INFINITESIMAL RENEWAL PROCESSES

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1. Introduction. Renewal theory (Smith [1]) is concerned primarily with the renewal function $H(x)$, which gives the expected number of partial sums $S_n = X_1 + X_2 + \dots + X_n$ of the independent and identically distributed random variables $\{X_i\}$ which satisfy the inequality $S_n \leq x$. Generally the X_i are non-negative, and we then call the sequence $\{X_i\}$ a renewal process; when we wish to drop the non-negativity restriction, i.e., consider unrestricted X_i, we can draw attention to this extra generality by speaking of an extended renewal process.

The renewal function $H(x)$ can be related to the following physical model concerned with the motion of a particle P on the real axis. At zero time P is assumed to be at the origin and P remains there until time $t = 1$, at which instant P moves instantaneously to the point $S_1 = X_1$ where it is to remain for a further time interval of unit duration. The motion of P then continues in an obvious way so that, for instance, P is at S_n during the time-interval $[n, n+1)$. In this model the renewal function $H(x)$ measures the expected amount of time which P spends in the interval $(-\infty, x]$. It is assumed here, as is usual in renewal theory, that $\mathcal{E} X_i > 0$ and it is known that in this case $H(x)$ is necessarily finite for all finite x.

One can now imagine the following development of our model of a renewal process. Suppose we change our time and distance scales so that the individual shifts in the position of P become, in general, very small but at the same time occur extremely frequently. We are led to consider a process

$x(t)$ with stationary independent increments for which $x(0) = 0$ and $\mu_1 = \; x(1) > 0$, and to consider the expected amount of time that this process satisfies the condition $x(t) \leq x$. We shall call this expected duration $H(x)$ and show that $H(x)$ has all the properties of a conventional renewal function. In particular we shall show that, <u>under suitable conditions</u>, the following three theorems hold.

<u>Blackwell's Theorem.</u> For any $\alpha > 0$,

$$H(x + \alpha) - H(x) \longrightarrow \alpha\, \mu_1^{-1}$$

as $x \longrightarrow +\infty$; the limit being zero if $\mu_1 = +\infty$.

<u>Second Renewal Theorem.</u> For a certain constant μ_2 (which is actually the second moment, assumed finite, of the renewal 'lifetimes' in a certain ordinary renewal process which is associated with the process $x(t)$; the details of this ordinary renewal process are given below)

$$H(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1 + o(1)$$

as $x \longrightarrow +\infty$.

<u>Renewal Density Theorem.</u> If $h(x) = H'(x)$ exists then

$$h(x) \longrightarrow \mu_1^{-1} \qquad \text{as } x \longrightarrow +\infty,$$
$$\longrightarrow 0 \qquad \text{as } x \longrightarrow -\infty.$$

The name for the second of these theorems is coined here, for the first time, as a convenience. It has only previously been proved for renewal processes satisfying the non-negativity assumption (Smith [2]); we give a proof of the second renewal theorem which is valid for extended renewal processes in section 3 below.

Because of the physical model which we have described, and which gives rise to the analogy between our present study and standard renewal theory, we shall speak of <u>infinitesimal renewal processes</u>.

In Theorems 1, 2 and 3 we, essentially, complete the program outlined above. In particular, when x(t) has a frequency function f(x; t), we shall see that

$$h(x) = \int_0^\infty f(x;t)dt .$$

Integrals of this kind are of particular interest in the theory of queues and dams, as we hope to show elsewhere, and it was in considering their asymptotic behavior that we were led to the present investigation. Unfortunately the most general sufficient conditions available for the renewal density theorem (those of Smith [3]) are, in the present application, bound up with the behavior of the density function

$$f(x) = \int_0^\infty e^{-t} f(x;t)dt .$$

One obstacle is that we need f(x) to belong to the class $L_{1+\delta}$ for some $\delta > 0$. Theorems 4 and 5 are concerned with providing sufficient restraints under which this condition will be automatically satisfied. We should point out incidentally that the renewal density theorem for the infinitesimal renewal process can apply to cases where x(t) is not absolutely continuous.

Finally we consider the particular process associated with the density

$$f(x,t) = \frac{e^{-x} x^{t-1}}{\Gamma(t)} , \quad t > 0 .$$

We shall see that none of our conditions covers this simple case even though the renewal density theorem does hold. In fact we shall deduce from this process a conventional renewal process for which the renewal density theorem holds, but for which $f(x)$ does not belong to any class $L_{1+\delta}$ for $\delta > 0$. This raises the intriguing question as to how the class of functions, considered in [3], for which the renewal density theorem holds, can be widened.

A heuristic discussion of the present ideas has already been given in Smith [1]. It is proposed to consider elsewhere the theory of infinitesimal renewal processes in relation to the particular homogeneous process which arises in the theory of dams. It will there be shown that the infinitesimal renewal density function $h(x)$ gives the expected number of up-crosses of the real number x by the process.

2. <u>The infinitesimal renewal function</u>. Let $\{X_i\}$ be a sequence of independent, identically distributed random variables such that $0 < \mathcal{E} X_i \leq +\infty$. Let $S_n = X_1 + X_2 + \dots + X_n$ and write $F_n(x)$ for the distribution function of S_n and $F(x)$ for $F_1(x)$. Then the renewal function corresponding to $\{X_i\}$ is determined by

$$H(x) = \sum_{k=1}^{\infty} F_k(x) , \tag{1}$$

(see, e.g., [1]).

We write $x(t)$ for a process with stationary independent increments such that $x(0) = 0$ and $0 < \mathcal{E} x(1) \leq \infty$ and suppose, wherever necessary, that $x(t)$ is both separable and measurable. (For a discussion of such processes we refer to Doob [4], Gnedenko and Kolmogorov [5], Lévy [6]). We shall also write $F(x;t) = \text{prob}\{x(t) \leq x\}$, defined for all $t \geq 0$.

The function $U(x)$ is defined by

$$U(x) = 1 \quad \text{for} \quad x \geq 0$$
$$= 0 \quad \text{for} \quad x < 0 \; .$$

Since $U(\cdot)$ is Borel-measurable, $U(x-x(t))$ will be a measurable process for any fixed x (Halmos [7], p. 81). Thus, by Fubini's theorem, we can introduce the random variable

(2) $$T(x) = \int_0^\infty U(x-x(t))dt \; .$$

Evidently $T(x)$ is the total amount of time for which $x(t) \leq x$. Further, since $\mathcal{E}\, U(x-x(t)) = F(x;t)$, we can deduce from (2) by a second appearl to Fubini's theorem that

(3) $$\mathcal{E}\, T(x) = \int_0^\infty F(x;t)dt \; ,$$

although the finiteness of $\mathcal{E}\, T(x)$ is not yet established.

Theorem 1. If we define the distribution function

(4) $$F(x) = \int_0^\infty e^{-t} F(x;t)dt \; ,$$

and the renewal function $H(x)$ by (1), in which of course $F_k(x)$ will be k-fold convolution of $F(x)$ with itself, then

$$H(x) = \mathcal{E}\, T(x) \; .$$

Consequently $\mathcal{E}\, T(x)$ is a finite and monotone non-decreasing function of x.

Proof. Let $\{Y_n\}$ be a sequence of independent, non-negative, and identically distributed random variables with the distribution function: $\text{prob}\{Y_n \leq y\} = U(y)[1 - e^{-y}]$, i.e., the Y_n are exponential variables. It is assumed that the variables $\{Y_n\}$ are independent of $x(t)$. Write $Z_n = Y_1 + Y_2 + \dots + Y_n$, for the n-th partial sum of the Y's. Then

$$\text{prob}\{x(Y_1) \leq x\} = \int_0^\infty e^{-y}\, \text{prob}\{x(y) \leq x\}\, dy$$

$$= F(x) , \quad \text{by (4).}$$

Consider the equation

$$x(Z_n) = \sum_{r=1}^{r=n} [x(Z_r) - x(Z_{r-1})] . \tag{5}$$

Since $x(t)$ is a process with stationary independent increments the n brackets in the summation in (5) enclose n independent and identically distributed random variables. For $[x(Z_r) - x(Z_{r-1})]$, say, is the change in the value of $x(t)$ during a period of time of length Y_r; thus all terms in the summation will have the same distribution function as $x(Y_1)$, namely $F(x)$. Hence $x(Z_n)$ will have $F_n(x)$ as its distribution function.

By a well-known property of the exponential distribution the random variable Z_n has the probability density function

$$\frac{e^{-x}\, x^{n-1}}{(n-1)!} , \quad \text{for } x \geq 0 .$$

Consequently we can infer that

$$\text{prob}\{x(Z_n) \leq x\} = \int_0^\infty \frac{e^{-y}\, y^{n-1}}{(n-1)!} F(x;y)dy .$$

Thus by equating the distribution functions we have obtained for the left- and right-hand sides of (5) we conclude that

$$F_n(x) = \int_0^\infty \frac{e^{-y}\, y^{n-1}}{(n-1)!}\; F(x;y)dy .$$

Hence, by a standard theorem from integration theory (Halmos [7], p. 112, Theorem B), we find that

$$H(x) = \sum_{n=1}^{\infty} F_n(x)$$

$$= \int_0^{\infty} \sum_{n=1}^{\infty} \frac{e^{-y} y^{n-1}}{(n-1)!} F(x;y)dy$$

$$= \int_0^{\infty} F(x;y)dy \quad .$$

This, by (3), proves $H(x) = \mathcal{E}\, T(x)$. The fact that $H(x)$ is finite and non-decreasing is well-known, since $H(x)$ is a conventional renewal function (see, e.g., Blackwell [8]).

We can call $F(x)$ the <u>associated distribution function</u>, and the random variables $\{X_n\}$ defined by $X_n = x(Z_n) - x(Z_{n-1})$ the <u>associated renewal process</u>.

The argument in which we have just indulged to prove Theorem 1 has glossed over a non-trivial point of rigor, namely the question of whether $x(Y_1)$, say, is a random variable. It can be shown fairly easily that this question has an affirmative answer, but as the necessary measure-theoretic elaborations would seem a trifle out of place in the present setting we shall not discuss this point further here; moreover Theorem 1 can equally well be proved without mentioning random variables, either by an argument using characteristic functions or by a somewhat tiresome manipulation of multiple integrals and Jacobians. However, we prefer the present demonstration as being more vivid.

Write

$$\mu_r = \int_{-\infty}^{+\infty} x^r dF(x) \; ,$$

and

$$m_r(t) = \int_{-\infty}^{+\infty} x^r \, d_x F(x;t) ,$$

whenever these moments are definable. Also write $m_r = m_r(1)$. Since, as is well-known, the cumulant generating function of $x(t)$ (given $x(0)=0$) is a direct multiple of t, the relations between the moments $\{m_r(t)\}$ and the moments $\{m_r\}$ are easy to determine. In particular, one finds that when the appropriate moments are finite

$$(6) \qquad \begin{aligned} m_1(t) &= m_1 t , \\ m_2(t) &= m_2 t + m_1^2 t(1-t) . \end{aligned}$$

Thus, by (4), we can find that

$$(7) \qquad \begin{aligned} \mu_1 &= m_1 \int_0^{\infty} e^{-t} \, t \, dt = m_1 \\ \mu_2 &= \int_0^{\infty} e^{-t} [m_2 t + m_1^2 t(1-t)] dt = m_2 + m_1^2 . \end{aligned}$$

Renewal processes may be <u>continuous</u> or <u>discrete</u> (see, e.g., [1]). For every theorem which is true for a continuous renewal process there is a closely analogous result for a discrete renewal process. <u>For the remainder of this section and for § 3 we shall assume that the associated renewal process is continuous</u>. Corresponding results for the case in which the associated renewal process is discrete may be proved on lines similar to those adopted here (by appealing to appropriate standard theorems for discrete, instead of continuous, renewal processes and by dealing with sums instead of integrals, etc.). We shall briefly discuss in § 4 the characterization of those stationary processes with independent increments which lead to discrete associated renewal processes.

Theorem 2. (i) For any $\alpha > 0$, as $x \longrightarrow \infty$

$$\mathcal{E}[T(x+\alpha) - T(x)] \longrightarrow \frac{\alpha}{m_1} \quad ,$$

where the limit is to be zero if $m_1 = +\infty$.

(ii) If $m_2 < \infty$, then as $x \longrightarrow +\infty$

$$\mathcal{E}\, T(x) = \frac{x}{m_1} + \frac{\sigma^2}{2m_1^2} + o(1) \quad ,$$

where $\sigma^2 = m_2 - m_1^2$.

Proof. Part (i) is simply a restatement of Blackwell's Theorem for the extended renewal process (Blackwell [8]). We have merely substituted $\mathcal{E}\, T(x)$ for $H(x)$ and, by (7), m_1 for μ_1.

Part (ii) follows immediately from the Second Renewal Theorem quoted in section 1 above. We have used (7) to express the result in terms of the moments $\{m_r\}$ instead of the moments $\{\mu_r\}$. However, the Second Renewal Theorem has nowhere been proved for the extended renewal process, so we must now prove this theorem.

3. Proof of the Second Renewal Theorem. Consider the function $Q(x)$ defined by

$$Q(x) = \mu_1 U(x) - \int_{-\infty}^{x} [U(y) - F(y)]dy \; . \tag{8}$$

Since

$$\int_0^\infty [1 - F(x)]dx = \int_0^\infty x dF(x)$$

and

$$\int_{-\infty}^0 F(x)dx = - \int_{-\infty}^0 x dF(x)$$

we can also write (8) as

$$(9)\quad Q(x) = \int_x^{\infty} [1 - F(y)]dy , \qquad \text{for } x \geq 0,$$

$$= - \int_{-\infty}^{x} F(y)dy \qquad , \qquad \text{for } x < 0.$$

Notice that, since we are supposing μ_2 to be finite, the first absolute moment of F must also be finite.

Let us write $F_+(x) = U(x)F(x)$, and μ_1^+ for the first moment of $F_+(x)$. Then for $x \geq 0$ we have, by (9), that

$$Q(x) = \mu_1^+ - \int_0^x [1 - F_+(y)]dy .$$

Then by the results of Smith [9] concerning 'derived distributions', or by direct computation, it follows that

$$\int_0^{\infty} Q(y)dy = \frac{1}{2} \int_0^{\infty} y^2 dF_+(y) .$$

A similar argument, coupled with this result, thus proves

<u>Lemma 1.</u> $\int_{-\infty}^{+\infty} Q(y)dy = \frac{1}{2}\mu_2$, <u>the integral converging absolutely.</u>

Given any $\epsilon > 0$ we can, by Lemma 1, find a large Δ so that

$$\int_{|y| \geq \Delta - 1} |Q(y)|dy < \epsilon .$$

Consider

$$I_+ = \int_{y-z \geq \Delta} Q(y-z)dH(z)$$

$$= \sum_{n=1}^{\infty} \int_{\Delta+n-1 \leq y-z < \Delta+n} Q(y-z)dH(z) .$$

Because $Q(\cdot)$ is non-increasing for positive values of its argument we therefore have

$$I_+ \leq \sum_{n=1}^{\infty} Q(\Delta + n - 1)[H(n + \Delta) - H(n + \Delta - 1)] .$$

But there is a finite β such that $H(x + 1) - H(x) < \beta$ for all x, for obviously $H(x + 1) - H(x) < 1 + H(1) - H(-1)$. Thus

$$I_+ \leq \beta \sum_{n=1}^{\infty} Q(\Delta + n - 1)$$

$$\leq \beta \int_{\Delta-1}^{\infty} Q(y)dy \quad ,$$

by the monotonicity of Q again. Hence I_+ can be made arbitrarily small for all y by choosing Δ large enough.

A similar argument can be applied to

$$I_- = \int_{y-z \leq \Delta} |Q(y-z)| dH(z) \quad ,$$

which, coupled with the result just obtained, shows that if

(10) $$I = \int_{|y-z| \geq \Delta} Q(y-z)dH(z)$$

then I can be made arbitrarily small, uniformly in y, by choosing Δ large enough.

Let $\mathcal{G}$ be the class of bounded step-functions each of which vanish outside some bounded interval (which may depend on the function) and have finitely many discontinuities in that interval. Then it is an easy deduction from Blackwell's Theorem for the extended renewal process that, as $x \longrightarrow +\infty$,

(11) $$\int_{-\infty}^{+\infty} G(x-z)dH(z) \longrightarrow \frac{1}{\mu_1} \int_{-\infty}^{+\infty} G(z)dz \ ,$$

whenever $G(\cdot) \in \mathcal{G}$. Since $Q(x)$ is monotone in $(-\infty, 0)$ and in $[0, +\infty)$ it is not difficult to see that for any $\epsilon > 0$ we can find $G^-(\cdot) \in \mathcal{G}$ and $G^+(\cdot) \in \mathcal{G}$ such that for all $|x| \leq \Delta$ we have

(12) $$G^-(x) \leq Q(x) \leq G^+(x) ,$$

and

(13) $$0 \leq \int_{|x|\leq\Delta} [G^+(x) - G^-(x)]dx \leq \epsilon .$$

Plainly

$$\int_{|x-z|\leq\Delta} G^-(x-z)dH(z) \leq \int_{|x-z|\leq\Delta} Q(x-z)dH(z)$$
$$\leq \int_{|x-z|\leq\Delta} G^+(x-z)dH(z) ,$$

and since (11) holds for all functions in $\mathcal{G}$ we discover that

$$\int_{|y|\leq\Delta} G^-(y)dy \leq \varliminf_{x=\infty} \int_{|x-z|\leq\Delta} Q(x-z)dH(z)$$
$$\leq \varlimsup_{x=\infty} \int_{|x-z|\leq\Delta} Q(x-z)dH(z)$$
$$\leq \int_{|y|\leq\Delta} G^+(y)dy .$$

This chain of inequalities in conjunction with (12), (13), and the arbitrariness of ϵ prove that as $x \longrightarrow \infty$

$$\int_{|x-z|\leq\Delta} Q(x-z)dH(z) \longrightarrow \int_{|y|\leq\Delta} Q(y)dy .$$

Because of Lemma 1 and of what we have already proved concerning the integral I of (10) we have thus established

<u>Lemma 2</u>. <u>As $x \longrightarrow +\infty$</u>,

$$\int_{-\infty}^{+\infty} Q(x-z)dH(z) = \frac{1}{2}\mu_2 + o(1) .$$

We shall presently make use of the elementary renewal theorem, which states that as $x \longrightarrow +\infty$, $H(x)/x \longrightarrow \mu_1^{-1}$. It will be found discussed in many places, in particular in [1]. The proof given in [1] can be adapted easily to cover the extended renewal process.

Since $\mu_2 < \infty$ it is an easy deduction that $z^2F(z) \longrightarrow 0$ as $z \longrightarrow -\infty$. Thus we can write

$$\int_{-\infty}^{x-z} F(y)dy = \frac{\rho(x-z)}{x-z}$$

where $\rho(x-z) \longrightarrow 0$ as $z \longrightarrow +\infty$, x fixed. Hence

$$\lim_{z=+\infty} H(z) \int_{-\infty}^{x-z} [U(y) - F(y)]dy = \lim_{z=+\infty} - H(z) \int_{-\infty}^{x-z} F(y)dy$$

$$= \lim_{z=+\infty} \frac{H(z)\rho(x-z)}{x-z} .$$

In view of the elementary renewal theorem we can thus conclude

(14) $$\lim_{z=+\infty} H(z) \int_{-\infty}^{x-z} [U(y) - F(y)]dy = 0 .$$

$H(z)$ must decrease to 0 as $z \longrightarrow -\infty$, or the strong law of large numbers would be contradicted. Thus

(15) $$\lim_{z=-\infty} H(z) \int_{-\infty}^{x-z} [U(y) - F(y)]dy = \lim_{z=-\infty} \mu_1 H(z) ,$$

$$= 0 ,$$

where we have used the fact that

$$\mu_1 = \int_{-\infty}^{+\infty} [U(y) - F(y)]dy$$

in the intermediate step.

If we now write

$$K = \int_{-\infty}^{+\infty} dH(z) \int_{-\infty}^{+\infty} [U(y) - F(y)]dy$$

and appeal to (14) and (15) then integration by parts yields

$$K = \int_{-\infty}^{+\infty} [U(x-z) - F(x-z)]H(z)dz$$

$$= \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} U(z-y)[U(x-z) - F(x-z)]dH(y) .$$

By Fubini's theorem, therefore,

$$K = \int_{-\infty}^{+\infty} dH(y) \int_{-\infty}^{+\infty} U(z-y)[(x-z) - F(x-z)]dz .$$

We change the variable of integration in the inner integral from z to u by putting $z = x + y - u$; this yields

$$K = \int_{-\infty}^{+\infty} dH(y) \int_{-\infty}^{+\infty} U(x-u)[U(u-y) - F(u-y)]du .$$

A further appeal to Fubini's Theorem then shows

$$K = \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} [U(u-y) - F(u-y)]dH(y) .$$

However, the integral equation of renewal theory (see, e.g., [1]) shows that the inner integral is simply F(u). Thus

(16) $$K = \int_{-\infty}^{x} F(u)du .$$

If we hark back to equation (8), which defines Q(•), we realize that (16) implies

Lemma 3.

$$\int_{-\infty}^{+\infty} Q(x-z)dH(z) = \mu_1 H(x) - \int_{-\infty}^{+\infty} F(u)du .$$

We are now in a position to finish our proof, for

$$\int_{-\infty}^{x} F(u)du = xU(x) - \mu_1 + o(1)$$

as $x \longrightarrow \infty$. Thus Lemma 2 and Lemma 3 jointly imply

$$H(x) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1 + o(1) ,$$

as $x \longrightarrow \infty$.

4. The renewal density theorem for infinitesimal processes. We prove first

Theorem 3. (i) If, (D1): $F(x) = pU(x) + q\hat{F}(x)$, where p and q are non-negative, $p + q = 1$, and $\hat{F}(x)$ is an absolutely continuous distribution function;

(D2): $\hat{f}(x)$ is a frequency function corresponding to $\hat{F}(x)$ and belongs to $L_{1+\delta}$ for some $\delta > 0$;

then $h(x) = H'(x)$ exists for all $x \neq 0$ and

(a) if $\hat{f}(x) \longrightarrow 0$ as $x \longrightarrow +\infty$, $h(x) \longrightarrow \mu_1^{-1}$ as $x \longrightarrow +\infty$;

(b) if $\hat{f}(x) \longrightarrow 0$ as $x \longrightarrow -\infty$, $h(x) \longrightarrow 0$ as $x \longrightarrow -\infty$.

(ii) A sufficient condition for $F(x)$ to satisfy (D1) is that

$$F(x,t) = p(t)U(x) + q(t)\hat{F}(x,t) , \tag{17}$$

where $p(\cdot)$ and $q(\cdot)$ are non-negative measurable functions such that $p(t) + q(t) = 1$, and $\hat{F}(x,t)$ is an absolutely continuous distribution function. In this case $p(t) = e^{-\gamma t}$ for some $\gamma > 0$.

(iii) <u>If $F(x,t)$ is of the form (17) then a sufficient condition for $F(x)$ to satisfy (D2) is that if we define</u>

$$I_{1+\delta}(t) = \int_{-\infty}^{+\infty} [\hat{f}(x,t)]^{1+\delta}\, dx$$

<u>then, for some $\delta > 0$, $e^{-t}\, I_{1+\delta}(t)$ shall be in $L_1(0, \infty)$.</u>

<u>Proof</u>. (i) If $F(x) = pU(x) + q\,\hat{F}(x)$ then, in an obvious notation, (1) shows that

$$H = \sum_{k=1}^{\infty} \sum_{r=1}^{k} \binom{k}{r} p^{k-r}\, q^r\, \hat{F}_r + \sum_{k=1}^{\infty} p^k\, U$$

$$= pq^{-1}U + \sum_{r=1}^{\infty} \left[q^r\, \hat{F}_r \sum_{s=0}^{\infty} \binom{r+s}{r} p^s \right]$$

$$= pq^{-1}U + \sum_{r=1}^{\infty} q^{-1}\, \hat{F}_r$$

i.e.

$$H(x) = pq^{-1}U(x) + q^{-1}\, \hat{H}(x) ,$$

so that for all $x \neq 0$

$$H'(x) = q^{-1}\, \hat{H}'(x) .$$

The renewal function $\hat{H}(x)$ is one to which the available renewal density theorem will apply (Smith [3]) and so 3(i) is proved. Notice that $p < 1$, since $F(x)$ is to have a strictly positive first moment.

(ii) If (17) holds then, by (4),

$$F(x) = U(x) \int_0^{\infty} e^{-t}\, p(t)dt + \int_0^{\infty} e^{-t}\, q(t)\, \hat{F}(x,t)dt . \tag{18}$$

But there will be a frequency function $\hat{f}(x,t)$ such that

$$\hat{F}(x,t) = \int_{-\infty}^{x} \hat{f}(z,t)dz ,$$

and so (18) leads, via Fubini's Theorem to

$$F(x) = U(x)\int_0^\infty e^{-t}\, p(t)dt + \int_{-\infty}^{x} dz \int_0^\infty e^{-t}\, q(t)\, \hat{f}(z,t)dt \; .$$

This shows F(x) must satisfy (D1).

If (17) holds, moreover, then prob $\{x(t_1 + t_2) = 0\}$ = prob $\{x(t_1) = 0\}$ x prob $\{x(t_1 + t_2) - x(t_1) = 0\}$, for any positive t_1, t_2. Thus $p(t_1 + t_2) = p(t_1)\, p(t_2)$, by the homogeneity of the x(t) process. Since p(t) must be monotone non-increasing, we must have $p(t) = e^{-\gamma t}$, as announced. The constant γ may not vanish for this would make F(x) = U(x) and then F(x) would have a zero mean value.

(iii) When (17) holds,

$$\hat{f}(x) = \int_0^\infty e^{-t}(1 - e^{-\gamma t})\, \hat{f}(x,t)dt \; .$$

Hölder's inequality then gives

$$[\hat{f}(x)]^{1+\delta} \leq \int_0^\infty e^{-t}(1 - e^{-\gamma t})[\hat{f}(x,t)]^{1+\delta}\, dt$$

so that, by Fubini's Theorem,

$$\int_{-\infty}^{+\infty} [\hat{f}(x)]^{1+\delta}\, dx \leq \int_0^\infty e^{-t}(1 - e^{-\gamma t}) I_{1+\delta}(t)dt \; .$$

This proves 3(iii).

<u>Corollary 3A.</u> <u>When F(x;t) has the form (17) then for all $x \neq 0$</u>

$$h(x) = \int_0^\infty (1 - e^{-\gamma t})\, \hat{f}(x,t)dt \; .$$

Notice that if F(x,t) is absolutely continuous for all t > 0, with a density function f(x,t) then (17) automatically holds (with p(t) = 0).

We can then take $\hat{f} = f$. The more "general" form (17) with $p(t) \neq 0$ can arise, however, when we have

$$F(x,t) = \sum_{n=0}^{\infty} \frac{e^{-\gamma t}(\gamma t)^n}{n!} A_n(x)$$

where $A(\cdot)$ is some distribution function and $A_n(\cdot)$ is its n-fold convollution.

It is desirable to obtain, if possible, conditions under which (D1) and (D2) are satisfied which relate to more intimate properties of the x(t) process. We shall now introduce the cumulant generating function $\psi(\theta)$ of x(1), i.e., we suppose

(19)
$$e^{i\theta x(t)} = e^{t\psi(\theta)} .$$

(Recall that $x(0) = 0$). Then since (19) is to be a characteristic function for all t we must have

(20)
$$\mathcal{R}\, \psi(\theta) \leq 0 , \quad \text{for all } \theta,$$

for otherwise the modulus of this characteristic function would exceed unity for large values of t.

If $\phi(\theta)$ is the characteristic function of F(x) then it follows from (4) and (19) that

(21)
$$\phi(\theta) = \int_0^\infty e^{-t+t\psi(\theta)}\, d\theta = \frac{1}{1-\psi(\theta)}$$

Theorem 4. (i) A necessary condition for F to satisfy (D1) is that there exist

$$\lim_{|\theta|=\infty} \psi(\theta) = \lambda \geq -\infty .$$

(ii) If λ is finite then $p = (1 - \lambda)^{-1} > 0$, and in this case a sufficient condition for F to satisfy (D1) and (D2) is that $[\psi\theta) - \lambda]$ be the Fourier Transform of a function in L_1 and $L_{1+\delta}$, for some $\delta > 0$.

(iii) If λ is infinite then $p = 0$, and in this case a sufficient condition for F to satisfy (D1) and (D2) is that for some $\epsilon > 0$

$$\underline{\lim_{|\theta|=\infty} \frac{|\psi(\theta)|}{|\theta|^{\frac{1}{2}+\epsilon}}} > 0 . \tag{22}$$

Proof. (i) If (D1) holds then $\phi(\theta) = p + q\,\hat{\phi}(\theta)$, where $\hat{\phi}(\theta)$ is the Fourier Transform of the density function $\hat{f}(x)$. Thus, by the Riemann-Lebesgue Lemma, $\phi(\theta) \longrightarrow p$ as $|\theta| \longrightarrow \infty$. Reference to (21) then proves 4(i).

(ii) From the previous remarks and (21) it is clear that if $\lambda > -\infty$ then $p = (1 - \lambda)^{-1} > 0$. From (21) it transpires that

$$q\,\hat{\phi}(\theta) = p\phi(\theta)[\psi(\theta) - \lambda] . \tag{23}$$

Suppose $g(x)$ is the function in L_1 and $L_{1+\delta}$ whose Fourier Transform is $[\psi(\theta) - \lambda]$. Then (23) implies that for almost all x

$$q\,\hat{f}(x) = p \int_{-\infty}^{+\infty} g(x-z)\, dF(z) ,$$

where we have appealed to the fact that $g \epsilon L_1$ in writing down the right-hand side. A familiar application of Hölder's inequality then gives

$$q^{1+\delta} \int_{-\infty}^{+\infty} [\hat{f}(x)]^{1+\delta}\, dx \leq p^{1+\delta} \int_{-\infty}^{+\infty} [g(x)]^{1+\delta} dx < \infty .$$

(iii) If (22) holds then for some $\eta > 0$ and all sufficiently large $|\theta|$

$$|\psi(\theta)| > \eta |\theta|^{1/2 + \epsilon} .$$

Thus, by (21), $\phi(\theta) = O(|\theta|^{-1/2-\epsilon})$. Hence $\phi(\theta) \in L_2$ and so, by Titchmarsh ([10], p. 6) there must exist a density f(x) which belongs to L_2.

It follows from the discussion of Lévy [6] that for the most general possible process with stationary independent increments one can write $\psi(\theta)$ in the form

$$
\begin{aligned}
(24) \qquad & i\theta\alpha - \frac{1}{2}\sigma^2\theta^2 + \lambda_1 \int_{-1-0}^{+1+0} \frac{e^{i\theta u} - 1 - i\theta u}{u^2} \, dN(u) \\
& + \lambda_2 \int_{|u|>1} (e^{i\theta u} - 1) dN(u) .
\end{aligned}
$$

In this expression σ^2, λ_1, λ_2 are arbitrary non-negative real numbers, α is an arbitrary real number, and $N(\cdot)$ is an arbitrary distribution function which is continuous at 0. The form (24) is not the most compact one given in the literature (see, e.g., Gnedenko and Kolmogorov [5]) but is more convenient for our present purposes. We shall say there is a drift α if $\alpha \neq 0$, and a Brownian component if $\sigma^2 > 0$.

If it turns out that the distribution function N in (24) is such that

$$\int_{-1}^{+1} \frac{1}{|u|} \, dN(u) < \infty$$

then $\psi(\theta)$ can be given the simpler and more convenient form

$$(25) \qquad i\theta\alpha - \frac{1}{2}\sigma^2\theta^2 + \lambda \int_{-\infty}^{+\infty} \frac{e^{i\theta u} - 1}{|u|} \, dM(u) ,$$

where, here, σ^2 and λ are arbitrary non-negative numbers, α is an arbitrary real number, and M is an arbitrary distribution function which is continuous at 0. We might call the third term in (25) the pure jump term.

Notice that if N is such that (24) can be thrown into the form (25) then the constant α will be modified in the conversion. This observation is relevant to Theorem 5(iii) below. Before we proceed to Theorem 5, however, the introduction of the canonical form for $\psi(\theta)$ provides us with an opportunity to give the brief discussion, promised in § 1, of conditions under which the associated renewal process will be discrete. For such a discrete process to arise there must be a real $\tilde{\omega} > 0$ such that, with probability one, $x(Y_1)$ is divisible by $\tilde{\omega}$. Plainly this requires that $\phi(2\pi/\tilde{\omega}) = 1$ and hence, by (21) that $\psi(2\pi/\tilde{\omega}) = 0$. The real part of (24) is, in general, strictly negative and can only vanish at $2\pi/\tilde{\omega}$ ($\neq 0$) if $\sigma^2 = 0$ and if the sole points of increase of $N(u)$ are where u is a positive or negative (but not zero) multiple of $\tilde{\omega}$. Thus $\psi(\theta)$ may be given the simpler form (25) with $\sigma^2 = 0$ and $M(u)$ the distribution function of a lattice variable. But Theorem 5(iii) below shows that we must have $\alpha = 0$, or else $\psi(\theta)$ would be the cumulant-generating function of an absolutely continuous distribution. We may therefore conclude that discrete renewal processes only arise if, for some $\tilde{\omega} > 0$ and some sequence $\{p_n\}$ such that $p_o = 0$ and $\Sigma_{-\infty}^{+\infty} p_n = 1$, we can write

$$\psi(\theta) = \lambda \sum_{n=-\infty}^{n=+\infty} \frac{e^{n\tilde{\omega}i\theta} - 1}{|n\tilde{\omega}|} p_n \quad .$$

<u>Theorem 5</u>. <u>Sufficient conditions for $F(\cdot)$ to satisfy (D1) and (D2) are that, in the expression (24) for $\psi(\theta)$, either</u>

(i) <u>$\sigma^2 > 0$, i.e., $x(t)$ has a Brownian component</u>

<u>or</u> (ii) <u>$\sigma^2 = 0$, but for some $\epsilon < 1/2$.</u>

$$\underline{\lim_{h=0+}} \frac{\lambda_1}{h^{1+\epsilon}} \int_{-h}^{+h} dN(u) > 0$$

<u>or</u> (iii) <u>$N(\cdot)$ is such that $\int_{-1}^{+1} (1/|u|)\, dN(u) < \infty$ and when $\psi(\Theta)$ is cast in the form (25) then $\alpha \neq 0$, i.e., there is a drift.</u>

Notice that sufficient conditions 5(ii) and 5(iii) might apply when x(t) is not absolutely continuous, in fact 5(iii) might apply when x(t) has a discrete distribution. Thus it is not necessary, apparently, for x(t) to be absolutely continuous before the infinitesimal renewal density theorem holds.

<u>Proof</u>. (i) Let G(x;t) be the distribution function of the process that would have the cumulant generating function (24), but with $\sigma^2 = 0$. Then, if F(x;t) corresponds to (24) as it stands, it is clear that F(x;t) is obtained by convoluting G(x;t) and a normal distribution function with zero mean and variance $\sigma^2 t$. Thus F(x;t) must be absolutely continuous, with a frequency function

$$f(x,t) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-z)^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}}\, dG(z,t) .$$

Thus, again by Hölder's inequality,

$$[f(x,t)]^2 \leq \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-z)^2}{\sigma^2 t}}}{2\sigma^2 \pi t}\, dG(z,t) .$$

A use of Fubini's Theorem yields, in the notation of Theorem 3(iii),

$$I_2(t) \leq \frac{1}{2\sigma\sqrt{\pi t}} .$$

Thus $e^{-t} I_2(t)$ is in L_1 and 5(i) follows from Theorem 3(iii).

(ii) If $\psi(\Theta)$ has the form (25), but with $\sigma^2 = 0$, then

$$\mathcal{R}\,\psi(\theta) = \lambda_1 \int_{-1-0}^{+1+0} \frac{\cos(\theta u) - 1}{u^2}\, dN(u)$$

$$+ \lambda_2 \int_{|u|>1} (\cos(\theta u) - 1)\, dN(u) ,$$

$$= J_1(\theta) + J_2(\theta), \text{ say.}$$

Trivially we have $|J_2(\theta)| \leq 2\lambda_2$. But

$$|J_1(\theta)| = 2\lambda_1 \int_{-1-0}^{+1+0} \frac{\sin^2(\frac{1}{2}u\theta)}{u^2}\, dN(u) .$$

Thus, for fixed $\eta > 0$, and $|\theta|$ sufficiently large

$$|J_1(\theta| > 2\lambda_1 \int_{-\eta/|\theta|}^{+\eta/|\theta|} \frac{\sin^2(\frac{1}{2}u\theta)}{u^2}\, dN(u)$$

$$> \frac{\lambda_1 \theta^2}{2} \int_{-\eta/|\theta|}^{+\eta/|\theta|} \frac{\sin^2(\frac{1}{2}u\theta)}{(\frac{1}{2}u\theta)^2}\, dN(u) .$$

If η is small enough $\sin^2 x/x^2 > \frac{1}{2}$ for all $|x| < \eta$. Thus

$$|J_1(\theta)| > \frac{\lambda_1 \theta^2}{4} \int_{-\eta/|\theta|}^{+\eta/|\theta|} dN(u) ,$$

and so

$$\underline{\lim}_{|\theta|=\infty} \frac{|J_1(\theta)|}{|\theta|^{1-\epsilon}} \geq \underline{\lim}_{|\theta|=\infty} \frac{\lambda_1 |(\theta)|^{1+\epsilon}}{4} \int_{-\eta/|\theta|}^{+\eta/|\theta|} dN(u)$$

$$> 0, \text{ by hypothesis.}$$

Since $\epsilon < \frac{1}{2}$ and $J_2(\theta)$ is bounded we can conclude that for some $\nu > 0$,

$$\underline{\lim}_{|\theta|=\infty} \frac{|\psi(\theta)|}{|\theta|^{\frac{1}{2}+\nu}} > 0 ,$$

for $|\psi(\theta)| \geq |\Re\,\psi(\theta)|$. An appeal to Theorem 4(iii) finishes the proof of 5(ii).

(iii) If $\psi(\theta)$ has the form (25) with $\alpha \neq 0$ then

$$\frac{\Im\,\psi(\theta)}{\theta} = \alpha + \lambda \int_{-\infty}^{+\infty} \frac{\sin(\theta u)}{\theta|u|}\, dM(u)\ . \tag{26}$$

If we can show that the integral on the right of (26) tends to zero as $|\theta| \longrightarrow \infty$ then it will appear that $|\Im\,\psi(\theta)| \sim |\alpha|\ |\theta|$ for large values of $|\theta|$. Since $|\psi(\theta)| \geq |\Im\,\psi(\theta)|$ we will then have

$$\frac{\lim}{|\theta|=\infty}\ \frac{|\psi(\theta)|}{|\theta|} > 0$$

and 5(iii) follows from Theorem 4(iii).

For any $\epsilon > 0$, by the Riemann-Lebesgue lemma,

$$\theta\, J_1 = \int_{|u|>\epsilon} \frac{\sin(\theta u)}{|u|}\, dM(u) \longrightarrow 0$$

as $|\theta| \longrightarrow \infty$. Thus J_1 approaches zero even more rapidly, and we need only consider

$$J_2 = \int_{|u|\leq\epsilon} \frac{\sin\,\theta u}{\theta|u|}\, dM(u)\ .$$

For all x, $|\sin x| \leq |x|$, so

$$|J_2| \leq \int_{|u|\leq\epsilon} dM(u)\ .$$

But M(u) is continuous at 0, so by choosing ϵ sufficiently small $|J_2|$ can be made arbitrarily small. This shows that the integral on the right of (26) tends to zero as $|\theta| \longrightarrow \infty$, and thereby completes the proof of 5(iii).

In connection with Theorem 5(ii) it is discouraging to note the following.

Theorem 6. If $w(\cdot)$ is any increasing function then there is a distribution function $N(\cdot)$ such that $N(0+) = 0$,

$$\int_0^1 \frac{1}{u}\, dN(u) = +\infty ,$$

but

$$\lim_{h=0+} w(\tfrac{1}{h}) \int_0^h dN(u) = 0 .$$

Proof. We have to show there is a strictly positive random variable X, say, such that $\mathcal{E} X^{-1} = +\infty$, but such that

$$\lim_{h=0+} w(\tfrac{1}{h}) \text{ prob}\left\{X \leq h\right\} = 0 .$$

Our task is made easier if rephrased in terms of $Y = X^{-1}$. We want to find a positive random variable Y with infinite expectation such that

$$\lim_{\ell=+\infty} w(\ell) \text{ prob}\left\{Y < \ell\right\} = 0 .$$

With no loss of generality we may suppose $w(x) > 1$ for $x > 1$. Let $\left\{\epsilon_n\right\}$ be any strictly decreasing sequence of strictly positive real numbers such that: $\epsilon_1 < 1$, and $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$. We define ξ_1, ξ_2, ..., and $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$, as follows:

$$\xi_1 = 1,$$

$$\xi_{n+1} = w(\zeta_n)/\epsilon_n, \quad \text{for } n > 1.$$

Thus

$$\xi_2 = \frac{w(\xi_1)}{\epsilon_1} = \frac{1}{\epsilon_1} > 1 = \xi_1$$

and for $n > 1$,

$$\xi_{n+1} > \frac{w(\zeta_{n-1})}{\epsilon_{n-1}} = \xi_n .$$

Therefore $\left\{\xi_n\right\}$ is a strictly increasing sequence and $\zeta_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

Define the function $G(x)$ as follows. In the interval $[\zeta_n, \zeta_{n+1})$, $n = 1,2,\ldots,$

$$1 - G(x) = \frac{\epsilon_n}{w(\zeta_n)} .$$

Then $G(x)$ increases to unity as x tends to infinity, and hence, if we define $G(x)$ to be zero for all $x < 1$, it will be a distribution function. However,

$$\int_{\zeta_n}^{\zeta_{n+1}} [1 - G(x)]dx = \frac{\epsilon_n}{w(\zeta_n)} \xi_{n+1} = 1$$

so that

$$\int_0^{\infty} [1 - G(x)]dx = \infty$$

and $G(x)$ therefore has an infinite mean value.

But

$$w(\zeta_n)[1 - G(\zeta_n)] = w(\zeta_n) \frac{\epsilon_n}{w(\zeta_n)} = \epsilon_n .$$

Since $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$ this implies that

$$\lim_{x=\infty} w(x) [1 - G(x)] = 0 ,$$

and the theorem is complete.

We close this section by drawing attention to a serious omission. We have been able to obtain some reasonably general conditions, in Theorems 4 and 5, which ensure that $F(x)$ satisfies (D1) and (D2). But to apply the density theorem (Theorem 3) we must also have that $\hat{f}(x)$ is vanishingly small at infinity. Indeed, this requirement is a <u>necessary</u> one for the validity of the density theorem. So far we have obtained no worthwhile results in this direction.

5. A special infinitesimal renewal process.

Consider the process given by the density function

$$f(x;t) = \frac{e^{-x}\, x^{t-1}}{\Gamma(t)}, \quad t > 0.$$

The associated cumulant generating function is

$$\psi(\theta) = -\log(1-i\theta)$$

so that, by (21),

$$\phi(\theta) = \frac{1}{1+\log(1-i\theta)} .$$

Plainly this characteristic function belongs to no class L_p, so there is no hope that the various conditions derived in the last section will be of any use in applying the density theorem to this special case.

However, by Corollary 3A,

$$h(x) = \frac{e^{-x}}{x}\int_0^\infty \frac{x^t}{\Gamma(t)}\, dt . \tag{27}$$

The integral on the right of (27) may be evaluated by a method similar to the one of steepest descents (see, e.g., Jeffreys and Jeffreys [11], p. 501 et seq.). However, the present case is not quite routine, so we shall give some details of the calculation.

Let A be a large positive constant, and write

$$h(x) = \left\{\int_0^{x-A\sqrt{x}} + \int_{x-A\sqrt{x}}^{x+A\sqrt{x}} + \int_{x+A\sqrt{x}}^\infty\right\} \frac{e^{-x}\, x^{t-1}}{\Gamma(t)}\, dt$$

$$= I_1 + I_2 + I_3, \text{ say} .$$

We first show that I_1 and I_3 are asymptotically negligible for large x, if A is large enough. If we use the inequality $\Gamma(t) > t^{t-\frac{1}{2}} e^{-t} \sqrt{(2\pi)}$ (see, e.g., Whittaker and Watson [12], p. 253) then we easily obtain

$$I_1 < \frac{1}{\sqrt{(2\pi x)}} \int_0^{x-A\sqrt{x}} e^{-(x-t)} \left(\frac{x}{t}\right)^t dt .$$

If we put u = x - t and use the further inequality

$$\left(\frac{x}{t}\right)^t < e^{u - \frac{u^2}{2x}}$$

which is valid for t < x, then we find

$$I_1 < \frac{1}{\sqrt{(2\pi x)}} \int_{A\sqrt{x}}^{x} e^{-\frac{u^2}{2x}} du .$$

Evidently I_1 can be made arbitrarily small by choosing A large enough.

Next we find, in a similar way, that

$$I_3 < \frac{1}{\sqrt{(2\pi x)}} \int_{x+A\sqrt{x}}^{\infty} e^{-(x-t)} \left(\frac{x}{t-\frac{1}{2}}\right)^{t-\frac{1}{2}} dt .$$

On putting $t - \frac{1}{2} = \tau$ and B = A - 1 we have, more tidily,

$$I_3 < \frac{e^{\frac{1}{2}}}{\sqrt{(2\pi x)}} \int_{x+B\sqrt{x}}^{\infty} e^{-(x-\tau)} \left(\frac{x}{\tau}\right)^{\tau} d\tau .$$

We now observe that for $\tau > x$ one has

$$\left(\frac{x}{\tau}\right)^{\tau} < e^{-(\tau-x) - \frac{(\tau-x)^2}{2\tau}} .$$

Thus

$$I_3 < \frac{e^{\frac{1}{2}}}{\sqrt{(2\pi x)}} \int_{x+B\sqrt{x}}^{\infty} e^{-\frac{(\tau-x)^2}{2\tau}} d\tau .$$

But in the range of integration

$$\frac{(\tau-x)^2}{2\tau} > \frac{1}{4} B^2 + \frac{B}{2\sqrt{x}} (\tau - x - B\sqrt{x}) .$$

Thus

$$I_3 < \frac{e^{\frac{1}{2} - \frac{1}{4}B^2}}{B} \sqrt{(\tfrac{2}{\pi})} \quad ,$$

and so I_3, also, will be arbitrarily small if A is sufficiently large.

Finally we consider I_2. For large x, in view of Stirling's asymptotic approximation to the gamma function, we can evidently write

$$I_2 \sim \int_{x-A\sqrt{x}}^{x+A\sqrt{x}} \frac{e^{-(x-t)}\, x^{t-1}}{t^{t-\frac{1}{2}}\sqrt{2\pi}}\, dt$$

$$\sim \frac{1}{\sqrt{(2\pi)}} \int_{x-A\sqrt{x}}^{x+A\sqrt{x}} e^{g(x,t)}\, dt$$

where $g(x,t) = t - x + (t-1)(\log x) - (t-\frac{1}{2})(\log t)$.

One finds that

$$\frac{\partial g}{\partial t} = \frac{1}{2t} + \log\left(\frac{x}{t}\right)$$

$$\frac{\partial^2 g}{\partial t^2} = -\frac{1}{t} - \frac{1}{2t^2}$$

$$\frac{\partial^3 g}{\partial t^3} = \frac{1}{t^2} + \frac{1}{t^3} \quad .$$

It is then simple to show that g has a unique maximum in the range of integration for I_2; it occurs at

$$t_o = x + \frac{1}{2} + O(\tfrac{1}{x}) \ .$$

Routine analysis will then show that

$$g(x, t_o) = -\frac{1}{2}\log x + O(\tfrac{\log x}{x})$$

$$\left.\frac{\partial^2 g}{\partial t^2}\right|_{t_o} = -\frac{1}{x} + O(\tfrac{1}{x^2})$$

and for all t in the range under consideration

$$\frac{\partial^3 g}{\partial t^3} = O(\frac{1}{x^2}) .$$

Thus, for all t in this range we have, by Taylor's Theorem,

$$g(x,t) = -\frac{1}{2}\log x - \frac{(t-x)^2}{2x} + O(\frac{1}{\sqrt{x}}) ;$$

in this expansion the final correction term is uniform in t. It follows from this last expansion that as x tends to infinity we have

$$I_2 \sim \int_{x-A\sqrt{x}}^{x+A\sqrt{x}} \frac{e^{-\frac{(t-x)^2}{2x}}}{\sqrt{(2\pi x)}}\, dt .$$

This final integral can be made arbitrarily near unity by choosing A large enough. Thus, as $x \longrightarrow \infty$, $h(x) \longrightarrow 1$ and the renewal density holds.

The density function

$$f(x) = \int_0^\infty \frac{e^{-t-x}\, x^{t-1}}{\Gamma(t)}\, dt$$

is therefore one which does not satisfy the conditions required by Smith [3], but it is yet one for which the density theorem is valid.

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