EFFICIENT FITTING OF LINEAR MODELS FOR CONTINUOUS STATIONARY TIME SERIES FROM DISCRETE DATA

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James Durbin

University of North Carolina
and
University of London

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J. Durbin
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1. Introduction

Recent statistical research in the field of stationary time series has been mainly concerned with the estimation of spectral density by non-parametric methods. We may refer to the books by Blackman & Tukey (1958) and Grenander & Rosenblatt (1957), to the Royal Statistical Society (1957) symposium on the spectral approach to time series and to the paper by Parzen (1958) for some representative work in this area. Valuable though this work is, it does not liberate us completely from the burden of developing useful parametric methods. However seductive the appeal of an infinity of parameters, there remain occasions when the underlying mechanism and the purpose of the investigation tempt us to undertake the rigours of parametric estimation.

For continuous stationary series the simplest and most useful model is that having a rational spectral density. The model is linear in the sense that the observations satisfy a linear stochastic differential equation (see section 5 below). It has the important property that its structure is invariant under linear operations of the kind found in electrical engineering

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and elsewhere (see, for example, Laning & Battin, 1956, Chapter 5).

In this paper we consider the problem of estimating the parameters of the rational spectral density function of a continuous process given n observations equi-spaced in time. The estimates are derived by assuming the observations to be normally distributed. However, it must not be thought that the properties of the estimates depend critically on this assumption. If the distribution of the observations is non-normal the estimation process can be thought of as arising from a sort of generalized least-squares method. By analogy with Mann & Wald's (1943) results for autoregressive series it appears likely that the large-sample distribution of the estimates will be rather insensitive to departures from normality.

Our method of attack will be to consider the joint distribution of the equi-spaced observations and to estimate first the relevant parameters of this distribution. We then convert these to estimates of the parameters of the spectral density following the route charted out by Phillips (1959). In section 5 we go on to consider the estimation of the parameters of the underlying stochastic differential-equation model.

The non-parametric estimation of the spectral density of a continuous series is bedevilled by the "aliasing" problem, discussed at length by Blackman & Tukey (1958), which arises from the fact that high frequency components cannot be identified from a sample of discrete data. It should be emphasized that with a parametric approach of the kind employed here this problem disappears completely.

2. Model Structure

We are concerned with a continuous stationary process having the rational spectral density

$$(1) \qquad f(\omega) = \text{constant} \times \left| \frac{1 + \beta_1(i\omega) + \ldots + \beta_{p-1}(i\omega)^{p-1}}{1 + \alpha_1(i\omega) + \ldots + \alpha_p\ (i\omega)^p} \right|^2 \quad (\alpha_p \neq 0)(-\infty \leq \omega \leq \infty)$$

where the α's and β's are all real and the roots $\nu_1, \ldots, \nu_{p-1}$ of the equation $1 + \beta_1 x + \ldots + \beta_{p-1}\ x^{p-1} = 0$ together with the roots $\lambda_1, \ldots, \lambda_p$ of the equation $1 + \alpha_1 x + \ldots + \alpha_p x^p = 0$ are all unequal with negative real part. Since complex roots occur in conjugate pairs (1) reduces to the form

$$(2) \qquad f(\omega) = \text{constant} \times \frac{\prod_{j=1}^{p-1} (\omega^2 + \nu_j^2)}{\prod_{j=1}^{p} (\omega^2 + \lambda_j^2)} .$$

The autocorrelation function of the process may be obtained as the Fourier transform of (2). It is well known that this is the sum of weighted exponentials

$$(3) \qquad \rho_r = \sum_{j=1}^{p} H_j\ e^{\lambda_j |r|} \qquad (-\infty \leq r \leq \infty)$$

(Doob, 1953, XI § 10; also Laning & Battin, 1956, Appendix D). Calculating $f(\omega)$ as the inverse transform we obtain (Laning & Battin, 1956, Appendix D)

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^{p} H_j\ e^{\lambda_j |r|}\ e^{-i\omega r}\ dr, \text{ i.e.}$$

$$(4) \qquad f(\omega) = -\frac{1}{\pi} \sum_{j=1}^{p} \frac{H_j \lambda_j}{\omega^2 + \lambda_j^2} \quad ,$$

which has the required form (2).

We now consider how the values of H_j, λ_j and hence, in virtue of (4), the spectral density, may be inferred from observations which are equi-spaced along the series. Our general approach follows that of Phillips (1959). Taking the time-interval between successive observations to be unity, which we may do without loss of generality, we obtain a discrete process having an autocorrelation function identical with (3) except that r is restricted to the values ..., -1, 0, 1, ... It is more convenient for the present purpose to write this in the form

$$\rho_r = \sum_{j=1}^{p} H_j \mu_j^{|r|} \quad (r = \ldots, -1, 0, 1, \ldots) \tag{5}$$

where $\log \mu_j = \lambda_j$. Since λ_j has negative real part, $|\mu_j| < 1$ $(j=1,\ldots,p)$.

Now (5) is the autocorrelation function of observations $\{u_t\}$ satisfying the stochastic difference equation

$$u_t + \gamma_1 u_{t-1} + \ldots + \gamma_p u_{t-p} = \epsilon_t + \delta_1 \epsilon_{t-1} + \ldots + \delta_{p-1} \epsilon_{t-p+1} \tag{6}$$

$$(t = \ldots, -1, 0, 1, \ldots)$$

(Walker, 1950). Here $\{\epsilon_t\}$ is a series of uncorrelated random variables with zero mean and constant variance σ^2, $\mu_1, \ldots, \mu_p$ are the roots of the equation $x^p + \gamma_1 x^{p-1} + \ldots + \gamma_p = 0$ and ρ_r is the coefficient of z^r in the expansion of the autocorrelation generating function

$$R(z) = \frac{\sigma^2}{\sigma_u^2} \frac{(1+\delta_1 z + \ldots + \delta_{p-1} z^{p-1})(1+\delta_1 z^{-1} + \ldots + \delta_{p-1} z^{-p+1})}{(1+\gamma_1 z + \ldots + \delta_p z^p)\,(1+\gamma_1 z^{-1} + \ldots + \gamma_p z^{-p})} \tag{7}$$

(see Walker, 1950). The spectral density of this process is

$$\frac{1}{2\pi} R(e^{i\omega}) \quad (-\pi\leq\omega\leq\pi).$$

It will be noted that this differs from the spectral density of the corresponding continuous process although the two autocorrelation functions are the same.

If the γ's and δ's were known the μ's could be calculated and hence $\lambda_1,\ldots,\lambda_p$. There remains the calculation of $H_1, \ldots, H_p$. Two methods for this will be suggested. For the first, we have from (5)

$$(8) \qquad R(z) = \sum_{j=1}^{p} H_j\left(\frac{1}{1-\mu_j z} + \frac{1}{1-\mu_j z^{-1}} -1\right).$$

Multiplying by $1 - \mu_j z$ and putting $z = \mu_j^{-1}$ we have

$$(9) \qquad H_j = (1-\mu_j z)\, R(z) \Big|_{z = \mu_j^{-1}}$$

Using (7) and remembering that $\sum_{j=1}^{p} H_j = 1$ the evaluation of (9) is straight forward except for the part

$$\frac{1 - \mu_j z}{1 + \gamma_1 z + \ldots + \gamma_p z^p}\Bigg|_{z = \mu_j^{-1}} .$$

By ordinary partial fraction methods this is seen to be equal to

$$\lim_{dz\to 0} \frac{1 - \mu_j(\mu_j^{-1} + dz)}{\gamma(\mu_j^{-1}) + \gamma'(\mu_j^{-1})dz} = -\frac{\mu_j}{\gamma'(\mu_j^{-1})},$$

where $\gamma'(z)$ is the derivative of $\gamma(z) = 1 + \gamma_1 z+\ldots+\gamma_p z^p$.

For the second method of calculating $H_1, \ldots, H_p$ we observe that (8) reduces to

$$R(z) = \sum_{j=1}^{p} \frac{H_j(1-\mu_j^2)}{1+\mu_j^2-\mu_j(z+z^{-1})} \tag{10}$$

Putting $z = e^{i\omega}, \ldots, e^{i\omega_p}$ in (7) and (10) where $\omega_1, \ldots, \omega_p$ are arbitrarily chosen and using the result $\sum H_j = 1$ we obtain $H_1, \ldots, H_p$ as the solution of a set of linear equations.

In the next section we consider the problem of obtaining efficient estimates $c_1, \ldots, c_p$ and $d_1, \ldots, d_{p-1}$ of the γ's and δ's. Using the parametric relations established in the present section these are converted into estimates ℓ_j and h_j of λ_j and H_j. The spectral density is then computed by substituting in (4).

3. Estimation Procedure

Our first objective is to estimate the coefficients of (6) from the sample of n successive observations. A procedure for doing this by autocorrelation methods has been suggested in an earlier paper (Durbin, 1959-b). The procedure we now develop is essentially a spectral form of the same method. However, the spectral approach not only seems more suited to the purpose in hand but also leads to some interesting results of wider application.

As in the earlier method we begin by considering the autoregressive process of order k, where k is large, which approximates (6) most closely, i.e. setting

$$u_t + \phi_1 u_{t-1} + \dots + \phi_k u_{t-k} = e_t,$$

we suppose that $\phi_1, \dots, \phi_k$ are chosen so as to minimize $E(e_t^2)$. Assuming $u_1, \dots, u_n$ to be normally distributed their joint distribution is, for sufficiently large k,

$$\text{(11)} \qquad dP = \frac{|V^{-1}|^{\frac{1}{2}}}{(2\pi)^{n/2}\sigma^n} \exp\left[-\frac{1}{2\sigma^2}\left\{ u'V^{-1}u + \sum_{t=k+1}^{n} (u_t + \phi_1 u_{t-1} + \dots + \phi_k u_{t-k})^2 \right\}\right] du_1 \dots du_n$$

as accurately as required (we assume that n is so large that we may always suppose that $k = o(n)$). (11) is obtained by taking $u_1, \dots, u_n$ to be approximately distributed as if generated by the autoregressive process

$$\text{(12)} \qquad u_t + \phi_1 u_{t-1} + \dots + \phi_k u_{t-k} = \epsilon_t$$

where ϵ_t is defined by (6). $\sigma^2 V$ is the variance matrix of k successive observations generated by this process; an explicit form for V^{-1} in terms of the ϕ's has been given elsewhere (Durbin, 1959 a). u denotes the column vector $\{u_1, \dots, u_k\}$.

Neglecting terms of order lower than n, the loglikelihood is

$$\text{(13)} \qquad \log L = -\frac{1}{2\sigma^2}\left\{ \sum_{s=0}^{k} \phi_s^2 \sum_{t=1}^{n} u_t^2 + 2 \sum_{s=0}^{k-1} \phi_s \phi_{s+1} \sum_{t=1}^{n-1} u_t u_{t+1} + \dots \right.$$

$$\left. \dots + 2\, \phi_0\, \phi_k \sum_{t=1}^{n-k} u_t u_{t+k} \right\}$$

where we have taken $\phi_o = 1$.

Let

(14) $$G_j = \sum_{r=-k}^{k} e^{\frac{2\pi rij}{2k+1}} \sum_{s=o}^{k-|r|} \phi_s \, \phi_{s+|r|} \quad (j=0,1,\ldots,k)$$

(15) $$s_j = \frac{1}{n} \sum_{r=-k}^{k} e^{\frac{2\pi rij}{2k+1}} \sum_{t=1}^{n-|r|} u_t u_{t+|r|} \quad (j=0,1,\ldots,k) .$$

Substituting in (13) we have

(16) $$\log L = - \frac{n}{2(2k+1)\sigma^2} \left\{ G_o s_o + 2 \sum_{j=1}^{k} G_j s_j \right\} .$$

Observing that $G_j = \left| 1 + \phi_1 e^{\frac{2\pi ij}{2k+1}} + \ldots + \phi_k e^{\frac{2\pi kij}{2k+1}} \right|^2$ and recalling that the spectral density of the autoregressive process (12) is $\left|1+\phi_1 e^{i\omega}+\ldots+\phi_k e^{ki\omega}\right|^2$ times a constant, we see that, for sufficiently large k,

(17) $$G_j = \text{constant} \times \left[g\left(\frac{2\pi j}{2k+1}\right) \right]^{-1} \quad (j=0,1,\ldots,k)$$

as accurately as required, $g(\omega)$ being $2\pi\sigma_u^2/\sigma^2$ times the spectral density of the process under consideration. For the special case of process (6) we have

(18) $$g(\omega) = \left| \frac{1+\delta_1 e^{i\omega}+\ldots+\delta_{p-1} e^{(p-1)i\omega}}{1+ \gamma_1 e^{i\omega}+\ldots+\gamma_p e^{pi\omega}} \right|^2 .$$

We note also the following simpler expression for s_j,

(19) $$s_j = v_o + 2v_1 \cos \frac{2\pi j}{2k+1} + \ldots + 2v_k \cos \frac{2\pi jk}{2k+1} \quad (j=0,1,\ldots,k),$$

where $v_r = \frac{1}{n} \sum_{t=1}^{n-r} u_t u_{t+r} \quad (r = 0,1,\ldots,k)$.

Substituting in (16) we see that maximizing the likelihood is equivalent to minimizing

$$S = \frac{s_o}{g_o} + 2 \sum_{j=1}^{k} \frac{s_j}{g_j} , \tag{20}$$

where g_j denotes $g\left(\frac{2\pi j}{2k+1}\right)$. Note that s_j is closely related to Bartlett's original smoothed estimate of the spectral density (Bartlett, 1948); its expectation is approximately $\sigma^2 g_j$, the bias being arbitrarily small for sufficiently large k and n (see Grenander & Rosenblatt, 1957, p. 149).

Thus our basic estimating equations are given by

$$\frac{s_o}{\hat{g}_o} + 2 \sum_{j=1}^{k} \frac{s_j}{\hat{g}_j} = \min. \tag{21}$$

the minimization being performed with respect to each of the relevant parameters.

(21) may be regarded as approximately equivalent to Whittle's (1953) fundamental estimation equations

$$\frac{1}{2\pi i} \int_c \frac{\lambda(z)}{\hat{M}(z)} \frac{dz}{z} = \min. \tag{22}$$

where $\lambda(z)$, $M(z)$ are the autocorrelation generating function for the sample and population respectively and $\int_c$ denotes integration round the unit circle. As k becomes larger one would expect (21) and (22) to move close together. Note that (21) was derived on the assumption that the process is a discrete Gaussian process which can be approximated to any required degree of accuracy by an autoregressive process of sufficiently high order k where $k = o(n)$; its validity is not restricted to processes satisfying (6).

Referring to (18) we see that for the model (6) the minimization with respect to $\gamma_1,\ldots,\gamma_p$ for given values of $\delta_1,\ldots,\delta_{p-1}$ is a simple matter since we obtain a set of equations which are linear in the estimates required. Minimization with respect to $\delta_1,\ldots,\delta_{p-1}$ is not so simple, however, since these parameters occur in the denominators of the terms of (20). We therefore seek to transform (20) into a form which is more suitable for the estimation of the δ's. This is achieved as follows.

Let s/g denote a typical term s_j/g_j and let $\sigma^2 g^*$ denote $E(s)$. The deviation $s - \sigma^2 g^* = ds$ say has variance $O(n^{-1})$, k being taken as fixed. We wish to consider the behavior of the likelihood function for parameter points differing from the true parameter point by deviations comparable with those expected of efficient estimates. Now efficient estimates have variances $O(n^{-1})$. Thus, arguing heuristically, both ds and the deviation $dg = g-g^*$ can be taken to be $O(n^{-\frac{1}{2}})$ and we have

$$(23) \quad \frac{s}{\sigma^2 g} + \frac{\sigma^2 g}{s} = \left(1 + \frac{ds}{\sigma^2 g^*}\right)\left(1 + \frac{dg}{g^*}\right)^{-1} + \left(1 + \frac{dg}{g^*}\right)\left(1 + \frac{ds}{\sigma^2 g^*}\right)^{-1}$$

$$= 2 + O\,(n^{-1}).$$

Consequently, on differentiating with respect to a parameter Θ, we have

$$\frac{s}{g^2}\;\frac{dg}{d\Theta} \sim \frac{1}{s}\;\frac{dg}{d\Theta}\,.$$

The approximation is better than one might have expected a priori owing to the cancelling out of the terms of $O(n^{-\frac{1}{2}})$ in (23). The effect is that instead of minimizing S we may minimize

$$(24) \qquad S^* = \frac{g_o}{s_o} + 2\sum_{j=1}^{k}\frac{g_j}{s_j}\,,$$

wherever this is more convenient, e.g. for parameters occurring in the numerator of $g(\omega)$ such as the δ's in (18). We therefore have in addition to (21) the alternative or complementary estimating equations

$$\frac{\hat{g}_0}{s_0} + 2 \sum_{j=1}^{k} \frac{\hat{g}_j}{s_j} = \text{min.} \tag{25}$$

We are able to ignore the σ^2 in (23) partly because it is constant for all j and partly because the efficient estimate of σ^2 has been shown by Whittle (1953) to be uncorrelated with efficient estimates of the other parameters.

(25) gives the spectral form of the method of fitting moving-average models proposed elsewhere (Durbin, 1959 a) and elaborated later (Durbin, 1959b) for models of type (6); these papers may be consulted for a justification of the transition from (21) to (25) which is quite different from that advanced here.

We now have the basis for an iterative method for estimating the parameters of (18). We use (21) for estimating the γ's for given values of δ's and (25) for estimating the δ's for given values of the γ's. What we require in addition is a set of starting values.

In this situation the simplest expedient is to begin by supposing that all the parameters in one set, say the δ's, are zero. However, this suggestion has the drawback that we start with an inconsistent set of estimates. Alternatively we may use the result

$$\rho_{p+s} + \gamma_1 \rho_{p+s-1} + \dots + \gamma_p \rho_s = 0 \qquad (s = 0, 1, \dots, p-1) \tag{26}$$

obtained by multiplying (6) by u_{t-p-s} and taking expectations. Substituting sample estimates of the ρ's in (26) gives estimates of the γ's which are consistent though not, unfortunately, very efficient.

A third procedure is to approximate (6) by a moving-average process of order h, where h is large, say

$$u_t = \epsilon_t + \psi_1 \epsilon_{t-1} + \dots + \psi_h \epsilon_{t-h};$$

on multiplying (6) by u_{t-s} $(s=0,1,\dots,p)$ and ϵ_{t-s} $(s=1,\dots,p)$ in turn and taking expectations we then obtain the equations

$$(27) \qquad \sum_{r=0}^{p} \gamma_r \rho_{|s-r|} = \kappa \sum_{r=0}^{p-s-1} \delta_{s+r} \psi_r \qquad (s=0,1,\dots,p)$$

$$(28) \qquad \sum_{r=0}^{p} \gamma_r \psi_{s-r} = \delta_s \qquad (s=1,\dots,p),$$

where $\kappa = \sigma^2/\sigma_u^2$, $\gamma_o = \delta_o = \psi_o = 1$ and $\psi_r = 0$ $(r < 0)$. Sample estimates of $\psi_1,\dots,\psi_p$ can be obtained by fitting a sufficiently long moving-average to the data using (25). Inserting these and estimates of $\rho_1,\dots,\rho_p$ in (27) and (28) we obtain consistent estimates of the parameters. These will probably be more efficient than the preceding ones but are not fully efficient.

Whatever starting values are used the iterative method proceeds as follows. Suppose that at the beginning of the m-th round of the iteration we have estimates $d_1^{m-1}, \dots, d_{p-1}^{m-1}$ of $\delta_1,\dots,\delta_{p-1}$ and wish to estimate the γ's next. Let $D_j^{m-1} = |1+d_1^{m-1} e^{i\omega_j} + \dots + d_{p-1}^{m-1} e^{(p-1)i\omega_j}|^2$ where $\omega_j = 2\pi j/(2k+1)$. On employing (21) for $\gamma_1,\dots,\gamma_p$ we obtain the equations

$$(29) \qquad c_1^m \sum \frac{s_j \cos(r-1)\omega_j}{D_j^{m-1}} + c_2^m \sum \frac{s_j \cos(r-2)\omega_j}{D_j^{m-1}} + \ldots$$

$$+ c_p^m \sum \frac{s_j \cos(r-p)\omega_j}{D_j^{m-1}} = -\sum \frac{s_j \cos r\omega_j}{D_j^{m-1}} \quad (r=1,\ldots,p),$$

for the mth round estimates $c_1^m, \ldots, c_p^m$ of $\gamma_1, \ldots, \gamma_p$, where $\sum w_j$ denotes $w_o + 2\,(w_1+\ldots+w_k)$.

Similarly the m-th round estimates $d_1^m, \ldots, d_{p-1}^m$ of the δ's are given by

$$(30) \qquad d_1^m \sum \frac{\cos(r-1)\omega_j}{s_j\ C_j^m} + \ldots + d_{p-1}^m \sum \frac{\cos(r-p+1)\omega_j}{s_j\ C_j^m} = -\sum \frac{\cos r\omega_j}{s_j\ C_j^m} (r=1,\ldots,p-1),$$

where $\quad C_j^m = |1+c_1^m e^{i\omega_j} + \ldots + c_p^m e^{pi\omega_j}|^2.$

Alternatively, in place of (30) it may be found advantageous to use the equations

$$(31) \qquad d_1^m \sum \frac{s_j C_j^m \cos(r-1)\omega_j}{(D_j^{m-1})^2} + \ldots + d_{p-1}^m \sum \frac{s_j C_j^m \cos(r-p+1)\omega_j}{(D_j^{m-1})^2} = -\sum \frac{s_j C_j^m \cos r\omega_j}{(D_j^{m-1})^2}$$

$$(r=1,\ldots,p-1).$$

obtained by using (21) for the δ's. (31) has the advantage that it is obtained directly from the original loglikelihood.

4. Complex roots.

A special word should perhaps be included about the consequences of the occurence of complex values of μ_j in (5). In theory the results of section 2 apply to complex values just as much as to real values. In practice, however, one does not calculate complex roots explicitly but works instead with the relevant quadratic factors. Taking a particular pair of conjugate-complex roots μ and $\bar{\mu}$ the contribution to (8) has the form

$$\frac{H}{1-\mu z} + \frac{\bar{H}}{1-\bar{\mu} z} + \frac{H}{1-\mu z^{-1}} + \frac{\bar{H}}{1-\bar{\mu} z^{-1}} .$$

Putting $H = x + iy$ and $\mu = re^{i\theta}$, the first two terms give

$$\frac{2\{x - (x\cos\theta + y\sin\theta)rz\}}{1 - 2rz\cos\theta + r^2z^2} .$$

The linear function of z in the numerator is obtained by replacing z^2 in $(1-2rz\cos\theta + r^2z^2)\,R(z)$ by $(2rz\cos\theta-1)/r^2$, or by Euclid's algorithm.

The values of λ corresponding to μ and $\bar{\mu}$ are $\log r + i\theta$ and $\log r - i\theta$; they contribute to (4) an amount

$$(32)\quad \begin{aligned} &-\frac{1}{\pi}\left\{\frac{(x+iy)(\log r+i\theta)}{\omega^2+(\log r+i\theta)^2} + \frac{(x-iy)(\log r-i\theta)}{\omega^2+(\log r-i\theta)^2}\right\} \\ &= -\frac{2\{x\log r\,(\log^2 r+\theta^2+\omega^2) + y\theta\,(\log^2 r+\theta^2-\omega^2)\}}{\pi\{\log^4 r+2\log^2 r(\theta^2+\omega^2) + (\theta^2-\omega^2)^2\}} .\end{aligned}$$

5. Estimation of the coefficients of the associated stochastic differential equation.

It is known that a series generated by the stochastic differential equation

$$(33)\qquad (1 + \alpha_1 D + \dots + \alpha_p D^p)\,u_t = (1 + \beta_1 D + \dots + \beta_{p-1} D^{p-1})\,\xi_t,$$

where D denotes the differentiation operator $\frac{d}{dt}$ and ξ_t is a white noise element such that $\int_{t_1}^{t_2} \xi_t dt$ has mean zero, variance $\sigma^2(t_2-t_1)$ and is uncorrelated with $\int_{t_3}^{t_4} \xi_t dt$ for $t_1 < t_2 < t_3 < t_4$, has the spectral density (1) (see Phillips, 1959). Conversely there is a stochastic differential-equation model (33) corresponding to any continuous series having spectral density (1). Let us therefore consider the calculation of the coefficients of (33) given the spectral density in the form (4).

The α's emerge immediately since $\lambda_1, \ldots, \lambda_p$ were defined in section 2 as the roots of the equation $1 + \alpha_1 x + \ldots + \alpha_p x^p = 0$. Thus

$$(34) \qquad 1 + \alpha_1 x + \ldots + \alpha_p x^p = \frac{\prod_{j=1}^{p} (\lambda_j - x)}{\prod_{j=1}^{p} \lambda_j} .$$

To obtain the β's express $\sum_{j=1}^{p} H_j \lambda_j \prod_{\ell \neq j}^{p} (\omega^2 + \lambda_\ell^2)$

as a polynomial in ω^2, say $h(\omega^2)$. Denote the roots of $h(\omega^2) = 0$ by $\omega_1^2, \ldots, \omega_{p-1}^2$. Let z_j be the solution of the equation $z_j^2 \omega_j^2 + 1 = 0$, which has negative real part. Then

$$(35) \qquad 1 + \beta_1 x + \ldots + \beta_{p-1} x^{p-1} = \prod_{j=1}^{p-1} (1 - z_j x).$$

Inserting estimates of the H's and λ's we obtain estimates of the α's and β's.

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