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Evaluation of Determinants, Characteristic Equations and Their Roots for a Class of Patterned Matrices

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EVALUATION OF DETERMINANTS, CHARACTERISTIC EQUATIONS AND THEIR ROOTS FOR A CLASS OF PATTERNED MATRICES

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1. Introduction. In previous papers [2], [3], [5], the authors have examined matrices with special but common patterns and noticed that they were amenable to simple and rapid methods for inversion. The present effort is concerned with the evaluation of determinants, characteristic equations, and characteristic roots for the same class of specially structured matrices. Some of these operations upon the matrix are not only mathematically related to the process of inversion, but are often required, in addition to the inversion process in such fields as analysis of variance, response-surface fitting and multivariate analysis, and on data under various types of models, "normal" and "non-normal."

It turns out that determinants of this class of patterned matrices can be evaluated more readily than is apparent at first glance. A by-product of this process is to demonstrate methods of reducing patterned matrices, such as by triangulation, to facilitate rapid evaluation of the determinants. It also turns out that for this entire class of patterned matrices considered here, the characteristic equations can be obtained more readily than apparent at first glance. Indeed, for many of these patterned matrices the characteristic equations can even be written down at sight. In some but not all of the cases considered here the latent roots are obtained as readily.

Unfortunately, according to our present knowledge, the latent roots in the more difficult cases appear to be determinable only by the numerical solution of the corresponding characteristic equations. However, even this would be an improvement over the methods of numerical evaluation of latent roots given by such workers as Aitken [1] and Hotelling [4], if those elegant methods, intended for general matrices, were applied to the patterned matrices considered here.

2. Method. The methods used for evaluating determinants are elementary and well-known. For example, by subtracting linear functions of columns from one another, the structure of the determinant can be modified and reduced to the triangular form. This allows immediate evaluation since the determinant of a triangular matrix is the product of the diagonal elements alone.

Another device for reducing a matrix to a more convenient form is that of partitioning. It is known that if

$$\Delta = \begin{vmatrix} P & Q \\ R & S \end{vmatrix}$$

then

$$\Delta = |S| \; |P - Q S^{-1} R| \; .$$

3. Evaluation of Determinants. (3.1) Consider the determinants $\Delta = |D_{a_i} + \alpha \, \underline{b} \, \underline{b}'|$, where D_{a_i} is a diagonal matrix with a_i as diagonal elements $(i=1,2,\ldots n)$ and $\underline{b}$ represents a vector $b_1 \, b_2 \ldots b_n$.

In its more conventional form, the determinant appears as

$$\Delta = \begin{vmatrix} a_1+\alpha b_1^2 & \alpha b_1 b_2 & \alpha b_1 b_3 & \cdots & \alpha b_1 b_n \\ \alpha b_2 b_1 & a_2+\alpha b_2^2 & \alpha b_2 b_3 & \cdots & \alpha b_2 b_n \\ \alpha b_3 b_1 & \alpha b_3 b_2 & a_3+\alpha b_3^2 & \cdots & \alpha b_3 b_n \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha b_n b_1 & \alpha b_n b_2 & \alpha b_n b_3 & \cdots & a_n+\alpha b_n^2 \end{vmatrix} .$$

Subtracting from the j^{th} column, the product of the first column times $\frac{b_j}{b_1}$ (where $j\geq 2$), one obtains

$$\Delta = \left|\begin{array}{c:cccc} a_1+\alpha b_1^2 & \frac{-a_1 b_2}{b_1} & \frac{-a_1 b_3}{b_1} & \cdots & \frac{-a_1 b_n}{b_1} \\ \hdashline \alpha b_2 b_1 & a_2 & 0 & \cdots & 0 \\ \alpha b_3 b_1 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha b_n b_1 & 0 & 0 & \cdots & a_n \end{array}\right|$$

$$= (a_2 a_3 \cdots a_n) \left| (a_1+\alpha b_1^2) + \alpha a_1 \begin{bmatrix} b_2 b_3 \cdots b_n \end{bmatrix} \begin{bmatrix} \frac{1}{a_2} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_3} & \cdots & 0 \\ 0 & 0 & & \frac{1}{a_n} \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \\ b_n \end{bmatrix} \right|$$

$$= (a_2 a_3 \cdots a_n) \left| (a_1+\alpha b_1^2) + \alpha a_1 \left(\frac{b_2^2}{a_2} + \frac{b_3^2}{a_3} + \cdots + \frac{b_n^2}{a_n}\right) \right|$$

$$= (a_1 a_2 a_3 \cdots a_n)\left(1 + \alpha\left(\frac{b_1^2}{a_1} + \frac{b_2^2}{a_2} + \cdots + \frac{b_n^2}{a_n}\right)\right)$$

$$= \left(1 + \alpha \sum_{i=1}^{n} \frac{b_i^2}{a_i}\right) \prod_{i=1}^{n} a_i \quad .$$

Example: In the multinomial distribution, the determinant of the variance-covariance matrix would appear in a form as follows:

$$\Delta = \begin{vmatrix} p_1(1-p_1) & -p_1p_2 & -p_1p_3 & \cdots & -p_1p_k \\ -p_2p_1 & p_2(1-p_2) & -p_2p_3 & \cdots & -p_2p_k \\ -p_3p_1 & -p_3p_2 & p_3(1-p_3) & \cdots & -p_3p_k \\ \vdots & \vdots & \vdots & & \vdots \\ -p_kp_1 & -p_kp_2 & -p_kp_3 & & p_k(1-p_k) \end{vmatrix} \quad .$$

In this case, $a_i = p_i$, $n = k$, $\alpha = -1$ and $b_i = p_i$. Thus,

$$\Delta = \left| D_{p_i} - \underline{p}\,\underline{p}' \right| = \left(1 - \sum_{i=1}^{k} p_i\right) \prod_{i=1}^{k} p_i \quad .$$

<u>Special cases</u>: (3.1.1) If in (3.1), $b_i = 1$, so that $\underline{b}$ represents $\underline{e}$, or a unit vector, then $\underline{e}\,\underline{e}' = J$, where J = a(nxn) matrix with all elements unity. Thus,

$$\Delta = \left| D_{a_i} + \alpha J \right| = \left(1 + \alpha \sum_{i=1}^{n} \frac{1}{a_i}\right) \prod_{i=1}^{n} a_i \quad .$$

Example. Consider the determinant of the matrix which occurs in the anlysis of variance for the one-way classification with unequal frequencies in the different classes.

$$\Delta = \begin{vmatrix} \frac{1}{n_1} + \frac{1}{n_k} & \frac{1}{n_k} & \frac{1}{n_k} & \cdots & \frac{1}{n_k} \\ \frac{1}{n_k} & \frac{1}{n_2} + \frac{1}{n_k} & \frac{1}{n_k} & \cdots & \frac{1}{n_k} \\ \frac{1}{n_k} & \frac{1}{n_k} & \frac{1}{n_3} + \frac{1}{n_k} & \cdots & \frac{1}{n_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n_k} & \frac{1}{n_k} & \frac{1}{n_k} & \cdots & \frac{1}{n_t} + \frac{1}{n_k} \end{vmatrix}$$

$$= \left| D_{\frac{1}{n_i}} + \frac{1}{n_k} J \right| = (1 + \frac{1}{n_k} \sum_{i=1}^{t} n_i) \prod_{i=1}^{t} \frac{1}{n_i} \quad .$$

(3.1.2) If in (3.1), $a_i = a$, for all i, then

$$\Delta = \left| aI + \alpha \underline{b}\underline{b}' \right| = a^n (1 + \frac{\alpha}{a} \sum_{i=1}^{n} b_i^2)$$

$$= a^{n-1} (a + \alpha \sum_{i=1}^{n} b_i^2) \quad .$$

(3.1.3) By combining (3.1.1) and (3.1.2), one obtains a form of matrix very frequently encountered in response-surface fitting as well as in the analysis of variance. That is, $a_i = \alpha$, $b_i = \underline{1}$, then

$$\Delta = \left| aI + \alpha J \right| = a^{n-1} (a + \alpha n) \quad .$$

Example: The determinant of the matrix occurring in the analysis of variance for the one-way classification with equal frequencies in the different classes is:

$$\Delta = \begin{vmatrix} 3 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 1 & \cdots & 1 \\ 1 & 1 & 3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 3 \end{vmatrix} = \left| 2I + J \right| .$$

Thus, $a = 2$, $\alpha = 1$, $n = n$, and

$$\Delta = 2^{n-1}(2 + n).$$

(3.2) <u>Determinants for the diagonal matrix of type 2.</u>

The diagonal matrix of type 2 has been defined elsewhere [2], and its determinant is as follows:

$$\Delta = \begin{vmatrix} a & b & c & d & \cdots & k & \ell \\ b & \mu b & \mu c & \mu d & \cdots & \mu k & \mu \ell \\ c & \mu c & \alpha c & \alpha d & \cdots & \alpha k & \alpha \ell \\ d & \mu d & \alpha d & \gamma d & \cdots & \gamma k & \gamma \ell \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ k & \mu k & \alpha k & \gamma k & \cdots & \pi k & \pi \ell \\ \ell & \mu \ell & \alpha \ell & \gamma \ell & \cdots & \pi \ell & \delta \ell \end{vmatrix} .$$

Multiplying the first column by μ and subtracting the result from the second column, multiplying the first column by α and subtracting this result from the third column, and so on, one obtains,

$$\Delta = \begin{vmatrix} a & b-a\mu & c-a\alpha & d-a\gamma & \cdots & k-a\pi & \ell-a\delta \\ b & 0 & \mu c-b\alpha & \mu d-b\gamma & \cdots & \mu k-b\pi & \mu\ell-b\delta \\ c & 0 & 0 & \alpha d-c\gamma & \cdots & \alpha k-c\pi & \alpha\ell-c\delta \\ d & 0 & 0 & 0 & \cdots & \gamma k-d\pi & \gamma\ell-d\delta \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k & 0 & 0 & 0 & & 0 & \pi\ell-k\delta \\ \ell & 0 & 0 & 0 & & 0 & 0 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a\mu-b & a\alpha-c & a\gamma-d & \cdots & a\pi-k & a\delta-\ell & a \\ 0 & b\alpha-\mu c & b\gamma-\mu d & \cdots & b\pi-\mu k & b\delta-\mu\ell & b \\ 0 & 0 & c\gamma-\alpha d & \cdots & c\pi-\alpha k & c\delta-\alpha\ell & c \\ 0 & 0 & 0 & \cdots & d\pi-\gamma k & d\delta-\gamma\ell & d \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & k\delta-\pi\ell & k \\ 0 & 0 & 0 & \cdots & 0 & 0 & \ell \end{vmatrix}$$

$$\Delta = (a\mu - b)(b\alpha - \mu c)(c\gamma - \alpha d) \cdots (k\delta - \pi\ell)\,\ell \ .$$

It may be noted that this determinant is the product of a series of (2x2) determinants obtained in the following way: Put on one line the elements of the first row (viz. $a, b, c, \ldots k, \ell$); on the second line immediately below, place the coefficients that make this a type 2 diagonal matrix (viz. $\mu, \alpha, \gamma, \ldots \pi, \delta$); make the $(n+1)^{\underline{th}}$ element equivalent to zero, and the $(n+1)^{\underline{th}}$ coefficient equal to unity. Thus, one obtains

$$\begin{array}{cccccccc} a & b & c & d & \cdots & k & \ell & 0 \\ \hline 1 & \mu & \alpha & \gamma & \cdots & \pi & \delta & 1 \end{array}$$

and

$$\Delta = (a\mu - b)(b\alpha - \mu c)(c\gamma - d\alpha) \ . \ . \ . \ (k\delta - n\ell)(\ell) \ .$$

It can be seen that the overall determinant will be zero if any one of the individual (2x2) determinants is zero. In other words, the matrix is singular if $a\mu = b$, $b\alpha = \mu c$, etc.

Example. Consider the value of the determinant resulting from the major part of the variance matrix of the order statistics in a sample from the rectangular population:

$$\Delta = \begin{vmatrix} n & n-1 & (n-2) & \cdots & 1 \\ n-1 & 2(n-1) & 2(n-2) & \cdots & 2 \\ n-2 & 2(n-2) & 3(n-2) & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & & n \end{vmatrix} .$$

In this case,

$$\begin{array}{ll} a = n & \mu = 2 \\ b = n-1 & \alpha = 3 \\ c = n-2 & \gamma = 4 \\ d = n-3 & \vdots \\ & \delta = n \\ \ell = 1 & \end{array}$$

Rewriting in the form of

$$\begin{array}{cccccc} n & n-1 & n-2 & n-3 \ . \ . \ . & 1 & 0 \\ 1 & 2 & 3 & 4 \ . \ . \ . & n & 1 \end{array}$$

$$\Delta = \left[2n - (n-1)\right]\left[3(n-1) - 2(n-2)\right]\left[4(n-2) - 3(n-3)\right] \ . \ . \ . \ 1$$

$$= (n+1)\,(n+1)\,(n+1) \ . \ . \ . \ (n+1)\ 1$$

$$= (n+1)^{n-1} \ .$$

(3.2.1) The determinant of the type 2 matrix given in (3.2) may be rewritten in another, more convenient form, as

$$\Delta = \begin{vmatrix} c_1^2 a_1 & c_1c_2a_1 & c_1c_3a_1 & \cdots & c_1c_na_1 \\ c_2c_1a_1 & c_2^2(a_1+a_2) & c_2c_3(a_1+a_2) & \cdots & c_2c_n(a_1+a_2) \\ c_3c_1a_1 & c_3c_2(a_1+a_2) & c_3^2(a_1+a_2+a_3) & \cdots & c_3c_n(a_1+a_2+a_3) \\ \vdots & \vdots & \vdots & & \vdots \\ c_nc_1a_1 & c_nc_2(a_1+a_2) & c_nc_3(a_1+a_2+a_3) & \cdots & c_n^2(a_1+a_2+a_3+\cdots+a_n) \end{vmatrix} .$$

The matrix related to this determinant was shown in [5] to be expressible as

$$D_{c_i} \, \tilde{T}_1 \, D_{a_i} \, \tilde{T}_1' \, D_{c_i}$$

where $\tilde{T}_1$ is a lower triangular matrix with all non-zero elements equal to unity and D_{c_i} and D_{a_i} are diagonal matrices as defined before. Each of the five matrices has a determinant equal to the product of its diagonal elements, so that $\Delta = \prod_{i=1}^{n} a_i c_i^2$. The value of this determinant is zero, of course, when any a_i or c_i is equal to zero.

This determinant could have been evaluated from the previous result given in (3.2). The transformations making this possible are as follows:

$$a = c_1^2 a_1$$

$$b = c_1 c_2 a_1 \qquad \mu = \frac{c_2}{c_1}\left(\frac{a_1+a_2}{a_1}\right)$$

$$c = c_1 c_3 a_1 \qquad \alpha = \frac{c_3}{c_1}\left(\frac{a_1+a_2+a_3}{a_1}\right)$$

$$\vdots \qquad\qquad \vdots$$

$$\ell = c_1 c_n a_1 \qquad \delta = \frac{c_n}{c_1}\left(\frac{a_1+a_2+a_3+\cdots+a_n}{a_1}\right) .$$

(3.3) Consider the determinant

$$\Delta = \begin{vmatrix} kI_n & aJ \\ aJ & mI_p \end{vmatrix} \quad \text{where } I_t = \text{identity matrix of rank } t.$$

Using the partitioning principle in Section 2, this may be written as

$$\Delta = \left| mI_p \right| \left| kI_n - a^2 J_{(nxp)} \left(\frac{1}{m} I_p\right) J_{(pxn)} \right|$$

$$\Delta = m^p \left| kI_n - \frac{a^2 p}{m} J_{(nxn)} \right|$$

The results from (3.1.3) can be used to evaluate directly the latter part of the right-hand side by letting $a = k$ and $\lambda = -\frac{a^2 p}{m}$.

Therefore,

$$\Delta = m^p \, k^{n-1} \left(k - \frac{a^2 p}{m} n\right) , \text{ or}$$

$$\Delta = m^{p-1} \, k^{n-1} \, (mk - a^2 np) .$$

(3.4) Consider the determinant of n rows and columns of the form such that

$$\Delta = \begin{vmatrix} g & b\underline{e}' \\ b\underline{e} & (c-d)I_{n-1} + dJ \end{vmatrix} .$$

Then,

$$\Delta = g \left| (c-d)I_{n-1} + dJ - \frac{b^2}{g} J \right|$$

$$\Delta = g \left| (c-d)I_{n-1} + (\frac{dg-b^2}{g}) J \right| .$$

Again, the results of (3.1.3) can be used where $a = c-d$, and $\lambda = \frac{dg-b^2}{g}$. Therefore

$$\Delta = g(c-d)^{n-2} \left[(c-d) + (\frac{dg-b^2}{g}) (n-1) \right]$$

$$\Delta = (c-d)^{n-2} \left[g(c-d) + (dg-b^2) (n-1) \right] .$$

Example. A typical matrix from a problem in response-surface estimation [2] may have a determinant as follows:

$$\Delta = \begin{vmatrix} 27 & 34 & 34 & 34 \\ 34 & \frac{161}{2} & \frac{113}{4} & \frac{113}{4} \\ 34 & \frac{113}{4} & \frac{161}{2} & \frac{113}{4} \\ 34 & \frac{113}{4} & \frac{113}{4} & \frac{161}{2} \end{vmatrix}$$

$$= \begin{vmatrix} 27 & 34\underline{e}' \\ 34\underline{e} & \frac{209}{4} I + \frac{113}{4} J \end{vmatrix}$$

Here, $g = 27$ $\quad c = \frac{161}{2}$ $\quad n = 4$

$b = 34$ $\quad d = \frac{113}{4}$.

Therefore

$$\Delta = \left[\left(\frac{209}{4}\right)^2\right]\left\{27\left(\frac{209}{4}\right) + 3\left[27\left(\frac{113}{4}\right) - (34)^2\right]\right\} = 630,644.4375$$

(3.4.1) A Special case. If the determinant considered in Section (3.4) is simplified by making b = d and g = c, we have

$$\Delta = \begin{vmatrix} g & b\underline{e}' \\ b\underline{e} & (g-b)\, I_{n-1} + b\, J \end{vmatrix}$$

$$\Delta = (g-b)^{n-2}\left\{g(g-b) + (bg-b^2)(n-1)\right\}$$

$$\Delta = (g-b)^{n-1}\left\{g + b(n-1)\right\} .$$

Example 1. Consider the infinite matrix used in Section (3.1.3) where the determinant was found to be $2^{n-1}(2+n)$.

This same determinant may be rewritten in the form,

$$\Delta = \begin{vmatrix} 3 & \underline{e}' \\ \underline{e} & 2I_{n-1} + J \end{vmatrix}$$

where g = 3 and b = 1 .

Then, $\Delta = 2^{n-1}(2+n)$. This can be generalized by substituting (g-b) for a, and b for λ, in the results from Section (3.1.3).

Example 2. The determinant considered in Section (3.4) is only a portion of the overall determinant from the particular problem considered. The original determinant was,

$$\Delta = \begin{vmatrix} 27 & 0 & 0 & 0 & 34 & 34 & 34 & 0 & 0 & 0 \\ 0 & 34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 34 & 0 & 0 & 0 & 0 & 0 & 0 \\ 34 & 0 & 0 & 0 & \frac{161}{2} & \frac{113}{4} & \frac{113}{4} & 0 & 0 & 0 \\ 34 & 0 & 0 & 0 & \frac{113}{4} & \frac{161}{2} & \frac{113}{4} & 0 & 0 & 0 \\ 34 & 0 & 0 & 0 & \frac{113}{4} & \frac{113}{4} & \frac{161}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{113}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{113}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{113}{4} \end{vmatrix}$$

$$\Delta = (-1)^{30} \left| \begin{array}{cccccc|cccc} 34 & & & & & & & & & \\ & 34 & & & 0 & & & & & \\ & & 34 & & & & & 0 & & \\ & & & \frac{113}{4} & & & & & & \\ & 0 & & & \frac{113}{4} & & & & & \\ & & & & & \frac{113}{4} & & & & \\ \hline & & & & & & 27 & 34 & 34 & 34 \\ & & & 0 & & & 34 & \frac{161}{2} & \frac{113}{4} & \frac{113}{4} \\ & & & & & & 34 & \frac{113}{4} & \frac{161}{2} & \frac{113}{4} \\ & & & & & & 34 & \frac{113}{4} & \frac{113}{4} & \frac{161}{2} \end{array} \right|$$

$$\Delta = \begin{vmatrix} D_{a_i} & 0 \\ 0 & M \end{vmatrix} = |D_{a_i}|\,|M| = (34)^3\left(\frac{113}{4}\right)^3 |M| = \left\{\frac{(34)(113)}{4}\right\}^3 630,644,4375$$

(3.5) The matrix which occurs in the analysis of variance of a latin square experiment presents a determinant of another interesting form as

$$\Delta = \begin{vmatrix} mI_k & J & J \\ J & mI_{k-1} & J \\ J & J & mI_{k-1} \end{vmatrix}$$

$$\Delta = |mI_k| \left| \begin{matrix} mI_{k-1} & J \\ J & mI_{k-1} \end{matrix} - \begin{bmatrix} J \\ J \end{bmatrix} \left(\frac{1}{m}I_k\right) \begin{bmatrix} J & J \end{bmatrix} \right|$$

$$\Delta = m^k \left| \begin{bmatrix} mI_{k-1} & J \\ J & mI_{k-1} \end{bmatrix} - \frac{1}{m}\begin{bmatrix} kJ & kJ \\ kJ & kJ \end{bmatrix} \right|$$

$$\Delta = m^k \begin{vmatrix} (mI_{k-1} - \frac{k}{m} J) & (1 - \frac{k}{m}) J \\ (1 - \frac{k}{m}) J & (mI_{k-1} - \frac{k}{m} J) \end{vmatrix}$$

$$\Delta = m^k \left| mI_{k-1} - \frac{k}{m}J \right| \left| (mI_{k-1} - \frac{k}{m}J) - (1 - \frac{k}{m})^2 J (mI_{k-1} - \frac{k}{m}J)^{-1} J \right|$$

By using the results of Roy and Sarhan [4], the matrix $(mI_{k-1} - \frac{k}{m} J)$ can be inverted and the first determinant evaluated by the use of Section (3.1.3) so that the result is

$$\Delta = (m^k)(m^{k-2}) \left\{ m - (\tfrac{k}{m})(k-1) \right\} \left| (mI_{k-1} - \tfrac{k}{m}J) - (\tfrac{m-k}{m})^2 J \left\{ \tfrac{1}{m} I_{k-1} + \frac{k}{m(m^2-k^2+k)} J \right\} J \right|$$

$$\Delta = m^{2k-3}(m^2-k^2+k) \left| mI_{k-1} - \tfrac{k}{m}J - (\tfrac{m-k}{m})^2 \left\{ \frac{k-1}{m}J + \frac{k(k-1)^2}{m(m^2-k^2+k)} J \right\} \right|$$

$$\Delta = m^{2k-3}(m^2-k^2+k) \left| mI_{k-1} - \frac{1}{m^3} \left\{ km^2 + (m-k)^2(k-1)\left[1 + \frac{k(k-1)}{m^2-k^2+k}\right] \right\} J \right|$$

Using the results from Section (3.1.3) again, this reduces to:

$$\Delta = m^{3k-5} (m^2-k^2+k) \left[m - \frac{k-1}{m^3} \left\{ km^2 + (m-k)^2 (k-1)\left(1+\frac{k(k-1)}{m^2-k^2+k} \right) \right\} \right]$$

$$= m^{3k-5} (m - k+1) \left\{ m^2 + m(k+1) - 2k(k-1) \right\}$$

Consider as a special case of (3.5) the determinant given by letting $m = k$, viz.

$$\Delta = \begin{vmatrix} kI_k & J & J \\ J & kI_{k-1} & J \\ J & J & kI_{k-1} \end{vmatrix}$$

The result is clearly seen to be

$$\Delta = k^{3k-4} .$$

(4) Characteristic Equations and Roots

The characteristic equation of a matrix A is defined by $|A - \lambda I_n| = 0$. The solution of this characteristic equation gives the characteristic roots which are represented by $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$.

Consider the matrix, M,

(4.1) $$M = \begin{bmatrix} a_1+\alpha b_1^2 & \alpha b_1 b_2 & \alpha b_1 b_3 & \cdots & \alpha b_1 b_n \\ \alpha b_2 b_1 & a_2+\alpha b_2^2 & \alpha b_2 b_3 & \cdots & \alpha b_2 b_n \\ \alpha b_3 b_1 & \alpha b_3 b_2 & a_3+\alpha b_3^2 & \cdots & \alpha b_3 b_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha b_n b_1 & \alpha b_n b_2 & \alpha b_n b_3 & \cdots & a_n+\alpha b_n^2 \end{bmatrix}$$

$$M = \left| D_{a_i} + \alpha \underline{b}\, \underline{b}' \right|$$

The characteristic equation is, therefore, given by

$$\left| D_{(a_i-\lambda)} + \alpha \underline{b}\, \underline{b}' \right| = 0 \quad .$$

By use of (3.1), this leads to the characteristic equation

(4.1.1) $$\left\{ 1 + \alpha \left(\sum_{i=1}^{n} \frac{b_i^2}{a_i-\lambda} \right) \right\} \prod_{i=1}^{n} (a_i - \lambda) = 0$$

Example: The characteristic equation for the example given in (3.1) is

$$\left\{ 1 - \alpha \sum_{i=1}^{k} \frac{p_i^2}{p_i-\lambda} \right\} \prod_{i=1}^{k} (p_i-\lambda) = 0 \quad .$$

As a special case of the matrix given in (4.1), let $\underline{b}$ = a unit vector. The characteristic equation then becomes

(4.1.2) $$\left\{ 1 + \alpha \left(\sum_{i=1}^{n} \frac{1}{a_i-\lambda} \right) \right\} \prod_{i=1}^{n} (a_i-\lambda) = 0 \ .$$

If all $a_i \equiv a$, (4.1.1) reduces to

$$(4.1.3)\quad \left\{1 + \frac{\alpha}{a-\lambda} \sum_{i=1}^{n} b_i^2\right\}(a-\lambda)^n = \left\{(a-\lambda) + \alpha \sum_{i=1}^{n} b_i^2\right\}(a-\lambda)^{n-1} = 0$$

Solution of (4.1.3) yields (n-1) characteristic roots all equal to a, and the $n^{\underline{th}}$ root is obtained by equating the quantity in the brackets to zero. Thus, $\lambda_n = a + \alpha \sum_{i=1}^{n} b_i^2$.

Suppose that only $a_1 = a_2$ (=a, let us say.) In this case, a is one of the characteristic roots and the equation can be reduced one degree to solve for the remaining (n-1) roots.

Suppose that $a_1=a_2=a_3$ (= a, let us say.) In this case , a is a repeated characteristic root and the equation can be reduced two degrees to solve for the remaining (n-2) roots.

By induction, if $a_1=a_2, \ldots, =a_n$, and the result in (4.1.3) follows.

(4.1.4) If all $a_i=a$, and $\underline{b}$ = a unit vector, then (4.1.3) reduces to

$$\left[(a-\lambda) + \alpha n\right](a-\lambda)^{n-1} = 0$$

Thus, there are (n-1) characteristic roots equal to a, and the $n^{\underline{th}}$ root is $(a+\alpha n)$.

Example: The characteristic roots of the matrix

$$M = \begin{bmatrix} 5 & 3 & 3 & 3 \\ 3 & 5 & 3 & 3 \\ 3 & 3 & 5 & 3 \\ 3 & 3 & 3 & 5 \end{bmatrix} = \left[2I_4 + 3J\right]$$

can be found by letting a=2, α=3, n=4. The four roots are therefore

2, 2, 2, and 14. Since the sum of the characteristic roots (viz. 2+2+2+14=20) must equal the sum of the diagonal elements (or trace), this is obviously confirmed. Furthermore, the product of the characteristic roots (viz. 2x2x2x14=112) must equal the determinant, and this is also confirmed.

(4.2) Consider the diagonal matrix of type 2 occurring in Section (3.2.1). In this case, the characteristic equation can be expressed as

$$\lambda^n - \lambda^{n-1}\, tr_1\, A + \lambda^{n-2}\, tr_2\, A - \cdots + \cdots + (-1)^{n-1}\, tr_n\, A = 0$$

where the usual definition is used for the traces of different orders. Thus,

$$tr_1 A = c_1^2 a_1 + c_2^2 (a_1+a_2) + c_3^2 (a_1+a_2+a_3) + \cdots + c_n^2 (a_1+a_2+\cdots+a_n)$$

and

$$tr_n A = |A| = \prod_{i=1}^{n} c_i^2 a_i \quad .$$

To evaluate the traces of other orders, the matrix is first rewritten in what was previously shown to be an equivalent form, viz.

$$D_{c_i}\, \tilde{T}_i\, D_{a_i}\, \tilde{T}_i'\, D_{c_i} \quad .$$

To obtain from this, for example, $tr_2 A$, it can be observed that $tr_2 A$ is the sum of the second order principal minors of this matrix. A typical second order principal minor formed by the rows i and j and columns i and j (i<j) of the original matrix is given by

$$c_i^{2}c_j^{2}\left|\begin{bmatrix}\sqrt{a_1} & \sqrt{a_2} & \cdots & \sqrt{a_i} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \sqrt{a_1} & \sqrt{a_2} & \cdots & \sqrt{a_i} & \sqrt{a_{i+1}} & \cdots & \sqrt{a_j} & 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}\sqrt{a_1} & \sqrt{a_1}\\ \sqrt{a_2} & \sqrt{a_2}\\ \vdots & \vdots\\ \sqrt{a_i} & \sqrt{a_i}\\ 0 & \sqrt{a_{i+1}}\\ \vdots & \vdots\\ 0 & \sqrt{a_j}\\ 0 & 0\\ 0 & 0\\ \vdots & \vdots\\ 0 & 0\end{bmatrix}\right|$$

This, in turn, is equivalent to

$$= c_i^{2}c_j^{2}\,(a_1 + a_2 + \cdots + a_i)(a_{i+1} + \cdots + a_j) \;.$$

Thus,

$$tr_2A = \sum_{j=2}^{n}\sum_{i=1}^{j-1} c_i^{2}c_j^{2}\,(a_1+a_2+\cdots+a_i)(a_{i+1}+\cdots+a_j)$$

In general, one can proceed to find

$$tr_tA = \sum_{i_t=t}^{n}\cdots\sum_{i_2=2}^{i_3-1}\sum_{i_1=1}^{i_2-1} c_{i_1}^{2}\,c_{i_2}^{2}\cdots c_{i_t}^{2}(a_1+a_2+\cdots+a_{i_1})$$

$$(a_{i_1+1}+\cdots+a_{i_2})(a_{i_2+1}+\cdots+a_{i_3})\cdots(a_{i_{t-1}+1}+\cdots+a_{i_t}) \;,$$

where $(i_1<i_2<\cdots<i_t)$.

For illustration, the case when $n=4$ may be considered by indicating the characteristic equation. It is as follows:

$$\lambda^4 - \lambda^3\left\{c_1^2a_1 + c_2^2(a_1+a_2) + c_3^2(a_1+a_2+a_3) + c_4^2(a_1+a_2+a_3+a_4)\right\}$$

$$+\lambda^2\left\{c_1^2c_2^2a_1a_2 + c_1^2c_3^2a_1(a_2+a_3) + c_1^2c_4^2a_1(a_2+a_3+a_4)\right.$$

$$c_2^2c_3^2(a_1+a_2)a_3 + c_2^2c_4^2(a_1+a_2)(a_3+a_4)$$

$$\left. + c_3^2c_4^2(a_1+a_2+a_3)a_4\right\}$$

$$-\lambda\left\{c_1^2c_2^2c_3^2a_1a_2a_3 + c_1^2c_2^2c_4^2a_1a_2(a_3+a_4)\right.$$

$$\left. + c_2^2c_3^2c_4^2(a_1+a_2)a_3a_4 + c_1^2c_3^2c_4^2a_1(a_2+a_3)a_4\right\}$$

$$+c_1^2c_2^2c_3^2c_4^2a_1a_2a_3a_4 = 0 \;.$$

Example: Consider the matrix of diagonal type 2 given by

$$V = \begin{bmatrix} 4 & 3 & 2 & 1 \\ & 6 & 4 & 2 \\ & & 6 & 3 \\ & & & 4 \end{bmatrix} .$$

The matrix V represents the variance matrix of order statistics for a sample of size 4 from the rectangular distribution. This form of the matrix is the same as that in Section (3.2.1) by putting

$$c_1 = 4 \qquad a_1 = 1/4$$
$$c_2 = 3 \qquad a_2 = 5/12$$
$$c_3 = 2 \qquad a_3 = 5/6$$
$$c_4 = 1 \qquad a_4 = 5/2 \;.$$

The characteristic equation for this matrix is

$$\lambda^4 - 20\lambda^3 + 105\lambda^2 - 200\lambda + 125 = 0 \quad .$$

(4.2.1) Special cases. For a general n, if all c_i's are equal to c and if all a_i's are equal to a, then

$$tr_t A = c^{2t} a^t \sum_{i_t=t}^{n} \cdots \sum_{i_2=2}^{i_3-1} \sum_{i_1=1}^{i_2-1} i_1(i_2-i_1)(i_3-i_2)\cdots(i_t-i_{t-1}) \quad .$$

Thus, for n=4, the characteristic equation reduces to

$$\lambda^4 - 10c^2 a\lambda^3 + 15c^4 a^2\lambda^2 - 7c^6 a^3\lambda + c^8 a^4 = 0 \quad .$$

If only the c_i's are all equal to c, and a_i remains as such, the case for n=4 will have a characteristic equation as follows:

$$\lambda^4 - \lambda^3 c^2\left[4a_1+3a_2+2a_3+a_4\right] + \lambda^2 c^4\left[3a_1a_2+4a_1a_3+3a_1a_4+2a_2a_3+2a_2a_4+a_3a_4\right]$$
$$- \lambda c^6\left[2a_1a_2a_3+2a_1a_2a_4+2a_1a_3a_4+a_2a_3a_4\right] + c^8 a_1a_2a_3a_4 = 0 \quad .$$

For example, consider the portion of the variance matrix of order statistics for a sample of size 4 from the exponential distribution. Let

$$A = \begin{bmatrix} 1/16 & 1/16 & 1/16 & 1/16 \\ & 1/16+1/9 & 1/16+1/9 & 1/16+1/9 \\ & & 1/16+1/9+1/4 & 1/16+1/9+1/4 \\ \text{(symmetric)} & & & 1/16+1/9+1/4+1 \end{bmatrix}$$

This has a pattern typical of the type 2 matrices and can be derived by equating

$c_1 = c_2 = c_3 = c_4 = 1$

$a_1 = 1/16, \; a_2 = 1/9, \; a_3 = 1/4, \; a_4 = 1$.

The characteristic equation is clearly

$$\lambda^4 - 25/12\,\lambda^3 + 115/144\,\lambda^2 - 11/144\,\lambda + 1/576 = 0 \;.$$

Alternatively, if the a_i's are all equal to a but the c_i's remain unaltered, the characteristic equation for n=4 is

$$\lambda^4 - \lambda^3 a\left[c_1^2+2c_2^2+3c_3^2+4c_4^2\right]+\lambda^2 a^2\left[c_1^2c_2^2 + 2c_1^2c_3^2 +3c_1^2c_4^2+2c_2^2c_3^2+4c_2^2c_4^2+3c_3^2c_4^2\right]- \lambda a^3\left[c_1^2c_2^2c_3^2 +2c_1^2c_2^2c_4^2+2c_2^2c_3^2c_4^2+2c_1^2c_3^2c_4^2\right]+ a^4c_1^2c_2^2c_3^2c_4^2 = 0$$

(4.3) Characteristic Equations and Roots of Simply Partitioned Matrices.

Consider the matrix A,

(4.3.1) $$A = \begin{bmatrix} kI_n & aJ \\ aJ & mI_p \end{bmatrix} .$$

The determinant of matrix A was shown in Section (3.3) to be equal to $k^{n-1}m^{p-1}(mk - a^2np)$.

By replacing k by $k-\lambda$ and m by $m-\lambda$, the characteristic equation is seen to be

$$(k-\lambda)^{n-1}(m-\lambda)^{p-1}\left[(m-\lambda)(k-\lambda) - a^2np\right] = 0 \;.$$

Therefore, there are (n-1) roots equal to k, and there are (p-1) roots equal to m, and the remaining two roots are obtainable from solution of $(m-\lambda)(k-\lambda) - a^2np = 0$, being thus equal to

$$\frac{(k+m) \pm \sqrt{(k-m)^2 + 4a^2np}}{2} .$$

(4.4) Characteristic Equations and Roots of Other Partitioned Matrices.

Consider the matrix A,

$$A = \begin{bmatrix} a & b\,\underline{e}' \\ b\,\underline{e} & (c-d)I_{n-1}+dJ \end{bmatrix}$$ Where $\underline{e}$ = unit vector.

In Section (3.4), the determinant was shown to be equal to

$$a(c-d)^{n-2} \left\{ c-d + (d - \frac{b^2}{a})(n-1) \right\} .$$

By replacing a by $a-\lambda$ and c by $c-\lambda$, the characteristic equation is

$$(a-\lambda)(c-d-\lambda)^{n-2} \left\{ (c-d-\lambda) + (d - \frac{b^2}{a-\lambda})(n-1) \right\} = 0$$

or

$$(c-d-\lambda)^{n-2} \left\{ (c-d-\lambda)(a-\lambda) + \left[d(a-\lambda)-b^2 \right] (n-1) \right\} = 0$$

There are (n-2) roots equal to (c-d) and the other two roots are obtainable by setting

$$(c-d-\lambda)(a-\lambda) + \left[d(a-\lambda)-b^2 \right] (n-1) = 0$$

Example: The characteristic roots of the matrix given in Section (3.4) can be found as follows:

$$\text{Let } A = \begin{bmatrix} 27 & 34\,\underline{e}' \\ 34\,\underline{e} & 209/4\; I_3 + 113/4\; J \end{bmatrix} .$$

Then, the characteristic equation is:

$$(209/4 - \lambda)^2 \left\{ (209/4 - \lambda)(27 - \lambda) + 3\left[113/4(27-\lambda) - (34)^2\right] \right\} = 0$$

Two roots are equal to 209/4, and the other two come from solution of

$$(209/4 - \lambda)(27-\lambda) + 3\left[113/4\,(27 - \lambda) - (34)^2\right] = 0 \quad .$$

As a special case, let b=d and a=c. Then

$$A = \begin{bmatrix} a & b\,\underline{e}' \\ b\,\underline{e} & (a-b)I_{n-1}+bJ \end{bmatrix} .$$

Working with the determinant of this matrix and replacing a by a−λ, or by going directly to the characteristic equation given earlier and substituting for c and d, the result is

$$(a-b-\lambda)^{n-1}\left[a + b(n-1) - \lambda\right] = 0 \quad .$$

Thus, (n−1) roots are equal to (a−b) and the $n^{\underline{th}}$ root is equal to $\left[a+b(n-1)\right]$.

Example: Consider the matrix from (3.4.1) where

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & \underline{e}' \\ \underline{e} & 2I_3 + J \end{bmatrix}$$

so that a=3, b=1, and n=4. Three roots are equal to 2, and the fourth root is 6.

Example: Consider the matrix given in Section (3.4.1).

$$A = \left[\begin{array}{cccccc|cccc} 34 & & & & & & & & & \\ & 34 & & & & & & & & \\ & & 34 & & & & & & & \\ & & & 113/4 & & & & & & \\ & & & & 113/4 & & & & & \\ & & & & & 113/4 & & & & \\ \hline & & & & & & 27 & 34 & 34 & 34 \\ & & & & & & 34 & 161/2 & 113/4 & 113/4 \\ & & & & & & 34 & 113/4 & 161/2 & 113/4 \\ & & & & & & 34 & 113/4 & 113/4 & 161/2 \end{array}\right]$$

or

$$A = \begin{bmatrix} D_{p_i} & 0 \\ 0 & T \end{bmatrix} .$$

Therefore,

$$|A| = |p_i||T| .$$

Six of the characteristic roots are given by the p_i's, viz. three roots equal to 34 and three equal to 113/4. The remaining four roots are those obtained previously in the first example of this Section.

(4.5) Consider the matrix of Section 3.5.

$$\text{Let } A = \begin{bmatrix} mI_k & J & J \\ J & mI_{k-1} & J \\ J & J & mI_{k-1} \end{bmatrix} .$$

The determinant was given as

$$m^{3k-5}(m-k+1)\{ m^2 + m(k-1) - 2k(k-1)\}$$

By replacing m by $m-\lambda$ the characteristic equation is

$$(m-\lambda)^{3k-5}(m-\lambda-k+1)\ (m-\lambda)^2 + (m-\lambda)(k-1) - 2k(k-1) = 0$$

There are $(3k-5)$ repeated roots equal to m, one root equal to $(m-k+1)$, and the remaining two roots obtained from solving

$$(m-\lambda)^2 + (m-\lambda)(k-1) - 2k(k-1) = 0$$

or

$$\lambda = \frac{(2m+k-1) \mp \sqrt{(k-1)(9k-1)}}{2}$$

If the special case is considered whereby $m=k$, there are $(3k-5)$ repeated roots equal to k, one root of unity, and two roots equal to

$$\lambda = \frac{(3k-1) \mp \sqrt{(k-1)(9k-1)}}{2}$$

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