A NOTE ON THE RENEWAL FUNCTION WHEN THE MEAN RENEWAL LIFETIME IS INFINITE

by

Walter L. Smith

University of North Carolina

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Institute of Statistics
Mimeograph Series No. 255
April, 1960

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1. Introduction.

To avoid elaboration we shall make use, without further explanation, of the notation and terminology of Smith (1958). The present remarks arise from the consideration of the asymptotic behaviour of the renewal function $H(x)$ when, as $x \to \infty$, either

$$1 - F(x) \sim \frac{1}{x} \quad , \tag{1.1}$$

or

$$1 - F(x) \sim \frac{1}{\log x} \quad . \tag{1.2}$$

Far from being of purely academic interest, such renewal processes arise in connection with certain problems in wireless telegraphy*.

The renewal processes envisaged by (1.1) and (1.2) are such that μ_1, the mean lifetime of a renewal, is infinite; the only general result available for this case is the not particularly informative one that $H(x) = o(x)$, as $x \to \infty$ (a consequence of the Elementary Renewal Theorem). More specific information is available if more is assumed about $F(x)$ than the bare fact that $\mu_1 = \infty$, however. To explain further we need to introduce

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*Private communication from D. S. Palmer, Mathematics and Systems Analysis Research Group, Marconi's Wireless Telegraph Company Limited.

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<u>functions of slow growth</u>. The function L(x), defined for positive x, is said to be a function of slow growth if for every fixed c > 0

$$\frac{L(cx)}{L(x)} \longrightarrow 1 \quad , \quad \text{as } x \longrightarrow \infty.$$

Typical functions of slow growth are log x, log log x, $(\log x)^2$ log log x. Having explained what is meant by a function of slow growth we can now state

<u>Theorem A.</u> <u>If $1 - F(x) = x^{-\alpha}L(x)$, $0 < \alpha < 1$, where L(x) is a function of slow growth, then as $x \longrightarrow \infty$,</u>

$$H(x) \sim \frac{x^{\alpha} \sin \alpha\pi}{\alpha\pi L(x)} \qquad (1.3)$$

This theorem is a slight specialization of a result given by Smith (1955, Lemma 4), but its proof is really due to Feller (1949)* and makes use of Tauborian theorems. Unfortunately the restriction placed on the index α prevents Theorem A from being applied to the cases (1.1) and (1.2) of present interest, for which α = 1 and α = 0, respectively.

In the present short note we shall first obtain three inequalities of general interest, which are valid for <u>any</u> renewal process. From these inequalities we shall deduce the following theorem, which compliments Theorem A by covering the cases α = 0 and α = 1.

<u>Theorem 1.</u> (i) <u>A necessary and sufficient condition for the validity of the asymptotic relation</u>

*Feller (1949) actually discusses the corresponding theorem for discrete-time renewal processes, and does not allow for the function L(x) of slow growth. Only routine modifications of his proof are needed, to establish Theorem A, however.

$$H(x) \sim \frac{x^{\nu}}{L(x)} \quad , \quad \text{as } x \to \infty, \tag{1.4}$$

where $\nu = 0$ or 1 and $L(x)$ is a function of slow growth, is that

$$\frac{1}{x} \int_{0}^{x} \left\{ 1 - F(z) \right\} dz = \frac{L(x)}{x^{\nu}} \quad , \text{ as } x \to \infty. \tag{1.5}$$

(ii) When $\nu = 0$ the necessary and sufficient condition (1.5) is equivalent to the simpler condition

$$\left\{ 1 - F(x) \right\} \sim L(x), \tag{1.6}$$

so that in this case $H(x) \sim 1 / \left\{ 1 - F(x) \right\}$.

(iii) When $\nu = 1$, a sufficient condition for $H(x) / x$ to be a function of slow growth is that $x \left\{ 1 - F(x) \right\}$ be a function of slow growth.

Clearly this theorem applies to the particular renewal processes for which (1.1) or (1.2) holds. When (1.1) is true we can infer that $H(x) \sim x / \log x$; when (1.2) is true we can infer that $H(x) \sim \log x$.

It is noteworthy that Theorem 1 can be proved by quite elementary arguments which provide necessary and sufficient conditions and which make no appeal to deep Tauberian theorems. The present elementary methods seem inadequate to deal with renewal processes covered by Theorem A, however; but it must be remarked that the Tauberian theorems used to prove Theorem A cannot deal with the present cases.

There is an interesting and instructive analogy between the present theorem, covering certain renewal processes with infinite mean μ_1, and the familiar Elementary Renewal Theorem for processes with μ_1 finite. If X_n is a typical variable with distribution function $F(x)$, let $X_n^{\dagger}$ be the associated truncated variable

$$X_n^{\dagger} = X_n \quad \text{if } X_n \leq x \quad , \qquad\qquad (1.7)$$
$$\quad = x \quad \text{otherwise} \quad .$$

If we write $\mu(x) = \mathcal{E} X_n^{\dagger}$ then

$$\mu(x) = \int_0^x \{1 - F(z)\}\, dz$$

and so, when (1.5) holds, one can write the conclusion (1.4) in the form

$$H(x) \sim \frac{x}{\mu(x)} \quad , \qquad\qquad (1.8)$$

which is valid for both the case $\nu = 0$ and the case $\nu = 1$. Relation (1.8) is to be compared with the relation $H(x) \sim x/\mu_1$ which holds when μ_1 is finite.

For a very full discussion of functions of slow growth the reader is referred to Karamata (1930). It seemed worthwhile to make the present note self-contained and so we do not appeal to any of Karamata's general theorems. This note could have been very slightly shortened in one place (the proof of Lemma 5) by such an appeal, but the special properties of the functions we deal with allow a simple proof from first principles, so we give this. Nevertheless we wish to acknowledge indebtedness for the understanding of functions of slow growth which we have gained by reading Karamata's paper.

2. Some Fundamental inequalities

In this section we prove certain inequalities which are valid for any renewal process and which will later be used in the proof of Theorem 1.

Lemma 1. For any renewal process

$$H(x) < \frac{1}{1 - F(x)} ,$$

for all x.

Proof. Let $\{X_n\}$ be the renewal process and suppose $F(x) < 1$, otherwise the lemma is trivial. Define N_x, as usual, as the maximum k such that $S_k = X_1 + X_2 + \ldots + X_k \leq x$, with the proviso that if $X_1 > x$ then N_x is defined to be zero. Define M_x as the smallest k for which $X_k > x$. By a familiar property of the geometric distribution ,

$$\mathcal{E} M_x = \frac{1}{1 - F(x)} . \tag{2.1}$$

Since $H(x) = \mathcal{E} N_x$ and, with probability one, $N_x < M_x$, the lemma follows from (2.1).

Lemma 2. For any renewal process

$$\liminf_{x = \infty} \frac{H(x)}{x} \int_0^x \{1 - F(z)\} \, dz \geq 1.$$

Proof: Let $\{X_n\}$ be the renewal process and $\{X_n^+\}$ the associated renewal process of truncated variables, as defined in (1.7). If an obvious notation is employed then (1.1) of Smith (1958) can be written

$$x + \mathcal{E} \zeta_x^+ = \{1 + H^+(x)\} \int_0^x \{1 - F(z)\} \, dz . \tag{2.2}$$

It is worth mentioning that (2.2) is a fairly easy deduction from the law of large numbers.

Evidently the truncation procedure cannot affect $H(t)$ for $t < x$, so that $H^{+}(t) = H(t)$ for all $t < x$. However, an effect of the truncation is to increase, by the amount $1 - F(x + 0)$, the probability of a renewal taking place at x. Thus, since renewal functions are customarily taken as continuous to the right,

$$H^{+}(x) = H(x) + \frac{1 - F(x + 0)}{1 - F(0 +)} \quad . \tag{2.3}$$

Since $\zeta_x^{+} \geq 0$ it obviously follows from (2.2) and (2.3) that

$$\frac{1}{x} \left\{ 1 + H(x) + \frac{1 - F(x+0)}{1 - F(0+)} \right\} \int_0^x \left\{ 1 - F(z) \right\} dz \geq 1. \tag{2.4}$$

The function $\{1 - F(x)\}$ decreases to zero as $x \to \infty$ so that $x^{-1} \int_0^x \{1 - F(z)\} \, dz$ also decreases to zero as $x \to \infty$. Thus Lemma 2 is an immediate consequence of (2.4).

<u>Lemma 3</u>. <u>If $0 < \epsilon < 1$, then for any renewal process</u>

$$\limsup_{x = \infty} \frac{H(x)}{x} \int_0^{\epsilon x} \left\{ 1 - F(z) \right\} dz \leq (1 + \epsilon).$$

<u>Proof</u>. Let X_n^{+} now represent variables truncated according to the rule

$$X_n^{+} = X_n \qquad \text{if } X_n \leq \epsilon x$$

$$= \epsilon x \qquad \text{otherwise.}$$

For the new truncated variables,

$$\mathcal{E} X_n^{+} = \int_0^{\epsilon x} \left\{ 1 - F(z) \right\} dz \, ,$$

so that (2.2) must be rewritten

$$x + \zeta_x^{+} = \{1 + H^{+}(x)\}\int_0^{\epsilon x} \{1 - F(z)\}\, dz. \tag{2.5}$$

Because the partial sums of the "truncated" renewal process $\{X_n^{+}\}$ never exceed those of the "untruncated" process $\{X_n\}$, it is clear that $H^{+}(x) \geq H(x)$. Furthermore it is plain that $\zeta_x^{+} \leq \epsilon x$. Thus (2.5) implies that

$$\frac{1 + H(x)}{x}\int_0^{\epsilon x} \{1 - F(z)\}\, dz \leq (1 + \epsilon) \quad . \tag{2.6}$$

On repeating the observation that $x^{-1}\int_0^x \{1 - F(z)\}\, dz$ decreases to zero as $x \to \infty$, it is seen that Lemma 3 follows directly from (2.6).

<u>Lemma 4.</u> <u>If $H(x)$ is any renewal function and α any number such that $H(\alpha) > 0$ then</u>

$$\int_\alpha^\infty \frac{dy}{H(y)} = \infty \quad . \tag{2.7}$$

<u>Proof.</u> Let $\{X_n\}$ be the renewal process yielding $H(x)$ and let $\{X_n^{+}\}$ be an associated "truncated" process defined, this time, by

$$X_n^{+} = X_n \text{ if } X_n \leq 1$$
$$= 1 \text{ otherwise.}$$

Clearly $\mathcal{E}X_n^{+} < \infty$ and by the Elementary Renewal Theorem, if $H^{+}(x)$ is the renewal function associated with $\{X_n^{+}\}$, $H^{+}(x) \sim x/\mathcal{E}X_n^{+}$ as $x \to \infty$. Lemma 4 follows from the remark that $H(x) \leq H^{+}(x)$, the justification for which has already been given in the proof of Lemma 3.

3. Proof of Theorem 1 for the case when $\nu = 0$.

We prove first

Lemma 5. *If the renewal function $H(x)$ is a function of slow growth and α is any constant such that $H(\alpha) > 0$ then*

$$\frac{1}{x}\int_{\alpha}^{x}\frac{dy}{H(y)} \sim \frac{1}{H(x)} \quad , \text{ as } x \to \infty \; . \tag{3.1}$$

Proof. Fix ϵ such that $0 < \epsilon < 1$. Since $H(x) \,/\, H(x\epsilon) \to 1$ as $x \to \infty$, and since, by Lemma 4, the relevant integrals diverge, it is easy to see that

$$\frac{\displaystyle\int_{\alpha}^{x}\frac{dy}{H(y)}}{\displaystyle\int_{\alpha/\epsilon}^{x}\frac{dy}{H(\epsilon y)}} \to 1 \quad , \text{ as } x \to \infty .$$

On changing the variable of integration in the denominator of the last expression it can be seen that

$$\frac{\displaystyle\int_{\alpha}^{\epsilon x}\frac{dy}{H(y)} + \int_{\epsilon x}^{x}\frac{dy}{H(y)}}{\displaystyle\int_{\alpha}^{\epsilon x}\frac{dy}{H(y)}} \to \frac{1}{\epsilon} \; ,$$

and therefore that

$$\int_{a}^{\epsilon x} \frac{dy}{H(y)} \sim \frac{\epsilon}{(1-\epsilon)} \int_{\epsilon x}^{x} \frac{dy}{H(y)} .$$

Plainly, this last asymptotic relation implies that

$$\int_{a}^{x} \frac{dy}{H(y)} \sim \frac{1}{(1-\epsilon)} \int_{\epsilon x}^{x} \frac{dy}{H(y)} ,$$

so that

$$\frac{H(x)}{x} \int_{a}^{x} \frac{dy}{H(y)} \sim \frac{H(x)}{(1-\epsilon)x} \int_{\epsilon x}^{x} \frac{dy}{H(y)}$$

$$\sim \frac{1}{(1-\epsilon)} \int_{\epsilon}^{1} \frac{H(x)}{H(xu)} du , \qquad (3.2)$$

after a change of variable. For fixed u, $H(x)/H(xu) \to 1$ as $x \to \infty$; but H(x) is monotonic increasing, so the integrand on the right of (3.2) is always dominated by $H(x) / H(\epsilon x)$; thus an appeal to Lebesgue's theorem on bounded convergence proved Lemma 5 from (3.2).

We turn now to the proof of the main theorem (for $\nu = 0$), and deal with the necessity part first. The starting assumption is that $H(x) \sim 1/L(x)$, so that H(x) is a function of slow growth. We may assume in what follows that $\int_0^\infty \{1 - F(z)\} dz = \infty$ since the Elementary Renewal Theorem is available otherwise. Furthermore, in view of Lemma 2 we have only to prove

$$\limsup_{x = \infty} \frac{H(x)}{x} \int_{0}^{x} \{1 - F(z)\} dz \leq 1 , \qquad (3.3)$$

for the necessity of condition (1.5) to be established. But Lemma 1 shows that for all sufficiently large x

$$\int_{\alpha}^{x} \{1 - F(z)\}\, dz < \int_{\alpha}^{x} \frac{dz}{H(z)} , \qquad (3.4)$$

if α is any constant such that $H(\alpha) > 0$, it being observed that the integral on the right of (3.4) diverges in accordance with Lemma 4. Thus, for all sufficiently large x,

$$\frac{H(x)}{x} \int_{\alpha}^{x} \{1 - F(z)\}\, dz < \frac{H(x)}{x} \int_{\alpha}^{x} \frac{dz}{H(z)} ,$$

and Lemma 5 shows (3.3) to be a consequence of this last inequality, since $\int_{0}^{\infty} \{1 - F(z)\}\, dz$ diverges and therefore $\int_{\alpha}^{x} \{1 - F(z)\}\, dz \sim \int_{0}^{x} \{1 - F(z)\}\, dz$.

The necessity of (1.6) is now to be proved.

Choose a large positive constant c. Then, by (1.5),

$$\frac{1}{x} \int_{0}^{x} \{1 - F(z)\}\, dz \sim \frac{1}{cx} \int_{0}^{cx} \{1 - F(z)\}\, dz$$

$$\sim \frac{1}{cx} \int_{0}^{x} \{1 - F(z)\}\, dz + \frac{1}{cx} \int_{x}^{cx} \{1 - F(z)\}\, dz$$

so that

$$\frac{1}{x} \int_{0}^{x} \{1 - F(z)\}\, dz \sim \frac{1}{(c-1)x} \int_{x}^{cx} \{1 - F(z)\}\, dz .$$

Thus

$$\limsup_{x=\infty} \frac{x^{-1}\int_0^x \{1 - F(z)\}\, dz}{\{1 - F(x)\}} = \limsup_{x=\infty} \frac{(c-1)^{-1}\; x^{-1}\int_x^{cx}\{1-F(z)\}\, dz}{\{1 - F(x)\}}$$

$$\leq 1.$$

Since, trivially,

$$\liminf_{x=\infty} \frac{x^{-1}\int_0^x\{1-F(z)\}\, dz}{\{1 - F(x)\}} \geq 1,$$

it follows that $x^{-1} \int_0^x \{1 - F(z)\}\, dz \sim \{1 - F(x)\}$ and the necessity of (1.6) is established.

We now turn to the sufficiency part of the proof, and show first that (1.6) implies (1.5). The argument needed here is identical with that used in Lemma 5 concerning the renewal function; the only property of the renewal function that was used in the proof of Lemma 5 was that $1/H(x)$ decreases, and $\{1 - F(x)\}$ also decreases. Thus (1.6) implies (1.5).

To close this section it must be shown that (1.5) implies (1.4). Because of Lemma 2, once again we need only establish the truth of (3.3). But, by Lemma 1,

$$\limsup_{x=\infty} H(x)\{1 - F(x)\} \leq 1 ;$$

and we have just seen that $\{1 - F(x)\} \sim x^{-1} \int_0^x \{1 - F(z)\}\, dz$. Thus (3.3) must hold.

4. Proof of Theorem 1 for the case when $\nu = 1$.

The necessity part will be proved first, so we start with the assumption that $H(x) \sim x/L(x)$. By Lemma 2 it follows that

$$\liminf_{x=\infty} \frac{1}{L(x)} \int_0^x \{1 - F(z)\}\, dz \geq 1, \tag{4.1}$$

and by Lemma 3, if $0 < \epsilon < 1$,

$$\limsup_{x=\infty} \frac{1}{L(x)} \int_0^{\epsilon x} \{1 - F(z)\}\, dz \leq (1 + \epsilon). \tag{4.2}$$

If ϵx is changed to x in (4.2), and it is observed that $L(x) \sim L(x\epsilon^{-1})$, as $x \to \infty$, then it follows that

$$\limsup_{x=\infty} \frac{1}{L(x)} \int_0^x \{1 - F(z)\}\, dz \leq (1 + \epsilon). \tag{4.3}$$

Since ϵ can be arbitraily small, (1.5) follows from (4.1) and (4.3). This completes the necessity proof.

Now suppose (1.5) given, i.e. it is given that $\int_0^x \{1 - F(z)\}\, dz$ is a function of slow growth. Then Lemma 3 and this slow growth property of $\int_0^x \{1 - F(z)\}\, dz$ combine to show that

$$\limsup_{x=\infty} \frac{H(x)}{x} \int_0^x \{1 - F(z)\}\, dz \leq (1 + \epsilon),$$

whenever $0 < \epsilon < 1$. By taking ϵ artibrarily small and appealing to Lemma 2 it is obvious that (1.4) is proved.

Lastly we must show the sufficiency of the condition stated as part (iii) of Theorem 1. If $x\{1 - F(x)\}$ is a function of slow growth it is evident that for any fixed c > 0

$$\frac{\{1 - F(cx)\}}{\{1 - F(x)\}} \longrightarrow \frac{1}{c} \quad , \text{ as } x \longrightarrow \infty \ .$$

Thus, if we assume the relevant integrals to diverge,

$$\frac{c \int_0^x \left\{1 - F(cz)\right\} dz}{\int_0^x \left\{1 - F(z)\right\} dz} \longrightarrow 1 \quad , \text{ as } x \longrightarrow \infty,$$

i.e. $$\int_0^{cx} \left\{1 - F(z)\right\} dz \sim \int_0^x \left\{1 - F(z)\right\} dz \ .$$

This shows $\int_0^x \left\{1 - F(z)\right\} dz$ to be a function of slow growth, i.e. (1.5) is true for some L(x), and so (1.4) follows.

If the integrals which we wished to diverge actually converge then μ_1 must be finite, and the desired conclusion is a simple consequence of the Elementary Renewal Theorem.

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