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ON CERTAIN ALTERNATIVE HYPOTHESES ON DISPERSION MATRICES

by

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ABSTRACT

For two multivariate nonsingular normal distributions, the familiar null hypothesis of equal dispersion matrices is considered against various alternatives stated in terms of certain characteristic roots. Based on two independent random samples from the two distributions, similar region tests are proposed for the null hypothesis against each of the alternative hypotheses. Also, for each case, conservative confidence bounds are obtained on one or more parametric functions which measure departure from the null hypothesis in the direction of the corresponding alternative. Finally, a physical interpretation is given for the alternative hypotheses considered.

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1. INTRODUCTION

For two nonsingular p-variate normal distributions, $N[\underline{\mu}_1, \Sigma_1]$ and $N[\underline{\mu}_2, \Sigma_2]$, the familiar null hypothesis $H_o: \Sigma_1 = \Sigma_2$ is considered. H_o can be stated also in the form, H_o: all $c\left(\Sigma_1 \Sigma_2^{-1}\right) = 1$, where $c(A)$ denotes the characteristic roots of the square matrix A. As alternatives, however, the following are considered: (i) H_1: all $c\left(\Sigma_1 \Sigma_2^{-1}\right) > 1$; (ii) H_2: all $c\left(\Sigma_1 \Sigma_2^{-1}\right) < 1$; (iii) H_3: all $c\left(\Sigma_1 \Sigma_2^{-1}\right) > 1$ or < 1; (iv) H_4: at least one $c\left(\Sigma_1 \Sigma_2^{-1}\right) > 1$; (v) H_5: at least one $c\left(\Sigma_1 \Sigma_2^{-1}\right) < 1$; (vi) H_6: at least one $c\left(\Sigma_1 \Sigma_2^{-1}\right) > 1$ and at least one $c\left(\Sigma_1 \Sigma_2^{-1}\right) < 1$; (vii) H_7: at least one $c\left(\Sigma_1 \Sigma_2^{-1}\right) > 1$ or < 1. It may be noted that (iii) is the union of (i) and (ii), (vi) is the intersection of (iv) and (v), (vii) is the union of (iv) and (v), and (vi) is the complement of (iii). Also, while each of the alternatives is mutually exclusive with H_o, yet, only (vii) is the complement of H_o, and it is only (vii) that has attracted attention heretofore [2,5,10]. The relations in logical structure between the various alternatives may be useful in understanding the forms of the tests proposed in Section 2 of this paper for H_o against each of the alternatives. Section 3 presents the confidence bounds associated with each of the tests. Section 4, the final one, aside from other things, discusses certain possible physical meanings of the alternatives considered here.

2. TESTS FOR H_o AGAINST EACH OF THE ALTERNATIVES OF SECTION 1

Let S_1 and S_2 be the two (pxp) sample dispersion matrices based, respectively, on idependent random samples of sizes (n_1+1) and (n_2+1) from the two populations. We assume that $p \leq$ the smaller of n_1 and n_2, so that, S_1 and S_2 are positive definite almost everywhere. Let $c_M(A)$ and $c_m(A)$ denote, respectively, the largest and the smallest $c(A)$. Then, using a heuristic argument similar to that in [5], the following tests are proposed, wherein $\omega(H)$ denotes the acceptance region for the hypothesis H and $\omega(I)$, where it occurs, denotes the region of indecision or no discrimination between the two hypotheses in question:

$$\begin{aligned}
&(i) \quad \omega(H_o): \ c_M\left(S_1S_2^{-1}\right) \leq \lambda_1; \ \omega(H_1): \ c_m\left(S_1S_2^{-1}\right) > \lambda_1; \\
&\qquad \omega(I): \ c_m\left(S_1S_2^{-1}\right) \leq \lambda_1 < c_M\left(S_1S_2^{-1}\right), \\
&(ii) \quad \omega(H_o): \ c_m\left(S_1S_2^{-1}\right) \geq \lambda_2; \ \omega(H_2): \ c_M\left(S_1S_2^{-1}\right) < \lambda_2; \\
&\qquad \omega(I): \ c_m\left(S_1S_2^{-1}\right) < \lambda_2 \leq c_M\left(S_1S_2^{-1}\right),
\end{aligned} \tag{2.1}$$

(Cont'd)

(iii) $\omega(H_o)$: $\lambda_3 \leq c_m\left(S_1 S_2^{-1}\right) \leq c_M\left(S_1 S_2^{-1}\right) \leq \lambda_3'$;

$\omega(H_3)$: $c_m\left(S_1 S_2^{-1}\right) > \lambda_3'$, or, $c_M\left(S_1 S_2^{-1}\right) < \lambda_3$;

$\omega(I)$: $c_m\left(S_1 S_2^{-1}\right) < \lambda_3 \leq c_M\left(S_1 S_2^{-1}\right)$ and/or

$c_m\left(S_1 S_2^{-1}\right) \leq \lambda_3' < c_M\left(S_1 S_2^{-1}\right)$,

(iv) $\omega(H_o)$: $c_M\left(S_1 S_2^{-1}\right) \leq \lambda_4$; $\omega(H_4)$: $c_M\left(S_1 S_2^{-1}\right) > \lambda_4$,

(v) $\omega(H_o)$: $c_m\left(S_1 S_2^{-1}\right) \geq \lambda_5$; $\omega(H_5)$: $c_m\left(S_1 S_2^{-1}\right) < \lambda_5$,

(vi) $\omega(H_o)$: $\lambda_6 \leq c_m\left(S_1 S_2^{-1}\right) \leq c_M\left(S_1 S_2^{-1}\right) \leq \lambda_6'$;

$\omega(H_6)$: $c_m\left(S_1 S_2^{-1}\right) < \lambda_6$ and $c_M\left(S_1 S_2^{-1}\right) > \lambda_6'$;

$\omega(I)$: $c_m\left(S_1 S_2^{-1}\right) < \lambda_6$ and $c_M\left(S_1 S_2^{-1}\right) \leq \lambda_6'$, or,

$c_m\left(S_1 S_2^{-1}\right) \geq \lambda_6$ and $c_M\left(S_1 S_2^{-1}\right) > \lambda_6'$,

(vii) $\omega(H_o)$: $\lambda_7 \leq c_m\left(S_1 S_2^{-1}\right) \leq c_M\left(S_1 S_2^{-1}\right) \leq \lambda_7'$;

$\omega(H_7)$: $c_m\left(S_1 S_2^{-1}\right) < \lambda_7$ and/or $c_M\left(S_1 S_2^{-1}\right) > \lambda_7'$.

(2.1)

For Case (i), given λ_1 the probabilities assigned to the three regions, $\omega(H_o)$, $\omega(H_1)$ and $\omega(I)$, under H_o can be determined. Likewise, given the probability assigned to the region $\omega(H_o)$ under H_o, λ_1 can be determined by the methods described in [3,4], and hence the probabilities assigned to $\omega(H_1)$ and $\omega(I)$ under H_o may be determined. It should be noted that the method of evaluating the probability assigned to the region $\omega(I)$ under H_o, for a given λ_1, has not been explicitly considered. The authors, however, feel that this will not present any essentially new difficulty and that the methods of [3,4] will be applicable to this problem also.

Similar remarks hold concerning the determination of the other λ's, in Cases (ii)-(vii), under (2.1). For Cases (iii), (vi) and (vii), where we have two constants to determine since the tests are two sided in each of these cases, in addition to the conditions of a given probability for $\omega(H_0)$ under H_0, we may impose the condition of local unbiasedness of each of these tests. These two conditions taken together will enable us to determine both constants involved uniquely. As discussed in [3,5,9], for Case (vii), the condition of local unbiasedness implies certain optimum power properties of the test for this case. For the other two cases, however, such implications of the condition of local unbiasedness are yet to be established. Further, regarding all the λ's in (2.1), it should be noted that, in addition to depending on the conditions discussed above, they are also functions of p, n_1 and n_2.

Case (vii), as noted in Section 1, with the test given under (vii) of (2.1), is the one that has been considered in great detail elsewhere [5,6,7,8] and is included here merely for completeness.

Finally, it can be seen that all the tests proposed under (2.1) are similar region tests.

3. ASSOCIATED CONFIDENCE BOUNDS

The confidence bounds obtained in this section are, in each case, on one or more parametric functions which are measures of departure from H_o in the direction of the corresponding alternative hypothesis. For instance, in Case (i), where the alternative hypothesis is that all $c\left(\Sigma_1\Sigma_2^{-1}\right) > 1$, the confidence bound sought is a lower bound on $c_m\left(\Sigma_1\Sigma_2^{-1}\right)$. If, on the other hand, one is interested in the alternative hypothesis that all $c\left(\Sigma_1\Sigma_2^{-1}\right) > 1$ and bounded above, then, the appropriate confidence bounds are a lower bound on $c_m\left(\Sigma_1\Sigma_2^{-1}\right)$ and an upper bound on $c_M\left(\Sigma_1\Sigma_2^{-1}\right)$. It may be noted that these are the bounds obtained in [8] associated with Case (vii), and that in the spirit of the current attempt the appropriate confidence bounds for Case (vii) are a lower bound on $c_M\left(\Sigma_1\Sigma_2^{-1}\right)$ and/or an upper bound on $c_m\left(\Sigma_1\Sigma_2^{-1}\right)$ and not those obtained in [8].

For Cases (i)-(iii) and Case (vii) as considered heretofore [8], the confidence bounds turn out to be in terms of the appropriate characteristic root of $\Sigma_1\Sigma_2^{-1}$. Unfortunately, to date, the authors have not been able to do the same for Cases (iv)-(vi). The results for these cases are in terms of ratios of appropriate characteristic roots of Σ_1 and Σ_2.

<u>Case (i)</u> Using the canonical form of the distribution of the observations and proceeding exactly as in Sections 5.1 and 5.2 of [6] and Sections 1 and 2 of [8], we obtain with a preassigned probability $1-\alpha$, the statement

$$c_M\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_2^{-1}\right) \leq \lambda_1 , \qquad (3.1)$$

where λ_1 is the constant under (2.1) such that $P\left[c_M\left(S_1S_2^{-1}\right) \leq \lambda_1 \mid H_o\right] = 1-\alpha$. Also, D_a denotes a diagonal matrix whose diagonal elements are $a_1, a_2, \ldots$, and $\gamma_1, \ldots, \gamma_p$ are $c\left(\Sigma_1\Sigma_2^{-1}\right)$. Now (3.1) is equivalent to $c_M\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_1^{-1}S_1S_2^{-1}\right) \leq \lambda_1$ which, in turn, is equivalent to the set of simultaneous confidence interval statements

$$\frac{\underline{a}'\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_1^{-1}\right)\underline{a}}{\underline{a}'S_2S_1^{-1}\underline{a}} \leq \lambda_1 \quad \text{for all nonnull vectors } \underline{a}(p\text{x}1), \qquad (3.2)$$

with a confidence coefficient 1-α. (3.2) may be rewritten as

$$\frac{\underline{a}'\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_1^{-1}\right)\underline{a}}{\underline{a}'\underline{a}} \leq \lambda_1 \frac{\underline{a}'S_2S_1^{-1}\underline{a}}{\underline{a}'\underline{a}} , \qquad (3.3)$$

for all nonnull vectors $\underline{a}$. Choosing $\underline{a}$ successively so as to maximize, one after the other, both sides of (3.3), it follows that (3.3) implies that

$$c_M\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_1^{-1}\right) \leq \lambda_1 c_M\left(S_2S_1^{-1}\right) . \qquad (3.4)$$

We shall be using the phrase "choosing $\underline{a}$ successively ..." repeatedly. What this precisely implies is the following. Choose $\underline{a}$ so as to maximize the left side of (3.3) and denote this value of $\underline{a}$ by $\underline{a}^*$. Then it follows that (3.3) implies

$$c_M\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_1^{-1}\right) \leq \lambda_1 \frac{\underline{a}^{*\prime} S_2 S_1^{-1} \underline{a}^*}{\underline{a}^{*\prime}\underline{a}^*} \quad .$$

But the right side of the last inequality can be further increased to $\lambda_1 c_M\left(S_2 S_1^{-1}\right)$, whence (3.4) follows. This line of reasoning has been repeatedly used in [8,9]. Returning to (3.4) and writing $S_1 = TT'$, where T is a triangular matrix with zeros above the diagonal, and remembering that any nonzero c(AB) = a nonzero c(BA), we obtain from (3.4),

$$c_M\left[T^{-1}D_{1/\sqrt{\gamma}}\, TT'D_{1/\sqrt{\gamma}}\, (T')^{-1}\right] \leq \lambda_1 c_M\left(S_2 S_1^{-1}\right) \quad . \qquad (3.5)$$

But if A is a real matrix with real c(A), then it is known that

$$c_m(AA') \leq [c_m(A)]^2 \leq [c_M(A)]^2 \leq c_M(AA') \quad . \qquad (3.6)$$

Hence, (3.5) implies that

$$\left[c_M\left(T^{-1}D_{1/\sqrt{\gamma}}\, T\right)\right]^2 = \left[c_M\left(D_{1/\sqrt{\gamma}}\right)\right]^2 = c_M\left(\Sigma_2\Sigma_1^{-1}\right) \leq \lambda_1 c_M\left(S_2 S_1^{-1}\right) \quad ,$$

or, equivalently,

$$c_m\left(\Sigma_1\Sigma_2^{-1}\right) \geq \mu_1 c_m\left(S_1S_2^{-1}\right) , \qquad (3.7)$$

where $\mu_1 = \frac{1}{\lambda_1}$. (3.7) is thus a confidence interval statement with a confidence coefficient $\geq 1-\alpha$.

Case (ii) Our starting point here is the probability statement, with a preassigned probability $1-\alpha$,

$$c_m\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_2^{-1}\right) \geq \lambda_2 , \qquad (3.8)$$

where λ_2 is the constant under (2.1) such that $P\left[c_m\left(S_1S_2^{-1}\right) \geq \lambda_2 \mid H_o\right] = 1-\alpha$. Arguing again as we did in obtaining (3.3), we obtain that (3.8) is equivalent to the set of simultaneous confidence interval statements

$$\frac{\underline{a}'\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_1^{-1}\right)\underline{a}}{\underline{a}'\underline{a}} \geq \lambda_2 \frac{\underline{a}'S_2S_1^{-1}\underline{a}}{\underline{a}'\underline{a}} , \qquad (3.9)$$

for all nonnull vectors $\underline{a}$, with a confidence coefficient $= 1-\alpha$. Choosing $\underline{a}$ successively so as to minimize, one after the other, both sides of (3.9), it follows that (3.9) implies that

$$c_m\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_1^{-1}\right) \geq \lambda_2 c_m\left(S_2S_1^{-1}\right) . \qquad (3.10)$$

As before, setting $S_1 = TT'$ and then using (3.6), we now obtain that (3.10) implies

$$c_m\left(\Sigma_2\Sigma_1^{-1}\right) \geq \lambda_2 c_m\left(S_2S_1^{-1}\right) ,$$

or, equivalently,

$$c_M\left(\Sigma_1\Sigma_2^{-1}\right) \leq \mu_2 c_M\left(S_1S_2^{-1}\right) , \qquad (3.11)$$

where $\mu_2 = \frac{1}{\lambda_2}$. (3.11) is thus a confidence interval statement with a confidence coefficient $\geq 1-\alpha$.

Case (iii) Our starting point here is the probability statement, with preassigned probability $1-\alpha$,

$$c_m\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_2^{-1}\right) \geq \lambda_3' \quad \text{or} \quad c_M\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_2^{-1}\right) \leq \lambda_3 , \qquad (3.12)$$

where $\lambda_3 < \lambda_3'$ are constants such that $P\left[c_m\left(S_1S_2^{-1}\right) \geq \lambda_3', \text{ or, } c_M\left(S_1S_2^{-1}\right) \leq \lambda_3 \mid H_o\right] = 1-\alpha$, and further such that the test (iii) under (2.1) is locally unbiased. As before, we notice that (3.12) is equivalent to the set of simultaneous confidence interval statements

$$\frac{\underline{a}'\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_1^{-1}\right)\underline{a}}{\underline{a}'\underline{a}} \geq \frac{\lambda_3'\underline{a}'S_2S_1^{-1}\underline{a}}{\underline{a}'\underline{a}} ,$$

or $\qquad (3.13)$

$$\frac{\underline{a}'\left(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}}\, S_1^{-1}\right)\underline{a}}{\underline{a}'\underline{a}} \leq \frac{\lambda_3'\underline{a}'S_2S_1^{-1}\underline{a}}{\underline{a}'\underline{a}} ,$$

for all nonnull vectors $\underset{\sim}{a}$, with a confidence coefficient = 1-α. Proceeding now exactly as in Case (i) with the second inequality in (3.13) and as in Case (ii) with the first inequality in (3.13), we obtain that (3.13) implies

$$c_m\left(\Sigma_1\Sigma_2^{-1}\right) \geq \mu_3 c_m\left(S_1S_2^{-1}\right) \quad \text{or} \quad c_M\left(\Sigma_1\Sigma_2^{-1}\right) \leq \mu_3' c_M\left(S_1S_2^{-1}\right), \tag{3.14}$$

where $\mu_3 = \frac{1}{\lambda_3}$ and $\mu_3' = \frac{1}{\lambda_3'}$ so that $\mu_3 > \mu_3'$. (3.14) is thus a confidence statement with a confidence coefficient $\geq$ 1-α.

<u>Case (iv)</u> Taking the approach of [1], for this case, we write $S_1^* = D_{1/\sqrt{\gamma_1}} \Lambda_1 S_1 \Lambda_1' D_{1/\sqrt{\gamma_1}}$ and $S_2^* = D_{1/\sqrt{\gamma_2}} \Lambda_2 S_2 \Lambda_2' D_{1/\sqrt{\gamma_2}}$, where γ_1's are $c(\Sigma_1)$, γ_2's are $c(\Sigma_2)$ and Λ_1, Λ_2 are orthogonal matrices defined by the transformations $\Sigma_i = \Lambda_i' D_{\gamma_i} \Lambda_i$, $(i = 1,2)$. We take as our starting point the probability statement

$$\frac{c_m(S_1^*)}{c_M(S_2^*)} \leq \lambda \ , \tag{3.15}$$

where λ is such that $P\left[\frac{c_m(S_1)}{c_M(S_2)} \leq \lambda \mid H_o\right] = 1-\alpha$. It is known that if A is positive definite and B is at least positive semidefinite, then

$$c_m(A)c_m(B) \leq c_m(AB) \leq c_M(AB) \leq c_M(A)c_M(B) \ . \qquad (3.16)$$

Using (3.16), we have

$c_m(S_1^*) \geq c_m(D_{1/\gamma_1})c_m(\Lambda_1 S_1 \Lambda_1') = c_m(S_1)/c_M(\Sigma_1)$, and

$c_M(S_2^*) \leq c_M(D_{1/\gamma_2})c_M(\Lambda_2 S_2 \Lambda_2') = c_M(S_2)/c_m(\Sigma_2)$. Hence, (3.15) implies

$$\frac{c_m(S_1)}{c_M(\Sigma_1)} \cdot \frac{c_m(\Sigma_2)}{c_M(S_2)} \leq \lambda \ ,$$

or, equivalently,

$$\frac{c_M(\Sigma_1)}{c_m(\Sigma_2)} \geq \nu \ \frac{c_m(S_1)}{c_M(S_2)} \ , \qquad (3.17)$$

where $\nu = \frac{1}{\lambda}$. (3.17) is thus a confidence interval statement with a confidence coefficient $\geq 1-\alpha$.

<u>Case (v)</u> Using the notation above for Case (iv), we take as our starting point the probability statement

$$\frac{c_M(S_1^*)}{c_m(S_2^*)} \geq \lambda' \ , \qquad (3.18)$$

where λ' is such that $P\left[\frac{c_M(S_1)}{c_m(S_2)} \geq \lambda' \mid H_o\right] = 1-\alpha$.

Again using (3.16) we have,

$c_M(S_1^*) \leq c_M(D_{1/\gamma_1})c_M(\Lambda_1 S_1 \Lambda_1') = c_M(S_1)/c_m(\Sigma_1)$, and

$c_m(S_2^*) \geq c_m(D_{1/\gamma_2})c_m(\Lambda_2 S_2 \Lambda_2') = c_m(S_2)/c_M(\Sigma_2)$. Hence,

(3.18) implies

$$\frac{c_M(S_1)}{c_m(\Sigma_1)} \cdot \frac{c_M(\Sigma_2)}{c_m(S_2)} \geq \lambda' ,$$

or, equivalently

$$\frac{c_m(\Sigma_1)}{c_M(\Sigma_2)} \leq \nu' \frac{c_M(S_1)}{c_m(S_2)} , \tag{3.19}$$

where $\nu' = \frac{1}{\lambda'}$. (3.19) is thus a confidence interval statement with a confidence coefficient $\geq 1-\alpha$.

<u>Case (vi)</u> Our starting point here is the probability statement,

$$\frac{c_m(S_1^*)}{c_M(S_2^*)} \leq \lambda^* \quad \text{and} \quad \frac{c_M(S_1^*)}{c_m(S_2^*)} \geq \lambda^{*\prime} , \tag{3.20}$$

where $\lambda^* < \lambda^{*\prime}$ are such that

$P\left[\frac{c_m(S_1)}{c_M(S_2)} \leq \lambda^* < \lambda^{*\prime} \leq \frac{c_M(S_1)}{c_m(S_2)} \mid H_o\right] = 1-\alpha.$ Now using (3.16) and arguing exactly as in Cases (iv) and (v), we obtain that (3.20) implies

$$\frac{c_m(\Sigma_1)}{c_M(\Sigma_2)} \leq \nu^{*\prime} \frac{c_M(S_1)}{c_m(S_2)} \quad \text{and} \quad \frac{c_M(\Sigma_1)}{c_m(\Sigma_2)} \geq \nu^* \frac{c_m(S_1)}{c_M(S_2)} , \tag{3.21}$$

where $\nu^* = \frac{1}{\lambda^*}$ and $\nu^{*\prime} = \frac{1}{\lambda^{*\prime}}$ so that $\nu^* > \nu^{*\prime}$. (3.21) is thus a confidence statement with a confidence coefficient $\geq 1-\alpha$.

Case (vii) For this case, we have from [8], the confidence statement

$$\mu_7 c_M\left(S_1 S_2^{-1}\right) \geq c_M\left(\Sigma_1 \Sigma_2^{-1}\right) > c_m\left(\Sigma_1 \Sigma_2^{-1}\right) \geq \mu_7' c_m\left(S_1 S_2^{-1}\right) , \tag{3.22}$$

where $\mu_7 = \frac{1}{\lambda_7}$, $\mu_7' = \frac{1}{\lambda_7'}$, and $\lambda_7 < \lambda_7'$ are constants under (2.1) such that

$P\left[\lambda_7 \leq c_m\left(S_1 S_2^{-1}\right) \leq c_M\left(S_1 S_2^{-1}\right) \leq \lambda_7' \mid H_o\right] = 1-\alpha$, and further such that the test (vii) in (2.1) is locally unbiased. The confidence coefficient of (3.22) is, of course, $\geq 1-\alpha$. As mentioned at the beginning of Section 3, in the spirit of the present paper, (3.22) is not meaningful when the alternative is H_7. It is meaningful when the alternative is $H_1(H_2)$ together with the specification that all $c\left(\Sigma_1\Sigma_2^{-1}\right)$ are bounded above (below). On the other hand, if we take the approach of Cases (iv)-(vi) above, we obtain with a confidence coefficient $\geq 1-\alpha$, the following confidence statement

$$\frac{c_m(\Sigma_1)}{c_M(\Sigma_2)} \leq \nu_o' \frac{c_M(S_1)}{c_m(S_2)} \quad \text{and/or} \quad \frac{c_M(\Sigma_1)}{c_m(\Sigma_2)} \geq \nu_o \frac{c_m(S_1)}{c_M(S_2)} , \tag{3.23}$$

where $\nu_o = \frac{1}{\lambda_o}$, $\nu_o' = \frac{1}{\lambda_o'}$ and $\lambda_o < \lambda_o'$ are constants such that $P\left[\frac{c_m(S_1)}{c_M(S_2)} \leq \lambda_o', \text{ and/or, } \frac{c_M(S_1)}{c_m(S_2)} \geq \lambda_o' \,\middle|\, H_o\right] = 1-\alpha$.

4. CONCLUDING REMARKS

While the parametric functions on which confidence bounds are set in Cases (iv)-(vii) are also measures of departure from H_o in the direction of the appropriate alternative hypothesis, yet they are not as neatly interpretable as those of Cases (i)-(iii). Also, the confidence regions for these cases have not been obtained by "inverting" the acceptance regions of the tests for these cases given in Section 2. Hence, for these cases, more satisfactory bounds are to be sought. It may be noted, however, that in each case the distribution problem in obtaining the confidence bounds is one related to the null hypothesis, this being a desirable feature.

We shall now consider one possible physical meaning of the alternatives considered in this paper. If $\underline{x}_1(p\times 1)$ is p-variate nonsingular $N[\underline{\mu}_1, \Sigma_1]$ and $\underline{x}_2(p\times 1)$ is p-variate nonsingular $N[\underline{\mu}_2, \Sigma_2]$ and the elements (variates) of $\underline{x}_1$ are physically of the same nature as those of $\underline{x}_2$ (i.e., for e.g., the first element in both is amount of steel produced, the second element is total farm produce, etc.), then, if $\underline{c}'(1\times p) = (c_1, c_2, \ldots, c_p)$ is a vector of nonstochastic utilitarian "weights" that go with the p-variates, the linear functions $\underline{c}'\underline{x}_1$, and $\underline{c}'\underline{x}_2$ are of utilitarian interest. It is well known that $\underline{c}'\underline{x}_1$ is univariate $N(\underline{c}'\underline{\mu}_1, \underline{c}'\Sigma_1\underline{c})$ and $\underline{c}'\underline{x}_2$ is univariate $N(\underline{c}'\underline{\mu}_2, \underline{c}'\Sigma_2\underline{c})$. If $\underline{c}'$ is known then a direct comparison of $\underline{c}'\underline{x}_1$ and $\underline{c}'\underline{x}_2$, for observed values of $\underline{x}_1$ and $\underline{x}_2$, using the usual univariate techniques would be quite appropriate. Thus, for instance, one may be interested in differences between the means $\underline{c}'\underline{\mu}_1$ and $\underline{c}'\underline{\mu}_2$, or in the ratio of the variances, $\underline{c}'\Sigma_1\underline{c}/\underline{c}'\Sigma_2\underline{c}$. For a known system of utilitarian weights then, one may, for instance, wish to test $H_o: c'\Sigma_1\underline{c}/\underline{c}'\Sigma_2\underline{c} = 1$, against $H_1: c'\Sigma_1\underline{c}/\underline{c}'\Sigma_2\underline{c} > 1$. The test is the well-known one-sided F-test. But now, if $\underline{c}'$ is not known or given, then one may want to obtain a weight-free solution by protecting oneself against the worst possible set of weights (in a sort of minimax sense) and pose the question as a test of $H_o: \dfrac{c'\Sigma_1\underline{c}}{c'\Sigma_2\underline{c}} = 1$ for all $\underline{c}$, against $H_1: c'\Sigma_1\underline{c}/\underline{c}'\Sigma_2\underline{c} > 1$ for all $\underline{c}$. This is exactly the null hypothesis of

H_o: all $c\left(\Sigma_1 \Sigma_2^{-1}\right) = 1$, against H_1: all $c\left(\Sigma_1 \Sigma_2^{-1}\right) > 1$. Similarly, the other alternatives considered in this paper may also be interpreted as being physically meaningful, each for one class of problems.

The tests proposed here are heuristic and investigations for the power properties such as unbiasedness and monotonicity are underway and so, also, the explicit determination of the probabilities like the one assigned to $\omega(I)$ under H_o for Case (i). Finally, the problem of "truncation" or partial statements as considered in [8] and a generalization of the results to the case of more than two dispersion matrices are also being investigated by the authors.

REFERENCES

[1] R. Gnanadesikan, "Equality of More Than Two Variances and of More Than Two Dispersion Matrices Against Certain Alternatives," Ann. Math. Stat., 30 (1959), pp. 177-184.

[2] S. Kullback, "An Application of Information Theory to Multivariate Analysis," Ann. Math. Stat., 23 (1952), pp. 88-102.

[3] K. V. Ramachandran, "On Certain Tests and the Monotonicity of Their Powers," Institute of Stat., Univ. of North Carolina, Mimeo, Series No. 120, (1954).

[4] S. N. Roy, "The Individual Sampling Distributions of the Maximum, the Minimum and Any Intermediate One of the p-Statistics on the Null Hypothesis," Sankhya, 7 (1945), pp. 133-158.

[5] S. N. Roy, "On a Heuristic Method of Test Construction and Its Uses in Multivariate Analysis," Ann. Math. Stat., 24 (1953), pp. 220-238.

[6] S. N. Roy and R. C. Bose, "Simultaneous Confidence Interval Estimation," Ann. Math. Stat., 24 (1953), pp. 513-536.

[7] S. N. Roy, "Some Further Results in Simultaneous Confidence Interval Estimation," Ann. Math. Stat., 25 (1954), pp. 852-861.

References (Cont'd) - 2

[8] S. N. Roy and R. Gnanadesikan, "Further Contributions to Multivariate Confidence Bounds," Biometrika, 44 (1957), pp. 399-410.

[9] S. N. Roy, Some Aspects of Multivariate Analysis, John Wiley & Sons, New York (1958).

[10] S. S. Wilks, "Certain Generalizations in the Analysis of Variance," Biometrika, 24 (1932), pp. 471-494.