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ON SOME METHODS OF CONSTRUCTION OF PARTIALLY BALANCED ARRAYS

by

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Methods of construction of partially balanced arrays are considered in this report. Two methods of construction are given. One of them derives partially balanced arrays from $(\lambda - \mu - \nu)$ configurations and the other is an extension of Bose-Shrikhande method of construction of orthogonal arrays.

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ON SOME METHODS OF CONSTRUCTION OF PARTIALLY BALANCED ARRAYS*

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0. Introduction

Suppose $A = ((a_{ij}))$ is a matrix $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots N$ and the elements a_{ij} of the matrix are symbols $0, 1, 2, \ldots, s-1$. Consider the s^t matrices $X^1 = (x_1, x_2, \ldots x_t)$ that can be formed by giving different values to x_i, s, $x_i = 0, 1, 2, \ldots, s-1$; $i = 1, 2, \ldots t$. Suppose associated with each matrix X there is a positive integer $\lambda\ (x_1, x_2, \ldots, x_t)$ which is invariant under permutations of $(x_1, x_s, \ldots, x_t)$. If for every t-rowed submatrix of A, the s^t matrices X occur as columns $\lambda(x_1, x_2, \ldots, x_t)$ times, then the matrix A is called a partially balanced array of strength t in N assemblies, m constraints (or factors), s symbols (or levels) and the specified $\lambda\ (x_1, x_2, \ldots, x_t)$ parameters. When $\lambda(x_1, x_2, \ldots x_t) = \lambda$ for all $(x_1, x_2, \ldots, x_t)$, the array is called an Orthogonal array.

Orthogonal arrays were defined in ([4], [5]) and construction of orthogonal arrays were considered in ([1], [2], [3], [4], [5]). Partially balanced arrays were defined in [6] where their use as multifactorial designs is also discussed.

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In this paper, some methods of construction of partially balanced arrays are considered. One of the methods is applicable when $s = 2$ and derives partially balanced arrays from the well-known $(\lambda - \mu - \nu)$ configurations. The other method is an extension of Bose-Shrikhande [2] method of construction of orthogonal arrays.

1. An example: An example of a partially balanced array of strength 2, $s = 3$, $m = 6$ constraints in $N = 15$ assemblies is given below

Partially balanced array (15, 6, 3, 2)

constraints \ assemblies	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0	0	0	0	1	1	1	1	1	2	2	2	2	2
2	0	2	1	1	2	0	0	1	2	2	0	0	2	1	1
3	1	0	2	2	1	0	2	0	1	2	1	2	0	0	1
4	1	1	0	2	2	2	0	2	0	1	2	1	0	1	0
5	2	2	1	0	1	1	2	2	0	0	0	1	1	0	2
6	2	1	2	1	0	2	1	0	2	0	1	0	1	2	0

This array has the $\lambda(x_1, x_2)$ parameters

$$\lambda(x_1,x_2) = 2 \qquad \text{if } x_1 \text{ and } x_2 \text{ are unlike}$$
$$= 1 \qquad \text{otherwise}$$

This array was constructed by cutting out 3 assemblies and omitting one row from the orthogonal array A (18, 7, 3, 2)

2. Construction of partially balanced arrays for $s = 2$ from $\lambda - \mu - \nu$ configurations.

Definition: A $\lambda - \mu - \nu$ configuration, of m elements is defined [7] as the configuration of m elements taken ν at a time so that each set of μ elements shall occur together in just λ of the sets.

Suppose there are N_o sets of ν elements each in the configuration. Let N_t denote the number of sets each containing a fixed subset of t elements. Then it is easily seen that

$$(2.1) \qquad N_t = \lambda \binom{m-t}{\mu-t} \Big/ \binom{\nu-t}{\mu-t} \qquad t = 0, 1, 2, \ldots \mu$$

Consider the matrix $A = ((a_{ij}))$ of form $(m \times N_o)$ derived from a $\lambda - \mu - \nu$ configuration of m elements in N_o sets in the following manner: Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ denote the m elements and $s_1, s_2, \ldots s_{N_o}$ denote the N_o sets of the configuration. Then let

$$\begin{aligned} a_{ij} &= 1 \quad \text{if } \alpha_i \text{ occurs in the set } s_j \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Consider a μ-rowed submatrix of A with elements a_{ij} as defined above. Amongst the N_o columns of the submatrix, a column matrix $X_{\mu,1}$ where its transpose $X'_{1,\mu} = (x_1, x_2, \ldots, x_\mu)$, $x_i = 0$ or 1 $i = 1, 2, \ldots, \mu$ occurs $\lambda(x_1, x_2, \ldots, x_\mu)$ times. Specifically, let $x_i = 1$ for $i = 1, 2, \ldots, r$; $x_i = 0$ for $i = r+1, \ldots, \mu$ in X. Then it is easy to show that for such an X

$$(2.2) \qquad \lambda(x_1, x_2, \ldots, x_\mu) = N_r - \binom{\mu-r}{1} N_{r+1} + \binom{\mu-r}{2} N_{r+2} - \cdots$$

$$= (-1)^{\mu-r} \Delta^{\mu-r} N_r$$

where Δ stands for the symbol of finite difference, viz.,

$$\Delta N_r = N_{r+1} - N_r .$$

Value of $\lambda(x_1, x_2, \ldots, x_\mu)$ depends only on the count r of unities in its argument and hence it is invariant under permutation of its arguments.

Now provided $\lambda(x_1, x_2, \ldots, x_\mu) > 0$ for all s^μ sets of X, we have

Theorem 2:1 : The existence of a $(\lambda - \mu - \nu)$ of m elements with $\lambda\,(x_1, x_2, \ldots, x_\mu)$ all positive, implies the existence of a partially balanced array of strength μ with parameters $s = 2$ and $\lambda(x_1, x_2, \ldots, x_\mu)$ as defined in (2.2).

Well-known examples of $\lambda - \mu - \nu$ configurations are the triple systems, quadruple systems, etc., which are defined in [7].

3. An extension of Bose-Shrikhande method of construction of orthogonal arrays and its use in the construction of partially balanced arrays.

Definition: A pairwise partially balanced design with parameters $(\nu, k_1, k_2, \ldots, k_m; b_1, b_2, \ldots, b_m; \lambda_1, \lambda_2, \ldots, \lambda_t; n_1, n_2, \ldots, n_t)$ is defined as an arrangement of ν varieties in blocks of m different sizes $k_1, k_2, \ldots, k_m$, there being b_i blocks of size k_i,

$$\sum_{i=1}^{m} b_i = b,$$ satisfying the following conditions:

(i) No block contains a single variety more than once.

(ii) With respect to any variety, the remaining $(\nu-1)$ varieties fall into t categories, there being n_i varieties in the i^{th} category, called the i^{th} associates of the variety; $\sum_{i=1}^{t} n_i = \nu - 1$

(iii) Two varieties which are i^{th} associates, occur together in λ_i blocks, $i = 1, 2, \ldots, t$.

Then the following relations among the parameters hold,

$$\sum_{i=1}^{m} b_i k_i (k_i - 1) = \sum_{i=1}^{t} n_i \nu \lambda_i = \nu \sum_{i=1}^{t} n_i \lambda_i \tag{3.1}$$

Suppose there exist the orthogonal arrays

$$A_i\ (\lambda k_i^2, q_i, k_i, 2) \qquad i = 1, 2, \ldots, m$$

of strength two and index λ and in k_i symbols. Consider the pairwise partially balanced design defined earlier. There are b_i blocks each of size k_i. These b_i blocks provide b_i sets of k_i symbols each. Using each set of k_i symbols once in the orthogonal array A_i, one gets b_i such orthogonal arrays. If all such orthogonal arrays are arranged side by side, then one gets a matrix A with number of columns $N = \lambda \sum_{i=1}^{m} b_i k_i^2$ and number of rows $q = \min (q_1, q_2, \ldots, q_m)$. In any two-rowed submatrix of matrix A, every ordered pair $\binom{t_u}{t_v}$ of two distinct symbols of varieties which are ith associates will occur $\lambda\lambda_i$

times and every ordered pair $\binom{t_j}{t_j}$ of two like symbols occur λr_j times, if the variety t_j occurs in r_j blocks of the pairwise partially balanced design. Hence the

Theorem 3.1: The existence of a pairwise partially balanced design with parameters $(\nu; k_1, k_2, \ldots k_m; b_1, b_2, \ldots b_m, \lambda_1, \lambda_2, \ldots \lambda_t, n_1, n_2, \ldots, n_t)$ and of the orthogonal arrays $A_i(\lambda k_i^2, q_i, k_i, 2)$ $i = 1, 2, \ldots, m$, imply the existence of the partially balanced array of strength two in ν symbols and $q = \min(q_1, q_2, \ldots, q_m)$ constraints and $\lambda(x_1, x_2) = \lambda\lambda_i$ where x_1, x_2 stand for two varieties which are ith associates and $\lambda(x,x) = \lambda r_j$ where the variety x occurs r_j times in the pairwise partially balanced design.

As an illustration, a partially balanced array which has been constructed using the method described above, is given below. This is a partially balanced array in $\nu = 6$ symbols, $N = 48$ assemblies $m = 5$ constraints and

$$\lambda(x_1,x_2) = 2 \text{ if } (x_1,x_2) \text{ are first associates}$$

$$= 1 \text{ if } (x_1,x_2) \text{ are second associates}$$

$$= 2 \text{ if } x_1 \text{ and } x_2 \text{ are like}$$

where x_i, $i = 1, 2, \ldots, 6$ are the variety symbols.

In constructing this array, the partially balanced design

($\nu = 6$, $r = 2$, $b = 3$, $k = 4$, $n_1 = 1$, $n_2 = 4$, $\lambda_1 = 2$, $\lambda_2 = 1$) in

three blocks

x_1, x_4, x_2, x_5

x_2, x_5, x_3, x_6

x_3, x_6, x_1, x_4

and the orthogonal array A (16, 5, 4, 2) have been used.

Orthogonal Array A (16, 5, 4, 2)

R	0	0	0	0	1	1	1	1	t	t	t	t	t^2	t^2	t^2	t^2
C	0	1	t	t^2	0	1	t	t^2	0	1	t	t^2	0	1	t	t^2
L_1	0	1	t	t^2	1	0	t^2	t	t	t^2	0	1	t^2	t	1	0
L_2	0	1	t	t^2	t	t^2	0	1	t^2	t	1	0	1	0	t^2	t
L_3	0	1	t	t^2	t^2	t	1	0	1	0	t^2	t	t	t^2	0	1

Making successively the identifications

(1)	(2)	(3)
$1 = x_1$	$1 = x_2$	$1 = x_1$
$t = x_2$	$t = x_3$	$t = x_3$
$t^2 = x_4$	$t^2 = x_5$	$t^2 = x_4$
$0 = x_5$	$0 = x_6$	$0 = x_6$

and using them on the above array in place of $(0, 1, t, t^2)$ one gets three arrays - say A_1, A_2, A_3. Then the array $A_0 = [A_1 \quad A_2 \quad A_3]$. is the desired partially balanced array in 6 symbols and 48 assemblies.

Let A^* denote the array derived from A by truncating the first row and the first four columns (as indicated by the horizontal and vertical lines). Then the arrays A_1^*, A_2^* and A_3^* are obtained from A^* using the three identifications of variety -symbols given above. Let E denote the array

$$E : \begin{bmatrix} x_1 & x_2 & \cdots & \cdots & x_6 \\ x_1 & x_2 & \cdots & \cdots & x_6 \\ x_1 & x_2 & \cdots & \cdots & x_6 \\ x_1 & x_2 & \cdots & \cdots & x_6 \\ x_1 & x_2 & \cdots & \cdots & x_6 \\ x_1 & x_2 & \cdots & \cdots & x_6 \end{bmatrix}$$

Then the array $A_o^* = [E \quad A_1^* \quad A_2^* \quad A_3^*]$ is a partially balanced array in $v = 6$ symbols, $N = 42$ assemblies, $m = 4$ constraints and $\lambda(x,x) = 1$, $\lambda(x_i,x_j) = 2$ if x_i and x_j are first associates and $\lambda(x_i,x_j) = 1$ if they are second associates.

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