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ON THE MONOTONIC CHARACTER OF THE POWER FUNCTIONS OF TWO MULTIVARIATE TESTS

by

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The power function of the largest root test of normal multivariate linear hypothesis or of independence between two sets of variates involves, in each case, aside from the degrees of freedom, certain nonnegative, noncentrality parameters. This report supplies a relatively simple and elegant proof that the power function monotonically increases as each parameter, separately, increases - a result that was conjectured and proved (but not published) by one of the authors several years ago by a very lengthy and laborious method.

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ON THE MONOTONIC CHARACTER OF THE POWER FUNCTIONS OF TWO MULTIVARIATE TESTS[1]

By S. N. Roy and W. F. Mikhail

1. Summary. The largest characteristic root test for multivariate analysis of variance or independence between two sets of variates has in each case a number of well known properties [1,2,3] including, in particular, the following. (i) The power function involves as arguments, aside from the degrees of freedom and the level of significance, a set of non-negative non-centrality parameters that are statistically meaningful, (ii) the power functions has a lower bound which is a monotonically increasing function of each of these parameters, separately, and (iii) it is possible, by using the distribution of the test statistic under the null hypothesis alone, to obtain, with a confidence coefficient greater than or equal to a preassigned level, simultaneous confidence bounds on a set of parametric functions that might be interpreted as measures of departure, respectively from the total hypothesis or from partial hypothesis defined, for multivariate analysis of variance, by cutting out one or more variates and one or more factor levels, and for independence between a p-set and a q-set, by cutting out one or more of the p-set and one or more of the q-set.

It is the purpose of the present paper to prove that, for each test, the power function is a monotonically increasing function of each non-centrality (or deviation) parameter separately, a fact which was stated

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without proof in [1]. There the remark was made that the very long and tedious proof available to the author of [1] was not being offered in the hope that a far simpler and more elegant proof would be forthcoming in the near future. The kind of proof looked for at that stage is being offered here.

2. Preliminaries for the multivariate analysis of variance situation [1]. Let u, s and n-r denote respectively the "effective" number of variates, the degrees of freedom of the hypothesis and the degrees of freedom for the error and let $t = \min(u,s)$. Then with

$$(2.1)\quad \underset{u\times s}{X} = \begin{bmatrix} x_{11} & \cdots & x_{1s} \\ \cdot & \cdots & \cdot \\ x_{u1} & \cdots & x_{us} \end{bmatrix} \qquad \underset{u\times(n-r)}{Y} = \begin{bmatrix} y_{11} & \cdots & y_{1,n-r} \\ \cdot & \cdots & \cdot \\ y_{u1} & \cdots & y_{u,n-r} \end{bmatrix}$$

the canonical form for the elementary distribution is

$$(2.2)\quad [1/(2\pi)^{\frac{u(s+n-r)}{2}}] \exp [-\tfrac{1}{2}\{\sum_{i=1}^{u}\sum_{j=1}^{n-r} y_{ij}^2 + \sum_{i=1}^{t}(x_{ii} - \sqrt{\gamma_i})^2 + \sum_{i=t+1}^{u} x_{ii}^2 + \sum_{i=1}^{u}\sum_{j\neq i=1}^{s} x_{ij}^2\}] \prod_{i=1}^{u}\prod_{j=1}^{s} dx_{ij} \prod_{i=1}^{u}\prod_{j=1}^{n-r} dy_{ij};$$

and the acceptance region (at α) for the linear hypothesis H_o of analysis of variance (under which γ's=0) can be expressed as

$$(2.3)\quad \mathcal{D}: \qquad ch_{max} [(XX')(YY')^{-1}] \le \mu,$$

where μ is given by

$$(2.4)\qquad P[ch_{max}\{(XX')(YY')^{-1}\} \le \mu \mid \gamma\text{'s} = 0] = 1 - \alpha.$$

If we denote the domain (2.3) by $\mathcal{D}$, then with an obvious notation for the differential elements, the probability of the second kind of error can be written as

$$(2.5) \quad \int_{\mathcal{D}} \text{const. exp} \left[-\tfrac{1}{2} \left\{ \sum_{i,j} y_{ij}^2 + \sum_{i=1}^{t} (x_{ii} - \sqrt{\gamma_i})^2 + \sum_{i=t+1}^{u} x_{ii}^2 + \sum_{i=1}^{u} \sum_{j \neq i=1}^{s} x_{ij}^2 \right\} \right] dXdY.$$

The problem now is to prove that the integral (2.5) is a monotonically decreasing function of each γ_i, separately. If, now, we regard the domain $\mathcal{D}$ As one in an Euclidean space of dimensionality u s (for X) and u(n-r) (for Y), then it is clear that we can rewrite (2.5) as

$$(2.6) \quad \int_{\mathcal{D}^*} \text{const. exp} \left[-\tfrac{1}{2} \left\{ \sum_{i=1}^{u} \sum_{j=1}^{n-r} y_{ij}^2 + \sum_{i=1}^{u} \sum_{j=1}^{s} x_{ij}^2 \right\} \right] dXdY,$$

where $\mathcal{D}^*$ is merely the domain $\mathcal{D}$ translated by $\sqrt{\gamma_i}$ along x_{ii}, that is, along the ii-th axis (with i=1,2,...,t). Notice that if, in the integral (2.6), we replace the domain $\mathcal{D}^*$ by $\mathcal{D}$, integral over the new domain becomes equal to $1-\alpha$, where α is the probability of the first kind of error. It is useful now to put

$$(2.7) \quad Y\,Y' = (\tilde{V}'\tilde{V})^{-1},$$

where $\tilde{V}$ is a uxu triangular matrix with zeroes above the diagonal, observe that

$$\text{ch}_{\max} \left[(XX')(YY')^{-1} \right] = \text{ch}_{\max} \left[XX')(\tilde{V}'\tilde{V}) \right]$$
$$= \text{ch}_{\max} \left[\tilde{V}X)(\tilde{V}X)' \right],$$

where

$$(2.8)\quad \tilde{V}X = \begin{bmatrix} v_{11} & 0 & \cdots & 0 \\ v_{21} & v_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ v_{u1} & v_{u2} & \cdots & v_{uu} \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1s} \\ \cdot & \cdots & \cdot \\ x_{u1} & \cdots & x_{us} \end{bmatrix}$$

and rewrite (2.3), that is, the domain $\mathcal{D}$ as

$$(2.9)\quad \mathcal{D}: \quad ch_{max} [\tilde{V}X(\tilde{V}X)'] \leq \mu .$$

Notice that the domain $\mathcal{D}^*$ is a shift of $\mathcal{D}$ by $\sqrt{\gamma_i}$ along x_{ii} with $i=1,2,\ldots,t$. The problem now can be rephrased in the following way. How does the integral

$$(2.10)\quad \int_{\mathcal{D}} \exp \left[-\tfrac{1}{2}\left\{\sum_{i=1}^{u}\sum_{j=1}^{n-r} y_{ij}^2 + \sum_{i=1}^{u}\sum_{j=1}^{s} x_{ij}^2\right\}\right] dX dY$$

over the domain $\mathcal{D}$ given by (2.9) change under successive translations of $\sqrt{\gamma_1}$ along x_{11}, of $\sqrt{\gamma_2}$ along x_{22}, ..., $\sqrt{\gamma_t}$ along x_{tt}? It is clear that the successive changes are cumulative and that at any stage, say the i-th, the change depends on $\sqrt{\gamma_i}$. It will be also seen from the mechanics of the demonstration that if we can prove that the integral decreases for the first shift of $\mathcal{D}$, namely, by $\sqrt{\gamma_1}$ along x_{11}, then the general theorem itself will be proved.

3. <u>Proof of the monotonicity property for the multivariate analysis of variance situation</u>. The proof is developed in three main steps discussed in the following subsections.

3.1 <u>The proof for the univariate case</u>. In this case, u=1 and we can drop the first subscript in X, Y. The domain $\mathcal{D}$ of (2.9) now takes on the form

$$(3.1.1)\quad \mathcal{D}:\qquad \sum_{j=1}^{s} x_j^2 \leq \mu \sum_{j=1}^{n-r} y_j^2 ,$$

and the integral (2.10) the form

$$(3.1.2)\qquad \int_{\mathcal{D}} \exp\left[-\tfrac{1}{2}\left(\sum_{j=1}^{n-r} y_j^2 + \sum_{j=1}^{s} x_j^2\right)\right] \prod_{j=1}^{n-r} dy_j \prod_{j=1}^{s} dx_j .$$

Notice that now $\mathcal{D}^*$ is just a shift of $\mathcal{D}$ along x_1 by $\sqrt{\gamma}$. It is evident from the form of (3.1.2) that the integral (3.1.2) decreases under this shift if, for any given set of y_j's ($j=1,2,\ldots, n-r$) and x_j's($j=2,3,\ldots,s$),

$$(3.1.3)\qquad \int_{-a+\sqrt{\gamma}}^{a+\sqrt{\gamma}} \exp\left[-\tfrac{1}{2}x_1^2\right] dx_1 < \int_{-a}^{a} \exp\left[-\tfrac{1}{2}x_1^2\right] dx_1 ,$$

where a is the positive square root of $\mu \sum_{j=2}^{s} x_j^2 / \sum_{j=1}^{n-r} y_j^2$; then it is clear that it doesn't matter whether we take $\sqrt{\gamma}$ to be positive or negative. It is easy to varify an even more general result than this, namely that

$$(3.1.4)\qquad \int_{-a+\lambda}^{a+\lambda} \phi(x)\, dx < \int_{-a}^{a} \phi(x)\, dx ,$$

where a is positive and λ might be taken to be either positive or negative and $\phi(x)$ is a continuous function of x, symetrical about 0 and monotonically decreasing with $|x|$. It is also clear from the nature of the proof that the left side of (3.1.3) steadily decreases with $|\sqrt{\gamma}|$.

3.2. <u>The nature of the multivariate domain (2.9)</u>. We go back now to the multivariate case and to the domain of (2.9). Let us investigate the nature of the domain in $x_{11}, x_{21}, \ldots, x_{u1}$ for a given set of values of μ, Y (that is, $\tilde{V}$) and of the elements of the matrix X except those in the first column. Toward this end, put $X^* = \tilde{V}X$ and observe that, if ν is any characteristic root of $X^*X^{*\prime}$, then denoting the matrix by $[s^*_{ij}]$, we have that

$$(3.2.1) \qquad \begin{vmatrix} s^*_{11}-\nu & s^*_{12} & \cdots & s^*_{1u} \\ s^*_{12} & s^*_{22}-\nu & \cdots & s^*_{2u} \\ \cdot & \cdot & \cdots & \cdot \\ s^*_{1u} & s^*_{2u} & \cdots & s^*_{uu}-\nu \end{vmatrix} = 0$$

or $|s^*_{ij}| - \nu$ (sum of the p-1 rowed principal minors of $[s^*_{ij}]$) + ν^2 x (sum of the p-2 rowed principal minors of $[s^*_{ij}]$) - ... + $(-1)^u\nu^u = 0$

But $|s^*_{ij}| = \left| \begin{bmatrix} x^*_{11} & \cdots & x^*_{1s} \\ \cdot & \cdots & \cdot \\ x^*_{u1} & \cdots & x^*_{us} \end{bmatrix} \begin{bmatrix} x^*_{11} & \cdots & x^*_{u1} \\ \cdot & \cdots & \cdot \\ x^*_{1s} & \cdots & x^*_{us} \end{bmatrix} \right|$, which, given x^*_{ij} s(i=1,2,...,u; j=2,...,s), is easily seen to be a homogeneous quadric function of $(x^*_{11}, \ldots, x^*_{u1})$ + a constant which is really a function of the other x^*_{ij}'s just mentioned. The coefficients of the quadric function are each polynomial functions of x^*_{ij}'s (i=1,2,...,u; j=2,3,...,s). Likewise, if we take any q-rowed principal minor of $[s^*_{ij}]$, say the one with rows and columns numbered (1,2,...,q), then that minor

$$= \left| \begin{bmatrix} x^*_{11} & \cdots & x^*_{1s} \\ \cdot & \cdots & \cdot \\ x^*_{q1} & \cdots & x^*_{qs} \end{bmatrix} \begin{bmatrix} x^*_{11} & \cdots & x^*_{q1} \\ \cdot & \cdots & \cdot \\ x^*_{1s} & \cdots & x^*_{qs} \end{bmatrix} \right| ,$$

which, given x^*_{ij}'s (i=1,2,...,q; j=2,...,s), is a homogeneous quadradic function of $(x^*_{11}, \ldots, x^*_{q1})$(in which the coefficients are polonomials in x^*_{ij}'s, with i=1,2,...,q and j=2,...,s) + a constant which is really a function of the other x^*_{ij}'s just mentioned. Thus, given ν and x^*_{ij}'s $(i=1,2,\ldots,u;\ j=2,\ldots,s)$, the equation (3.2.1) in ν, yields a homogeneous quadric surface in $x^*_{11}, \ldots, x^*_{u1}$. Now recall from (2.8) that, given y_{ij}'s, that is $\tilde{V}$, the $(x^*_{11}, \ldots, x^*_{u1})$ are linear functions of $(x_{11},\ldots,x_{u1})$ and likewise $(x^*_{1j},\ldots,x^*_{uj})$ are linear functions of $(x_{1j}, \ldots, x_{uj})$(with $j=2,\ldots,s$). Thus, given ν, x_{ij}'s $(i=1,\ldots,u;\ j=2,\ldots,s)$, the equation (3.2.1) yields a homogeneous quadric surface in $(x_{11},\ldots,x_{u1})$ in which the coefficients and the constant term are all functions of ν, Y and the other x_{ij}'s already referred to. This is for any characteristic root ν.

Keeping this result in mind, let us examine more closely the nature of the domain (2.9) which is the same as the one given by (2.3). Let us rewrite (2.3) in the equivalent form

$$(3.2.2)\quad \operatorname{Sup}_{\underline{a}} \frac{(x_{11}+a_2x_{21}+\ldots a_ux_{u1})^2 + \ldots + (x_{1s}+a_2x_{2s}+\ldots a_ux_{us})^2}{(y_{11}+a_2y_{21}+\ldots a_uy_{u1})^2 + \ldots + (y_{1,n-r}+a_2y_{2,n-r}+\ldots a_uy_{u,n-r})^2} \leq \mu,$$

where $\underline{a}'$ stands for the (u-1)-dimensional vector $(a_2,\ldots,a_u)$. Now, given μ, Y and x_{ij}'s $(i=1,\ldots,u;\ j=2,\ldots,s)$, (3.2.2) represents the domain for $(x_{11},\ldots,x_{u1})$ in an u-dimensional Euclidean space, the boundary being given by the sur⸱ ce defined by the equality sign. An equivalent form of the same surface is the homogeneous quadric associated with (3.2.1) after ν is replaced by μ. Next, (3.2.2) tells us the following:

(i) if a particular point $(x_{11},\ldots,x_{u1})$ belongs to the domain just mentioned, then $c(x_{11},\ldots,x_{u1})$, where $0 < c \leq 1$ also belongs to the domain;

(ii) $(0,\ldots,0)$ belongs to the domain;

(iii) radiating from $(0,\ldots,)$ along any given direction there is only a finite length belonging to the domain.

Thus, given μ,Y and the other x_{ij}'s (already described), (2.3) or (2.9) represents a domain for $(x_{11},\ldots,x_{u1})$ which is the interior of an u-dimensional ellipsoid whose boundary is given by (3.2.1) after μ is susbstituted for ν. It is well known that there is an orthogonal transformation by which the ellipsoid can be referred to principal axes, or in other words, the transformed equation to the surface becomes free from the product terms in the transformed variables and involves only the square terms with positive coefficients. Let $\underset{1\times u}{x_1'} = [x_{11},\ldots,x_{u1}]$ and let

$$(3.2.3) \qquad \underset{u\times 1}{z} = \underset{u\times u}{L} \; \underset{u\times 1}{x_1} \; ,$$

where L is the orthogonal matrix that transforms the ellipsoid into principal axes. This L can be determined and the rows of L, say $(\ell_{i1},\ldots,\ell_{ij})$ $(i=1,2,\ldots,u)$ are the direction cosines of the different principal axes. Note that $\underline{z}'\underline{z} = \underline{x}_1'\underline{x}_1$. It would be useful to rewrite (2.6), after substitution of $\mathcal{D}$ for $\mathcal{D}^*$ and omission of the constant, in the form

$$(3.2.4) \qquad \int_{\mathcal{D}} \exp \left[-\tfrac{1}{2}\left\{\sum_{i=1}^{u}\sum_{j=1}^{n-r} y_{ij}^2 + \sum_{i=1}^{u}\sum_{j=2}^{s} x_{ij}^2 + \sum_{i=1}^{u} z_i^2\right\}\right] \times dY \prod_{i=1}^{u}\prod_{j=2}^{s} dx_{ij} \prod_{i=1}^{u} dz_i \; ,$$

where, given μ, Y and the x_{ij}'s, the domain $\mathcal{D}$, as a domain in $(z_1,\ldots,z_u)$, forms the interior of an ellipsoid referred to principal axes (that is, in a form which is free from the product terms of z's and invovles only the square terms with positive coefficients. In other words, $\mathcal{D}$ is symmetrical about the origin in each z_i separately. A displacement $\sqrt{\gamma_1}$ along the dirrection of x_{11} might be regarded as the resultant of a displacement

$\ell_{11}\sqrt{\gamma_1}$ along z_1, that is, along the direction with cosines $(\ell_{11},\ell_{12},\ldots,\ell_{1u})$ a displacement $\ell_{21}\sqrt{\gamma_1}$ along z_2, that is, along the direction with cosines $(\ell_{21}, \ell_{22}, \ldots, \ell_{2u})$, and so on, and finally a displacement $\ell_{u1}\sqrt{\gamma_1}$ along z_u, that is, along the direction with cosines $(\ell_{u1},\ldots,\ell_{uu})$. It should be remembered that these ℓ_{ij}'s are functions of μ, Y and the x_{ij}'s of (3.2.4)

3.3 <u>The final step in the proof of the monotonicity property</u>. Looking at (3.2.4) and using (3.1.4) we observe that a displacement of $\mathcal{D}$ by $\ell_{11}\sqrt{\gamma_1}$ aong z_1 will decrease the integral under (3.2.4), because, for any given set μ, Y, x_{ij}'s and $z_2, z_3, \ldots, z_u$,

$$(3.3.1)\quad \int_{-a+\ell_{11}\sqrt{\gamma_1}}^{a+\ell_{11}\sqrt{\gamma_1}} \exp\left[-\tfrac{1}{2} z_1^2\right] dz_1 < \int_{-a}^{a} \exp\left[-\tfrac{1}{2} z_1^2\right] dz_1,$$

where a and $\ell_{11}\sqrt{\gamma_1}$, without any loss of generality, can be assumed to be positive, a being a function of μ, Y, x_{ij}'s and $z_2,\ldots,z_u$. Using the same argument for successive displacements by ℓ_{21} along z_2, by $\ell_{31}\sqrt{\gamma_1}$ along z_3, and so on, and finally by $\ell_{u1}\sqrt{\gamma_1}$ along z_u we have a successive decrease of the integral. In other words, the resultant displacement which is along x_{11} and by $\sqrt{\gamma_1}$ decreases the integral. At this point we go back to (2.10), forget about the z_i's, use the result just stated about a displacement by $\sqrt{\gamma_1}$ along x_{11}, apply successive displacements by $\sqrt{\gamma_2}$ along x_{22}, $\sqrt{\gamma_3}$ along x_{33} and so on, and finally $\sqrt{\gamma_t}$ along x_{tt} and eventually obtain an integral over the displaced domain $\mathcal{D}^*$ which is less than the one over the original domain $\mathcal{D}$. It is also clear from the mechanics of the proof that the integral over $\mathcal{D}^*$ decreases as each $\left|\sqrt{\gamma_i}\right|$ $(i=1,2,\ldots,t)$ increases separately. This proves the monotonicity property.

4. The case of the test for independence between two sets of variates.

With a (p+q) set ($p \leq q$) of variables let us assume, for a sample of size n+1 (>p+q), the canonical distribution law [1]

$$(4.1)\quad \left[1/(2\pi)^{\frac{(p+q)n}{2}} \prod_{i=1}^{p} (1-\rho_i^2)^{\frac{n}{2}}\right] \exp\left[-\tfrac{1}{2}\left\{ \sum_{i=1}^{p} \frac{1}{1-\rho_i^2} \sum_{j=1}^{n} (x_{ij}^2+y_{ij}^2-2\rho_i x_{ij}y_{ij}) + \sum_{i=p+1}^{q} \sum_{j=1}^{n} y_{ij}^2 \right\}\right] \prod_{i=1}^{p} \prod_{j=1}^{n} dx_{ij} \prod_{i=1}^{q} \prod_{j=1}^{n} dy_{ij},$$

where ρ_i's are the population canonical correlation coefficients, the hypothesis of independence H_o is equivalent to the hypothesis that ρ_i's = 0, the acceptance region for H_o is

$$(4.2)\ \mathcal{D} : \ ch_{max} \left[(XX')^{-1}(XY')(YY')^{-1}(YX')\right] \leq \mu,$$

μ is given by

$$P\left[ch_{max} \leq \mu \mid H_o\right] = 1 - \alpha,$$

and X and Y are given by

$$\underset{p\times n}{X} = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \cdot & \cdots & \cdot \\ x_{p1} & \cdots & x_{pn} \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \cdot & \cdots & \cdot \\ y_{q1} & \cdots & y_{qn} \end{bmatrix} .$$

It would be profitable to reduce still further both the canonical distribution law (4.1) and the domain $\mathcal{D}$ of (4.2). Toward this end put

$$(4.3)\qquad \underset{p\times q}{Y} = \underset{q\times q}{\tilde{T}} \ \underset{q\times n}{L} \ ,$$

where $\tilde{T}$ is a triangular matrix of the type already described after (2.7) and L is an orthonormal matrix or in other words, $LL' = I\ (q)$. Next complete L (qxn) into an orthogonal matrix

$$(4.4)\qquad \underset{n}{\begin{bmatrix} L \\ M \end{bmatrix}}\begin{matrix} q \\ n-q \end{matrix}, \text{ so that } \begin{bmatrix} L \\ M \end{bmatrix}\begin{bmatrix} L' & \vdots & M' \end{bmatrix} = \begin{bmatrix} L' & \vdots & M' \end{bmatrix}\begin{bmatrix} L \\ M \end{bmatrix} = I(n).$$

Now put

$$(4.5)\qquad \underset{p\times n}{X^*} = \underset{p\times n}{X}\begin{bmatrix} \underset{q}{L'} & \vdots & \underset{n-q}{M'} \end{bmatrix} n$$

$$= \begin{bmatrix} x^*_{11} & \cdots & x^*_{1q} & x^*_{1,q+1} & \cdots & x^*_{1n} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ x^*_{p1} & \cdots & x^*_{pq} & x^*_{p,q+1} & \cdots & x^*_{pn} \end{bmatrix} \text{ (say)}$$

$$= \begin{bmatrix} \underset{q}{U^*} & \vdots & \underset{n-q}{V^*} \end{bmatrix} p \text{ (say)}.$$

Thus

$$X = \begin{bmatrix} U^* & \vdots & V^* \end{bmatrix}\begin{bmatrix} L \\ M \end{bmatrix} = U^*L + V^*M.$$

The Jacobian of the transformation given by (4.3) and the first line of (4.5) is [1]

$$(4.6)\qquad 2\prod_{i=1}^{q} t_{ii}^{n-i} \Big/ \left| \frac{\partial(LL')}{\partial(L_D)} \right|_{L_I},$$

where t_{ij}'s are the elements of $\tilde{T}$ and L_D and L_I are as in [1]. Thus, under this transformation, we have

$$(4.7) \qquad dXdY \longrightarrow \text{const.} \prod_{i=1}^{q} t_{ii}^{n-i}\, d\tilde{T}\, \frac{dL_I}{\left|\frac{\partial(LL')}{\partial(L_D)}\right|_{L_I}}\, dU^*dV^*,$$

Next observe that

$$(4.8) \qquad \sum_{i,j} x_{ij}^2 = \text{tr}\, XX' = \text{tr}\,(U^*U^{*\prime}+V^*V^{*\prime})$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} u_{ij}^{*2} + \sum_{i=1}^{p} \sum_{j=q+1}^{n} V_{ij}^{*2}\ ;$$

$$\sum_{i=1}^{p} \sum_{j=1}^{n} y_{ij}^2 = \sum_{i=1}^{p} \sum_{j=1}^{i} t_{ij}^2\ , \quad \sum_{i=p+1}^{q} \sum_{j=1}^{n} y_{ij}^2 = \sum_{i=p+1}^{q} \sum_{j=1}^{i} t_{ij}^2;$$

$\sum_{j=1}^{n} y_{ij}\, x_{ij}$ = the ii-th element of $X\,Y'$= the ii-th element of

$(U^*L+V^*M)L'\tilde{T}'$

$= \sum_{j=1}^{i} u^*_{ij}\, t_{ij}.$

We have now, for $\tilde{T}$, L_I, U^* and V^*, the distribution

$$\frac{\text{const.}}{\prod_{i=1}^{p}(1-\rho_i^2)^{\frac{n}{2}}} \exp\Big[-\tfrac{1}{2}\Big\{\sum_{i=1}^{p} \frac{1}{1-\rho_i^2}\Big(\sum_{j=1}^{q} u_{ij}^{*2} + \sum_{j=1}^{i} t_{ij}^2 - 2\rho_i \sum_{j=1}^{i} u_{ij}^* t_{ij}\Big)$$

$$(4.9) \qquad + \sum_{i=1}^{p} \frac{1}{1-\rho_i^2} \sum_{j=q+1}^{n} v_{ij}^{*2} + \sum_{i=p+1}^{q} \sum_{j=1}^{i} t_{ij}^2\Big\}\Big]\, dU^*dV^*$$

$$\times \prod_{i=1}^{q} t_{ii}^{n-i}\, d\tilde{T}\, dL_I \Big/ \left|\frac{\partial(LL')}{\partial(L_D)}\right|_{L_I} .$$

To express the domain $\mathcal{D}$ of (4.2) in terms of the transformed variables we observe that

$$\mathrm{ch}_{\max}\, [(XX')^{-1}(XY')(YY')^{-1}(YX')]$$
$$= \mathrm{ch}_{\max}\, [(X^*X^{*\prime})^{-1}\, X^* \binom{L}{M} L'\, \tilde{T}'\, (\tilde{T}\tilde{T}')^{-1}\, \tilde{T}L(L' \vdots M')\, X^{*\prime}]$$
$$= \mathrm{ch}_{\max}\, [(X^*X^{*\prime})^{-1}\, X^* \binom{I(q)}{0} (I(q) \vdots 0)\, X^{*\prime}]$$
$$= \mathrm{ch}_{\max}\, [(U^*U^{*\prime}+V^*V^{*\prime})^{-1}(U^*U^{*\prime})].$$

Hence we can rewrite (4.2) as

$$(4.10) \quad \mathcal{D}: \quad \mathrm{ch}_{\max}\, [(U^*U^{*\prime})\,(V^*V^{*\prime})^{-1}] \leq \frac{\mu}{1-\mu}\,.$$

Starting from (4.9) we can also integrate out over L_I [1] and have, for $\tilde{T}$, U^* and V^* the distribution

$$(4.11) \quad \frac{\text{const}}{\prod_{i=1}^{p} (1-\rho_i^2)^{\frac{n}{2}}} \exp \Big[-\tfrac{1}{2}\Big\{ \sum_{i=1}^{p} \frac{1}{1-\rho_i^2} \sum_{j=1}^{q} (u^*_{ij} - \rho^*_{ij} t_{ij})^2 + \sum_{i=1}^{q} \sum_{j=1}^{i} t^2_{ij}$$
$$+ \frac{1}{1-\rho_i^2} \sum_{i=1}^{p} \sum_{j=q+1}^{n} v^{*2}_{ij} \Big\}\Big]\ dU^*dV^* \prod_{i=1}^{q} t_{ii}^{n-i}\, d\tilde{T}\,,$$

where $\rho^*_{ij} = \rho_i$, for $j=1,2,\ldots,i$ and $i=1,2,\ldots,p$; and $= 0$, otherwise.

At this stage set

$$(4.12) \quad \underset{p\times q}{U} = \underset{p\times p}{D_1} / \sqrt{1-\rho_i^2}\ \underset{p\times q}{U^*}\,, \quad \underset{p\times(n-q)}{V} = \underset{p\times p}{D_1} / \sqrt{1-\rho_i^2}\ \underset{p\times(n-q)}{V^*}\,,$$

$$\rho^*_{ij} / \sqrt{1-\rho_i^2} = \gamma_{ij}\,,$$

where $\gamma_{ij} = \rho_i / \sqrt{1-\rho_i^2} = \gamma_i$ (say), for $j=1,2,\ldots,i$ and $i=1,2,\ldots,p$;
and $= 0$, otherwise.

Next observe that under this transformation [1] the characteristic roots of the matrix in (4.10) stay invariant, so that we can rewrite (4.10) as

(4.13) $\mathcal{D}$: $ch_{max} [(UU')(VV')^{-1}] \leq \frac{\mu}{1-\mu}$, ie, $\leq \mu^*$ (say).

We have now for U, V and $\tilde{T}$, the distribution

$$(4.14) \qquad \text{const. } \exp \left[-\tfrac{1}{2} \left(\sum_{i=1}^{p} \sum_{j=1}^{q} (u_{ij}-\gamma_{ij}t_{ij})^2 + \sum_{i=1}^{q} \sum_{j=1}^{i} t_{ij}^2 + \sum_{i=1}^{p} \sum_{j=q+1}^{n} v_{ij}^2\right)\right] \times dU\, dV \prod_{i=1}^{q} t_{ii}^{n-i}\, d\tilde{T}.$$

The probability of the second kind of error is given by integrating (4.14) over the domain (4.13). It is easy to see that, aside from the positive constant factor, this is equivalent to

$$(4.15) \qquad \int_{\mathcal{D}^*} \exp \left[-\tfrac{1}{2}\left(\sum_{i=1}^{q} \sum_{j=1}^{i} t_{ij}^2 + \sum_{i=1}^{p} \sum_{j=1}^{q} u_{ij}^2 + \sum_{i=1}^{p} \sum_{j=q+1}^{n} v_{ij}^2\right)\right] dU dV \times \prod_{i=1}^{q} t_{ii}^{n-i}\, d\tilde{T},$$

where, for any given set of $\tilde{T}$ and V, $\mathcal{D}$ * is just $\mathcal{D}$ displaced by $\gamma_1 t_{11}$ along u_{11}, by $\gamma_2 t_{21}$ along u_{21} and $\gamma_2 t_{22}$ along u_{22}, and so on, and finally by $\gamma_p t_{p1}$ along u_{p1}, $\gamma_p t_{p2}$ along $u_{p2}, \ldots, \gamma_p t_{pp}$ along u_{pp}. Notice that when H_o is true, that is, when all γ_i's = 0, we should have $\mathcal{D}$ * replaced by $\mathcal{D}$ in the integral (4.15). Using the same kind of argument as in section 3 it follows that, for any given $\tilde{T}$, the integral

$$(4.16) \qquad \int_{\mathcal{D}} \exp \left[-\tfrac{1}{2}\left(\sum_{i=1}^{p} \sum_{j=1}^{q} u_{ij}^2 + \sum_{i=1}^{p} \sum_{j=q+1}^{n} v_{ij}^2\right)\right] dU dV$$

decreases as $\mathcal{D}$ is displaced by $\gamma_1 t_{11}$ along u_{11} and we can continue using the same reasoning for the other displacements. Finally, introducing the density function of $\tilde{T}$ and going back to (4.15), it is easy to see from considerations of symmetry that the integral (4.15) monotonically decreases as each $|\gamma_i|$, that is each $|\rho_i|$ separately increases. The proves the monotonicity property of the power function of the test for independence between two sets of variates.

Concluding remarks. The power functions of the λ-criteria for the multivariate linear hypothesis and for the test of independence between two sets of variates have also somewhat similar monotonicity properties that will be discussed in a subsequent paper.

References

[1] S. N. Roy, "Some aspects of multivariate analysis," John Wiley and Sons (1958), Chapters 6, 10, 11, 14, and Chapters 4, 6 and 7 of the appendix.

[2] S. N. Roy and R. Gnanadesikan, "Further contributions to multivariate confidence bounds", Biometrika, Vol. 44 (1957), pp. 399-410.

[3] S. N. Roy and R. Gnanadesikan, "Some contributions to ANOVA in one or more discussions - Parts I and II," Annals of Mathematical Statistics, Vol. 30 (1959), pp. 304-340