block.

A very simple way of construction is that by means of those balanced incomplete block designs which are resolvable, i.e., the blocks of which can be grouped in such a way that each group contains the same number of complete replicates. One considers the treatments in such a design as the rare (frequent) treatments if the number of blocks in such a group is larger (smaller) than the number of replicates of that design. To every block of the same group one adds the same frequent (rare) treatment, the number of which is equal to the number of groups.

Example 6. Consider the treatments of design 11.1 as the frequent treatments and add one of the three rare treatments to each of the pairs of blocks (1) (2), (3) (4) and (5) (6), which form a replicate. The parameters are:

$$\lambda_{11}=0 \quad r_1=2 \quad v_1=3 \quad b=6 \quad k=3$$
$$\lambda_{12}=1$$
$$\lambda_{22}=1 \quad r_2=3 \quad v_2=4$$

Example 7. Consider the treatments of design 11.2 as the rare treatments and add one frequent treatment to the blocks (1) (2) (3) (4) (5) and another to the blocks (6) (7) (8) (9) (10). The parameters are:

$$\lambda_{11}=1 \quad r_1=4 \quad v_1=5 \quad b=10 \quad k=3$$
$$\lambda_{12}=2$$
$$\lambda_{22}=0 \quad r_2=5 \quad v_2=2$$

Another way of finding such designs is one of the general ways of constructing designs of type E, namely deleting one block of a balanced incomplete block design. In the resulting design the treatments which occurred in the deleted block have one replicate less than the other treatment, while λ_{22} and λ_{12} are equal to the corresponding balanced design parameter, and λ_{11} is one unity less. If the group of treatments in the deleted block happens to be equally represented in all the other blocks of the balanced design, we shall have $r_1r_2 = \lambda_{12}b$, and thus a design of the desired type.

Example 8. Delete one of the blocks of 11.7 and call the treatments occurring in that block rare. The parameters will be:

$\lambda_{11}=1 \quad r_1=3 \quad v_1=3 \quad b=6 \quad k=4$

$\lambda_{12}=2$

$\lambda_{22}=2 \quad r_2=4 \quad v_2=4$

Similarly we can use 11.8 and the design consisting of the ten possible combinations of five treatments in groups of size three.

A third method consists of combining the blocks of two balanced incomplete block designs, both with the same number of blocks, the one with the rare treatments and the other with the frequent treatments, into a design with the same number of blocks, and such that every pair consisting of a rare and a frequent treatment occurs equally often together in those new blocks. For that purpose it is necessary that one can find r_1 blocks in the balanced design with the frequent treatments such that they constitute a certain multiple of all the frequent treatments, and this in v_1 different ways. Similarly in the design with the rare treatments one must find r_2 blocks containing the same multiple as before of all the rare treatments, and this in v_2 different ways. The number of replicates in such block groups will, if the design exists, be equal to $\lambda_{12}=r_1r_2/b$ which of course must be an integer.

Example 9. Wanting to combine the designs 11.2 and 11.4, we observe that in 11.4 it is possible to take together the blocks (1) (3) (8) (9), (2) (4) (6) (10), (2) (3) (7) (10), (2) (5) (6) (9), and (4) (5) (7) (8) each group of which contains two replicates of the frequent treatments; similarly in 11.2 the block groups (1) (2) (3) (4) (5), (1) (2) (8) (9) (10), (3) (4) (6) (8) (10),

(4) (5) (7) (9) (10), (1) (5) (6) (7) (8) and (2) (3) (6) (7) (9) each contain two replicates of the rare treatments. The following design could be constructed:

1	2	5	:	7	8
1	2	6	:	9	10
1	3	4	:	7	9
1	3	6	:	8	11
1	4	5	:	10	11
2	3	4	:	8	10
2	3	5	:	9	11
2	4	6	:	7	11
3	5	6	:	7	10
4	5	6	:	8	9

with the parameters:

$\lambda_{11}=1 \quad r_1=4 \quad v_1=5 \quad b=10 \quad k=5$

$\lambda_{12}=2$

$\lambda_{22}=2 \quad r_2=5 \quad v_2=6$

Another design of the same kind obtained from 11.3 and 11.16 is:

1	2	3	4	:	11	12
1	2	5	6	:	13	14
1	3	7	8	:	13	15
1	4	9	10	:	14	15
1	5	7	9	:	11	16
1	6	8	10	:	12	16
2	3	6	9	:	15	16
2	4	7	10	:	13	16
2	5	8	10	:	11	15
2	7	8	9	:	12	14
3	5	9	10	:	12	13
3	6	7	10	:	11	14
3	4	5	8	:	14	16
4	5	6	7	:	12	15
4	6	8	9	:	11	13

with the parameters:

$\lambda_{11}=1 \quad r_1=5 \quad v_1=6 \quad b=15 \quad k=6$

$\lambda_{12}=2$

$\lambda_{22}=2 \quad r_2=6 \quad v_2=10$

Finally we give a design derived from 11.7 and 11.18:

1	2	4	5	8	9	:	13	14	15	16
5	6	7	8	9	10	:	12	14	15	16
2	4	5	6	9	10	:	11	12	13	16
1	2	4	6	7	8	:	11	12	15	16
3	4	7	8	9	10	:	11	13	15	16
2	3	4	6	8	10	:	12	13	14	15
1	2	6	7	9	10	:	11	13	14	15
1	3	5	6	8	9	:	11	12	13	15
1	2	3	8	9	10	:	11	12	14	16
2	3	4	5	7	9	:	11	12	14	15
1	4	5	7	8	10	:	11	12	13	14
1	2	3	5	7	10	:	12	13	15	16
2	3	5	6	7	8	:	11	13	14	16
1	3	4	5	6	10	:	11	14	15	16
1	3	4	6	7	9	:	12	13	14	16

which has the parameters:

$\lambda_{11}=5 \quad r_1=9 \quad v_1=10 \quad b=15 \quad k=10$

$\lambda_{12}=6$

$\lambda_{22}=6 \quad r_2=10 \quad v_2=6$

Type D: From the above discussion it follows that in this case there are r_1 blocks in each of which the rare treatments and in which λ_{12} replicates of the frequent treatments are arranged, while there are other blocks containing only $r_2-\lambda_{12}$ replicates of the frequent treatments.

The easiest way of construction one can think of is to choose some balanced incomplete block design with the frequent treatments together with blocks each containing all the frequent treatments; then to every block of the balanced design one adds an appropriate number of rare treatments such that all blocks have the same size. The number of complete blocks must be so large that the treatments in those blocks indeed will be frequent.

Example 10. Add two rare treatments to each of the blocks of 11.1 and take at least four, say four, blocks with the four treatments each. The parameters of the resulting design are:

$$\lambda_{11}=6 \quad r_1=6 \quad v_1=2 \quad b=10 \quad k=4$$
$$\lambda_{12}=3$$
$$\lambda_{22}=5 \quad r_2=7 \quad v_2=4$$

Sometimes one can add only one rare treatment.

Example 11. Consider the treatments of the design consisting of the four possible combinations of four treatments in groups of size three as frequent. Add the rare treatment to each of the blocks, and finally add at least two, say two, blocks each containing the four frequent treatments. The parameters, of which λ_{11} has no meaning in this case will be:

$$r_1=4 \quad v_1=1 \quad b=6 \quad k=3$$
$$\lambda_{12}=3$$
$$\lambda_{22}=4 \quad r_2=5 \quad v_2=4$$

Another extreme is that in which $\lambda_{12}=1$ so that the balanced incomplete block with the frequent treatments degenerates into blocks each containing one of the frequent treatments.

Example 12. Form $r_1=v_2=k$ blocks each containing all the v_1 rare treatments and one of the v_2 frequent treatments. Add at least r_1 blocks each containing the v_2 frequent treatments. Because $v_1=k-1$ we have $k \geq 2$. The parameters are:

$\lambda_{11}=r_1=v_2=k$ $r_1=v_2=k$ $v_1=k-1$ $b=r_1+r_2-1$ $k=r_1=v_2$

$\lambda_{12}=1$

$\lambda_{22}=r_2-1$ r_2 $v_2=r_1=k=v_1+1$

Analogously to what we remarked under type B the blocks without rare treatments could form a balanced incomplete design instead of a set of complete blocks. Now one needs two different balanced incomplete block designs with the same number of treatments such that their total number of replicates is larger than the number of blocks in that with the smallest block size.

Example 13. Consider the seven treatments of 11.7 as frequent, and add to each of its seven blocks the three rare treatments. Finally add the seven possible combinations of the seven frequent treatments in groups of size six as seven additional blocks. The resulting design has the parameters:

$\lambda_{11}=7$ $r_1=7$ $v_1=3$ $b=14$ $k=6$

$\lambda_{12}=3$

$\lambda_{22}=6$ $r_2=9$ $v_2=7$

Type E: One of the ways of construction for this case has previously been indicated under type C in connection with example 8, namely by deleting a block from a balanced incomplete block design.

Example 14. Delete one block from 11.1. The parameters will be:

$\lambda_{11}=0$ $r_1=2$ $v_1=2$ $b=5$ $k=2$

$\lambda_{12}=1$

$\lambda_{22}=1$ $r_2=3$ $v_2=2$

Instead of deleting one block one may also strike out a set of blocks, provided they form a balanced incomplete block design.

Example 15. Consider design 11.9 and strike out the blocks occurring in 11.1, namely (1) (2) (5) (9) (14) and (18). The parameters of the resulting design are:

$\lambda_{11}=0 \quad r_1=4 \quad v_1=4 \quad b=22 \quad k=2$

$\lambda_{12}=1$

$\lambda_{22}=1 \quad r_2=7 \quad v_2=4$

Another way of construction is analogous to the first method that we discussed under type C in connection with the examples 6 and 7. We start by following the same procedure. Hereafter, however, we add an appropriate balanced incomplete block design with the same number of treatments as in the balanced design with which we started. This added design will not be a complete block design in general.

Example 16. Take the design that we constructed in example 6 and add the four possible combinations of the four (now still more) frequent treatments in groups of three as four additional blocks. The parameters will be:

$\lambda_{11}=0 \quad r_1=2 \quad v_1=3 \quad b=10 \quad k=3$

$\lambda_{12}=1$

$\lambda_{22}=3 \quad r_2=6 \quad v_2=4$

4. Variances of treatment comparisons.

Putting the best estimates of the effects of the rare treatments t_{1i} $(i=1,\ldots,v_1)$, and those of the frequent treatments t_{2j} $(j=1,\ldots,v_2)$, the normal equations for the treatment effects are according to standard methods:

$$\begin{cases} r_1(1-\frac{1}{k})t_{1i} - \frac{\lambda_{11}}{k}\sum_{i'\neq i} t_{1i'} - \frac{\lambda_{12}}{k}\sum_j t_{2j} = q_{1i} \quad (i=1,\dots,v_1) \\ -\frac{\lambda_{12}}{k}\sum_i t_{1i} + r_2(1-\frac{1}{k})t_{2j} - \frac{\lambda_{22}}{k}\sum_{j'\neq j} t_{2j'} = q_{2j} \quad (j=1,\dots,v_2) \end{cases}$$

where q_{1i} is the sum of the observations with the i-th rare treatment diminished by the sum of the means of the blocks in which this treatment occurs, and q_{2j} is a similar expression for the j-th frequent treatment. Adding the first v_1 equations and inserting the restriction $r_1 \sum_i t_{1i} + r_2 \sum_j t_{2j} = 0$ we get after some reduction and using the relations (1):

$$\sum_i t_{1i} = \frac{r_2}{\lambda_{12}b} \sum_i q_{1i}$$

Because $\sum_i q_{1i} + \sum_j q_{2j} = 0$ we find: $\sum_j t_{2j} = \frac{r_1}{\lambda_{12}b} \sum_j q_{2j}$.

Now the solution of the equations is straightforward:

$$t_{1i} = \frac{k}{r_1(k-1)+\lambda_{11}} \left(q_{1i} + \frac{\lambda_{11}r_2-\lambda_{12}r_1}{\lambda_{12}n} \sum_i q_{1i}\right)$$

$$t_{2j} = \frac{k}{r_2(k-1)+\lambda_{22}} \left(q_{2j} + \frac{\lambda_{22}r_1-\lambda_{12}r_2}{\lambda_{12}n} \sum_j q_{2j}\right) \quad .$$

Applying the well-known fact that the variance of t_i-t_j is $(t_{ii}-t_{ij}-t_{ji}+t_{jj})\sigma^2$ where t_{ij} is the coefficient of the sum of the observations of the j-th treatment in any solution of t_i, and σ^2 the variance of the uncorrelated observations, we find that the variance of the difference between the effects of two rare treatments

$$\text{var}\ (t_{1i}-t_{1i'}) = \frac{2k\sigma^2}{r_1(k-1)+\lambda_{11}}$$, for two frequent treatments:

$$\text{var}\ (t_{2j}-t_{2j'}) = \frac{2k\sigma^2}{r_2(k-1)+\lambda_{22}}$$, and for a rare and a frequent treatment:

$$\text{var}\ (t_{1i}-t_{2j}) = \frac{k\sigma^2}{r_1(k-1)+\lambda_{11}}\ (1+\frac{\lambda_{11}r_2-\lambda_{12}r_1}{\lambda_{12}n}) + \frac{k\sigma^2}{r_2(k-1)+\lambda_{22}}\ (1+\frac{\lambda_{22}r_1-\lambda_{12}r_2}{\lambda_{12}n})$$

which by means of the relations (1) can be reduced to:

$$\frac{k\sigma^2}{r_1(k-1)+\lambda_{11}} + \frac{k\sigma^2}{r_2(k-1)+\lambda_{22}} + \frac{(\lambda_{11}\lambda_{22}-\lambda_{12}^2)k\sigma^2}{\lambda_{12}[r_1(k-1)+\lambda_{11}]\ [r_2(k-1)+\lambda_{22}]}\ .$$

Putting $E_1 = \frac{r_1(k-1)+\lambda_{11}}{kr_1}$ and $E_2 = \frac{r_2(k-1)+\lambda_{22}}{kr_2}$ we have:

$$\text{var}(t_{1i}-t_{1i'}) = \frac{2\sigma^2}{E_1r_1}$$

$$\text{var}(t_{2j}-t_{2j'}) = \frac{2\sigma^2}{E_2r_2}$$

$$\text{var}(t_{1i}-t_{2j}) = \sigma^2\ (\frac{1}{E_1r_1} + \frac{1}{E_2r_2} + \frac{\lambda_{11}\lambda_{22}-\lambda_{12}^2}{\lambda_{12}k}\cdot\frac{1}{E_1r_1}\cdot\frac{1}{E_2r_2})\ .$$

In the special case that there is only one frequent treatment the solutions evidently are:

$$t_{1i} = \frac{k}{r_1(k-1)+\lambda_{11}}\ (q_{1i} + \frac{\lambda_{11}r_2-\lambda_{12}r_1}{\lambda_{12}n}\ \sum_i\ q_{1i})\ \text{and}\ t_{2j} = \frac{r_1}{\lambda_{12}b}\ q_{2j}$$

Utilizing the same procedure as above we find:

$$\text{var}(t_{1i}-t_{1i'}) = \frac{2\sigma^2}{E_1r_1}\ \text{and}\ \text{var}(t_{1i}-t_{2j}) = \frac{\lambda_{11}+\lambda_{12}}{\lambda_{12}}\cdot\frac{\sigma^2}{E_1r_1}\ .$$

Similarly, if $v_1=1$ (only one rare treatment):

$$t_{1i} = \frac{r_2}{\lambda_{12}b} q_{1i} \text{ and } t_{2j} = \frac{k}{r_2(k-1)+\lambda_{22}} \left(q_{2j} + \frac{\lambda_{22}r_1-\lambda_{12}r_2}{\lambda_{12}n} \sum_j q_{2j}\right)$$

$$\mathrm{var}(t_{1i}-t_{2j}) = \frac{\lambda_{12}+\lambda_{22}}{\lambda_{12}} \cdot \frac{\sigma^2}{E_2 r_2} \text{ and } \mathrm{var}(t_{2j}-t_{2j'}) = \frac{2\sigma^2}{E_2 r_2} .$$

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