UNIVERSITY OF NORTH CAROLINA
Department of Statistics
Chapel Hill, N. C.

Mathematical Sciences Directorate
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INFERENCE ON TREATMENT EFFECTS AND DESIGN OF EXPERIMENTS IN RELATION TO SUCH INFERENCE

by

S. N. Roy

and

J. N. Shrivastava

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In this paper inference procedures and designs are discussed that are more appropriate to those situations in ANOVA or MANOVA where, against the customary or standard null hypothesis, we are interested in certain non-standard alternatives or, in other words, we are interested in increasing the discrimination along certain (non-standard) directions of deviation, even at some cost to the discrimination along other directions of deviation. It is indicated how, for several such situations, it is possible to improve upon both the customary designs and the customary inference procedure.

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INFERENCE ON TREATMENT EFFECTS AND DESIGN* OF EXPERIMENTS IN RELATION TO SUCH INFERENCE

By S. N. Roy and J. N. Shrivastava

1. Summary. This paper starts from the customary model, and the customary null hypothesis on any set of treatment effects, both for univariate and for multivariate response, and considers such a null hypothesis not against the perfectly general customary alternative, but against certain more specific alternatives that might be more natural and meaningful in some situations. Such situations occur quite often in practice, as, for example, when we are in possession of some prior information on the nature of the treatments either in terms of one or more characteristics other than the response we are looking for, that puts the treatments into a natural relationship or ordering, or even in terms of some very crude knowledge of this response itself, and furthermore this prior information is such as to focus our interest and attention on certain specific alternatives rather than the perfectly general one. It is well known that the usual F test for analysis of variance has some optimum properties, and it has also been known for some time that each of the several tests for multivariate analysis of variance has certain good properties, the optimum and the good being all in relation to the general alternative usually considered. If, however, one is interested in some more natural and specific alternative, then, for any design, it will be indicated in this

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paper how and in what sense an inference procedure based on a finite set of unions and intersections would, in general, be better than the usual F test for univariate problems or better than the analogues of F for multivariate problems. It will be also indicated that if, at this point, one decides to stay with this finite union-intersection procedure, then any customary design can often be replaced by another design that would do better. Thus, one has a two-fold development, one in the direction of inference and one in the direction of design. The same kind of improvement is reflected in the confidence intervals when we invert the test procedures to make confidence statements. This paper discusses a sample of such specific alternatives together with the corresponding inference procedures and designs, that are by no means exhaustive but merely illustrative, and the discussion will be expository rather than one involving full proofs of the various statements made. Furthermore, the statements in this paper will be more concerned with the univariate than with the multivariate aspects of the response. The follow-up papers will discuss the proofs of the statements made herein and also more statements, together with proofs, of the multivariate aspects.

2. Notation and preliminaries. As usual b will stand for the number of blocks, v for the number of treatments, k for the number of plots, ie, experimental units per block (assumed to be the same for each block); r_j for the number of replications of

the j-th treatment $(j = 1,2,\ldots,v)$ and $\lambda_{jj'}$ the number of times the j-th and j'-th treatments $(j \neq j' = 1,2,\ldots,v)$ occur together (in a block). With slight modification we take the customary model I of ANOVA

(2.1) $x_{ij} = \beta_i + \tau_j + \epsilon_{ij}$,

where x_{ij} is the observed (univariate) response from the j-th treatment in the i-th block, β_i the (unknown) block contribution to this response, σ_j the (unknown) treatment contribution and ϵ_{ij} the error, such that ϵ_{ij}'s are assumed to be independent $N(o,\sigma^2)$ $(i = 1,2,\ldots,b;\ j = 1,2,\ldots,v)$. The corresponding model I of MANOVA is

(2.2) $\underset{p\times 1}{\underline{x}_{ij}} = \underset{p\times 1}{\underline{\beta}_i} + \underset{p\times 1}{\underline{\tau}_j} + \underset{p\times 1}{\underline{\epsilon}_{ij}}$,

where everything is a p-dimensional vector and where $\underline{\epsilon}_{ij}$'s are assumed to be independent $N(\underset{p\times 1}{\underline{0}}, \underset{p\times p}{\underline{\Sigma}})$.

The customary null hypotheses on the treatment effects, for the univariate and the multivariate case, are written respectively as

(2.3) $\mathcal{H}$ (univariate): $\tau_1 = \tau_2 = \ldots = \tau_v$

and

(2.4) $\mathcal{H}$ (multivariate): $\underset{p\times 1}{\underline{\tau}_1} = \underset{p\times 1}{\underline{\tau}_2} = \ldots = \underset{p\times 1}{\underline{\tau}_v}$

Let us now express the of (2.3) in four different alternative forms

(2.5) (i) $\mathcal{H}_0: \bigcap_{\text{all } \underline{c}} \left[\sum_{j=1}^{v} c_j \tau_j = 0;\ \sum_j c_j = 0;\ \sum_j c_j^2 = 1\right]$,

(ii) $\mathcal{H}_1: \bigcap_{j_1=1}^{v-1} \left[\sum_{j=1}^{v} c_{j_1 j} \tau_j = 0;\ \sum_{j=1}^{v} c_{j_1 j} c_{j_2 j} = \delta_{j_1 j_2}\right.$ (Kronecker δ)

$\left.\sum_j c_{j_1 j} = 0\right]$,

where $j_1 \neq j_2 = 1,2,\ldots,v-1$ and where $c_{j_1 j}$'s $(j_1 = 1,2,\ldots,v-1)$ are associated with v-1 specific orthonormal contrasts,

$$\text{(iii)}\ \mathcal{H}_2 : \bigcap_{j \neq j'=1}^{v} \left[\tau_j = \tau_{j'}\right]$$

and

$$\text{(iv)}\ \mathcal{H}_3 : \bigcap_{j=1}^{v-1} \left[\tau_j = \tau_v\right].$$

If we denote by $\underline{\tau}'_{v\times 1}$ the vector $\tau_1, \tau_2, \ldots, \tau_v$, by $\underline{J}'$ the vector $1,1,\ldots,1$ and by $\underline{C}_{(v-1)\times v}$ the matrix whose elements are $c_{j_1 j}$, then it will be convenient to rewrite (i) and (ii) respectively as

$$\text{(v)}\ \mathcal{H}_D : \bigcap_{\text{all } \underline{c}} \left[\underline{c}'\ \underline{\tau} = 0;\ \underline{c}'\underline{J} = 0;\ \underline{c}'\underline{c} = 1\right]$$

and

$$\text{(vi)}\ \mathcal{H}_1 : \underset{(v-1)\times v}{\underline{C}}\ \underset{v\times 1}{\underline{\tau}} = \underset{(v-1)\times 1}{\underline{o}} \left[\underline{C}\ \underset{v\times 1}{\underline{j}} = \underset{(v-1)\times 1}{\underline{o}};\ \underline{C}\ \underline{C}' = \underline{I}(v-1)\right].$$

The motivation behind expressing in the four different alternative forms (i) - (iv) is that, for (i) we are looking for a test procedure that should, if possible, secure equal discrimination (in terms of power) against all alternatives (or contrasts) $\underline{\tau}'\ \underline{c}$, for (ii) one that should, if possible, secure equal discrimination against the (v-1) specific contrasts $\underset{v-1\times v}{\underline{C}}\ \underset{v\times 1}{\underline{\tau}}$, such that this discrimination might presumably be sharper than in other directions, ie, against other contrasts, for (iii) one that should, if possible, secure equal discrimination in the directions $\tau_j - \tau_{j'}$, that might presumably be sharper than against other contrasts and for (iv) one that should, if possible, secure equal discrimination in the direction $\tau_j - \tau_v$ (where v stands for some kind of a standard treatment and j for any other treatment), that might conceivably be sharper than in other directions. If the null hypothesis is expressed in forms like $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, etc., with particular ends in view (in terms of the power functions of the respective test-procedures), then $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ will be said to correspond to the structures S_1, S_2, S_3, etc. and each will be said to be a structured hypothesis. For the sake of a complete notation we shall use the symbols $\mathcal{H}_0$ and S_o to indicate what from our viewpoint would be the unstructured case. One extension to multivariate response would follow if in (i)-(iv) we replace τ_j's by $\underset{p\times 1}{\underline{\tau}_j}$'s and if in (v)-(vi) we replace the vector $\underset{v\times 1}{\underline{\tau}}$ by the matrix $\underset{v\times p}{\underline{\tau}}$. Other possible multivariate extensions will also be discussed later.

Let us now set up the test procedures for the univariate $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$. It is known that the usual F-test for the univariate has an acceptance region (at a level of significance, say) which can be expressed in the form

$$(2.6) \qquad \bigcap_{\text{all } \underline{c}} \left[\frac{(\underline{c}'\,\hat{\underline{\tau}})^2}{\hat{\sigma}^2_{nu}} \leq \mu_\alpha^{(o)} \quad ; \underline{c}'\underline{j} = o, \; \underline{c}'\underline{c} = 1 \right],$$

where $\hat{\underline{\tau}}$ is the customary least squares estimate of $\underline{\tau}$, $\hat{\sigma}^2_{nu}$ the customary estimate of the variance of the estimate $\underline{c}'\,\hat{\underline{\tau}}$ in the numerator and μ_α is the 100 o/o point of the F-distribution with a degree of freedom (v-1) due to the hypothesis, and another due to the error that depends upon the design used. It may be remarked that $\frac{(\underline{c}'\hat{\underline{\tau}})^2}{\hat{\sigma}^2_{nu}}$ itself has the distribution of t^2 with the error degree of freedom of the F-distribution. It should be noticed that the $\mu_\alpha^{(o)}$ of (2.6) will be the 100 o/o point of the t^2-distribution, when $\alpha^* > \alpha$. We shall let this region (2.6) correspond to S_0 and $\mathcal{H}_0$ and shall call it $\mathcal{D}_0$. Corresponding to $(\mathcal{H}_1, S_1)$, $(\mathcal{H}_2, S_2)$ and $(\mathcal{H}_3, S_3)$ let us set up test procedures with the acceptance regions, say $\mathcal{D}_1, \mathcal{D}_2$ (corresponding to the Tukey test [6]) and $\mathcal{D}_3$ given by

$$(2.7)\ \mathcal{D}_1: \quad \bigcap_{j_1=1}^{v-1} \left[\left(\sum_{j=1}^{v} c_{j_1 j} \hat{\tau}_j\right)^2 / \hat{\sigma}_{nu}^2 \leq \mu_\alpha^{(1)};\ \sum_{j=1}^{v} c_{j_1 j} = 0;\ \sum_{j=1}^{v} c_{j_1 j} c_{j_2 j} = \delta_{j_1 j_2} \right],$$

$$(2.8)\ \mathcal{D}_2: \quad \bigcap_{j \neq j'=1}^{v} \left[(\hat{\tau}_j - \hat{\tau}_{j'})^2 / \hat{\sigma}_{nu}^2 \equiv t_{jj'}^2 \leq \mu_\alpha^{(2)} \right],$$

and

$$(2.9)\ \mathcal{D}_3: \quad \bigcap_{j=1}^{v-1} \left[(\hat{\tau}_j - \hat{\tau}_v)^2 / \hat{\sigma}_{nu}^2 \equiv t_{jv}^2 \leq \mu_\alpha^{(3)} \right],$$

where the μ's are given respectively by

$$(2.10)\ 1 - \alpha = P[\mathcal{D}_1 | \mathcal{H}_1] = P[\mathcal{D}_2 | \mathcal{H}_2] = P[\mathcal{D}_3 | \mathcal{H}_3].$$

It should be noticed that for the above purpose $\mathcal{H} = \mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$.

We shall denote the power function of any test procedure by ψ with suitable agruments to indicate dependence on parameters and degrees of freedom and with suitable subscripts and superscripts, etc. to indicate the test procedure it is based on. In addition to the structures and the associated test procedures already mentioned a few other structures and the associated test procedures also will be discussed in sections 4 and 5, with integers larger than 3 being used as subscripts to $\mathcal{H}$, S and $\mathcal{D}$. This is not a very expressive and logical notation but would suffice for our present purpose. The different designs to be mentioned and discussed in this paper will be sometimes called, for convenience of reference, D_0, D_1, D_2, etc. with the caution that no correspondence is necessarily implied at this stage between the subscripts for the structured hypotheses and the subscripts for the designs.

Turning now to the case of multivariate response, we propose for the multivariate analogues of $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, to be called $\mathcal{H}_0^*, \mathcal{H}_1^*, \mathcal{H}_2^*, \mathcal{H}_3^*$, tests with acceptance regions given by

$$(2.11)\ \mathcal{D}_0^*: \bigcap_{\text{all } \underline{c} \text{ and } \underline{a}} \Big[(\underset{1\times v}{\underline{c}'} \underset{v\times p}{\hat{\tau}} \underset{p\times 1}{\underline{a}})^2 / (\underline{a}' \hat{\underline{\Sigma}}_{nu} \underline{a}) \leq \mu_\alpha^{(o)x}; \quad \underline{c}'\underline{j} = 0;\ \underline{c}'\underline{c}=1;\ \underline{a} \text{ is non-null} \Big],$$

$$\Longleftrightarrow \left\{ ch_{max} \big[\underline{S}\ \underline{S}_e^2 \big] \leq \mu_\alpha^{(o)x} \right\},$$

where $\hat{\Sigma}_{nu}$ is the usual estimate of the dispersion matrix of the estimate $\underline{c}'\tau$ in the numerator and $\underset{p\times p}{\underline{S}_e}$ is the "sample dispersion matrix due to the error", $\underset{p\times p}{\underline{S}}$ is the sample dispersion matrix due to the "hypothesis", $\underset{v\times p}{\hat{\underline{\tau}}}$ is the usual maximum likelihood estimate of $\underset{v\times p}{\tau}$ and $\mu_\alpha^{(o)*}$ is the 100 α % point of the largest characteristic root, with appropriate degrees of freedom;

$$(2.12)\ \mathcal{D}_1^*: \bigcap_{j_1=1}^{v-1} \bigcap_{\text{all } \underline{a}} \Big[(\underset{1\times v}{\underline{c}'_{j_1}} \underset{v\times p}{\hat{\underline{\tau}}} \underset{p\times 1}{\underline{a}})^2 / (\underline{a}' \hat{\underline{\Sigma}}_{nu} \underline{a}) \leq \underline{\mu}_{\underline{\alpha}}^{(1)*};$$

$\underline{a}$ is non-null and $\underset{1\times v}{\underline{c}'_{j_1}} \underset{v\times 1}{\underline{J}} = 0$; $\underline{c}'_{j_1} \underline{c}_{j_2} = \delta_{j_1 j_2} \Big]$;

$$(2.13)\ \mathcal{D}_2^*: \bigcap_{j\neq j'=1}^{v} \bigcap_{\text{all } \underline{a}} \Big[(\underset{1\times p}{\hat{\underline{\tau}}'_j} - \underset{1\times p}{\hat{\underline{\tau}}'_{j'}}) \underset{p\times 1}{\underline{a}}\ ^2 / (\underline{a}' \hat{\underline{\Sigma}}_{nu} \underline{a}) \leq \mu_\alpha^{(2)*}; \underline{a} \text{ is non-null} \Big]$$

$$\Longleftrightarrow \bigcap_{j\neq j'=1}^{v} \Big[T_{jj'}^2 \leq \mu_\alpha^{(2)*} \Big],$$

where $T^2_{jj'}$ is Hotelling's T^2 with appropriate degrees of freedom, the mean vectors being the maximum likelihood estimates of the two vector responses $\underline{\tau}_j$ and $\underline{\tau}_{j'}$ from the j-th and j'-th treatments and the error dispersion matrix being based on the overall residual;

$$(2.14)\ \mathcal{D}^*_3:\ \bigcap_{j=1}^{v-1}\ \bigcap_{\text{all } \underline{a}} \left[(\underset{1\times p}{\hat{\underline{\tau}}'_j} - \underset{1\times p}{\hat{\underline{\tau}}'_v})\underset{p\times 1}{\underline{a}} \right]^2 / (\underline{a}'\ \hat{\underline{\Sigma}}_{nu}\ \underline{a}) \le \mu^{(3)*}_\alpha\ ;\underline{a} \text{ is non-null}]$$

$$\iff \bigcap_{j=1}^{v-1} [T^2_{jv} \le \mu^{(3)*}_\alpha].$$

It should be noticed that $\mu^{(1)*}_\alpha$, $\mu^{(2)*}_\alpha$ and $\mu^{(3)*}_\alpha$ are defined by

$$(2.15)\quad 1 - \alpha = P[\mathcal{D}^*_1 | \mathcal{H}^*_1] = P[\mathcal{D}^*_2 | \mathcal{H}^*_2] = P[\mathcal{D}^*_3 | \mathcal{H}^*_3],$$

with the understanding, as before, that for the above purpose we have

$$\mathcal{H}^* = \mathcal{H}^*_0 = \mathcal{H}^*_1 = \mathcal{H}^*_2 = \mathcal{H}^*_3 .$$

3. Remarks on the power properties (under three different designs) of the test procedures appropriate to the four different structured hypotheses. With v treatments and r replications for each treatment (assumed to be the same for all treatments) let us consider in the first instance an RBD and a comparable BIBD with b, k and λ. Notice that $k < v$ and $r < b$. Let us denote these two designs by D_o and D_1. Then for D_1, the σ^2 of (2.1) depends on k and should be written as $\sigma_k^{\ 2}$ and likewise Σ of (2.2) as $\Sigma_k^{\ 2}$; similarly for D_o we should write σ^2 as $\sigma_r^{\ 2}$ and Σ as $\Sigma_r^{\ 2}$ (since $k = r$ for D_o). Notice that, in general, $\sigma_k^{\ 2} < \sigma_r^{\ 2}$. We observe, furthermore, that, for D_o and D_1, the respective degrees of freedom due to the error are $(rv - r - v+1)$ and $(rv - b - v+1)$, and,

for both D_o and D_1, the common degrees of freedom due to the hypothesis $\mathcal{H}$ (in the customary F-test) are (v-1). For D_o the $\hat{\sigma}^2_{nu}$ occurring in the $\mathcal{D}_o, \mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ of (2.6)-(2.9) are given respectively by

(3.1) (i) $s_e^{(o)^2}/r$, (ii) $s_e^{(o)^2}/r$, (iii) $s_e^{(o)^2}/2r$, (iv) $s_e^{(o)^2}/2r$,

and for D_1, the same are given by

(3.2) (i) $s_e^{(1)^2}/\frac{\lambda v}{k}$, (ii) $s_e^{(1)^2}/\frac{\lambda v}{k}$, (iii) $s_e^{(1)^2}/\frac{2\lambda v}{k}$, (iv) $s_e^{(1)^2}/\frac{2\lambda v}{k}$.

Similarly, the $\hat{\underline{\Sigma}}_{nu}$ occurring in $\mathcal{D}^*_o, \mathcal{D}^*_1, \mathcal{D}^*_2$ and $\mathcal{D}^*_3$ of (2.11)-(2.14) are given, for D_o and D_1, by

(3.3) (i) $S_e^{(o)}/r$, (ii) $S_e^{(o)}/r$, (iii) $S_e^{(o)}/2r$, (iv) $\underline{S}_e^{(o)}/2r$

and

(3.4) (i) $\underline{S}_e^{(1)}/\frac{\lambda v}{k}$, (ii) $\underline{S}_e^{(1)}/\frac{\lambda v}{k}$, (iii) $\underline{S}_e^{(1)}/\frac{2\lambda v}{k}$ and (iv) $\underline{S}_e^{(1)}/\frac{2\lambda v}{k}$.

We shall now consider in a symbolic form the power functions of the critical regions of the different test procedures, ie. of the complements $\bar{\mathcal{D}}_o, \bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2$ and $\bar{\mathcal{D}}_3$ for D_o and D_1, exhibiting, in particular, the arguments involved in the power functions. For D_o these are

(3.5) (i) $\bar{\mathcal{D}}_o : \psi_o(\alpha;\ v-1;\ vr - v - r+1;\ r \sum_{j=1}^{v} (\tau_j - \bar{\tau})^2/\sigma_r^2)$,

where $\bar{\tau} = \sum_{j=1}^{v} \tau_j/v$,

(ii) $\bar{\mathcal{D}}_1 : \psi_1(\alpha; 1, 1, \ldots, 1;\ vr - v - r+1;\ r\eta_1^2/\sigma_r^2, \ldots, r\eta_{v-1}^2/\sigma_r^2)$, where $\sum_{j'=1}^{v} c_{jj'}\tau_{j'} = \eta_j$ ($j=1,2,\ldots,v-1$) and where, it should be remarked that $\sum_{j=1}^{v-1} \eta_j^2 = \sum_{j=1}^{v} (\tau_j - \bar{\tau})^2$ of (i),

(iii) $\bar{\mathcal{D}}_2$: $\psi_2(\alpha;1,1,\ldots,;\ vr - v - r+1;\ r\eta_{1v}^2/\sigma_r^2,\ldots,r\eta_{v-1,v}^2/\sigma_r^2)$

and

(iv) $\bar{\mathcal{D}}_3$: $\psi_3(\alpha;1,\ldots,1;vr - v- r+1;\ r\eta_{1v}^2/\sigma_r^2,\ldots,r\eta_{v-1,v}^2/\sigma_r^2)$.

The symbolic form (1,1,...,1) in both (ii) and (iv) indicates that the corresponding power functions are related to regions such that each is the complement of the intersection of (v-1) acceptance regions each based on a "t" with degrees of freedom vr - v - r+1; and in (iii) this symbolic form indicates, instead of (v-1), $\binom{v}{2}$ such acceptance regions. Further in both (iv) and (iii), $\eta_{jv}(j=1,2,\ldots,v-1) = \tau_j - \tau_v$; in (iv) these are natural "deviation parameters," but in (iii) these are not natural, in that any particular treatment, say τ_v, can be taken as the standard, and all $\binom{v}{2}$ pairwise contrasts $\tau_j - \tau_{j'}$ can be expressed in terms of the (v-1) linearly independent contrast $\tau_j - \tau_v$ ($j=1,2,\ldots,v-1$), and thus the power functions can be expressed in terms of these (v-1) deviation parameters. Notice that ψ_2 and ψ_3 are entirely different functions.

For D_1, the corresponding power functions can be expressed as

$$(3.6)\ (i)\ \bar{\mathcal{D}}_0:\ \psi_0(\alpha;\ v\text{-}1;vr\text{-}v\text{-}b\text{+}1;\ \frac{\lambda v}{k}\sum_{j=1}^{v}(\tau_j-\bar{\tau})^2/\sigma_k^2),$$

$$(ii)\ \bar{\mathcal{D}}_1:\ \psi_1(\alpha;1,\ldots,1;vr\text{-}v\text{-}b\text{+}1;\ \frac{\lambda v}{k}\eta_1^2/\sigma_k^2,\ldots,\frac{\lambda v}{k}\eta_{v-1}^2/\sigma_k^2),$$

$$(iii)\ \bar{\mathcal{D}}_2:\ \psi_2(\alpha,1,\ldots,1;vr\text{-}v\text{+}b\text{+}1;\frac{\lambda v}{k}\eta_{1v}^2/\sigma_k^2,\ldots,\frac{\lambda v}{k}\eta_{v-1,v}^2/\sigma_k^2),$$

$$(iv)\ \bar{\mathcal{D}}_3:\ \psi_3(\alpha;1,\ldots,1;vr\text{-}v\text{+}b\text{+}1,\frac{\lambda v}{k}\eta_{1v}^2/\sigma_k^2,\ldots,\frac{\lambda v}{k}\eta_{v-1,v}^2/\sigma_k^2).$$

By way of comparison between (i)-(iv) of (3.5) we shall now state without proof a number of results proved elsewhere. But as a preliminary to that we recall that $\sum_{j=1}^{v}(\tau_j-\bar{\tau})^2$ of (i) can be expressed as the sum of squares of any (v-1) orthonormal contrasts, including, in particular, the specific set of (v-1) considered (ii), and also that it can be expressed (though not as a sum of squares) in terms of η_{jv}^2's of (iii) or (iv). Furthermore, the specific contrast set of (iii) can also be expressed in terms of η_{jv}'s and vice versa, and, finally, all pairwise contrasts implicit in (iii) can be expressed in terms of η_{jv}'s. With this remark we state that

(a) if in (ii) we put all η_j's = 0, except any particular one, then, in general, $\psi_1 > \psi_0, \psi_2, \psi_3$;

(b) if in (iii) we put $\eta_{jj'}(\equiv \tau_j - \tau_{j'},\ j\neq j') \neq 0$ for any particular pair (j,j') and equate to zero all other pairwise contrasts that we could so equate consistently with this particular pair being non-zero, then, in general, $\psi_2 > \psi_o, \psi_1$; but if $j' \neq v$, then $\psi_2 > \psi_3$, while, if $j' = v$, then $\psi_3 > \psi_2 > \psi_o, \psi_1$ (in general). In other words, this means that in the directions $\eta_{jj'}$'s ($j' \neq v$), $_2$ is better than the others, in the directions of a specific orthonormal set considered in (ii), $_1$ is better than the others, while in other directions $_o$ might be better or worse than the rest. Indeed, there are directions in which $_o$ would be definitely better than the rest. We have an exactly similar situation as among (i) - (iv) of (3.6). Next, as between (3.5) and (3.6) we observe that, in general, $\sigma_k^2 < \sigma_r^2$, so that if the former is sufficiently smaller than the latter (which, in fact, is achieved in many experimental situations) then ψ_i of (3.6) > ψ_i (3.5) (i=0,1,2,3). It is obvious, however, that the l.u.b. on σ_k^2/σ_r^2 to accomplish this would not be the same for i=0,1,2,3. Another point to note is that for both D_0 and D_1, or in other words, for all designs that are pairwise symmetrical (with respect to all treatments), $\mathcal{D}_o$ (or the F-test) is equally sensitive with regard to all normalized contrasts, $\mathcal{D}_1$ equally sensitive with regard to each of the specific set of (v-1) orthonormal contrasts, $\mathcal{D}_2$ equally so with regard to all pairs and $\mathcal{D}_3$ with regard to η_{jv}'s(j=1,2,...,v-1). However, this equal sensitivity, as will be presently seen, will not carry over to designs that are not pairwise symmetrical, as, for example, any PBIBD.

To illustrate this point let us take, for a given v and r, any combinatorially possible PBIBD with, say, m associate classes, and let us write for this design

(3. $\bar{\ }$) (i) $\mathcal{D}_o : \psi_o(\alpha; v-1; vr-b-v+1; \underline{\tau}' \ \underline{C}^* \ \underline{\tau}/\sigma_{k^*}^2)$,

where $\sigma_{k^*}^2$ (depending upon the block size k^* for this case) is what occurs in (2.1), and $\underset{v\times v}{\underline{C}^*}$ is such that $C^*_{jj} = r(k-1)/k$ and $C^*_{jj'}(j\neq j') = -\lambda e/k$, if the j-th and j'-th treatments are 1-th associates. It is clear that in this case the power function ψ_0 of the F-test already favors certain directions of deviation more than other directions, and it might be mentioned here, without expressing ψ_1, ψ_2 and ψ_3 in a symbolic manner as in the previous cases, that, in general, this differential discrimination will be further sharpened under ψ_1, ψ_2 and ψ_3 along the same general lines as indicated in the previous cases. We shall not pursue this in any detail here, since we are going to consider in the next two sections certain other useful and interesting types of structured hypotheses and the associated designs. It may be remarked that in this paper we shall not consider the general power properties and aspects of the corresponding multivariate or multiresponse test procedures and their possible impact upon designs. This will be discussed in a later communication.

4. Some other types of structured hypotheses with the appropriate test procedures (with power properties)

For concreteness of illustration, let us consider nine treatments with hypothetical effects $\tau_1, \ldots, \tau_9$. On the general lines indicated in section 1, assume that the treatments have been divided into two groups $(1, \ldots, 5)$ and $(6, \ldots, 9)$, and that, although we would be interested in all contrasts we might be much more interested in any one (and one alone) of the following types of contrasts or structured hypotheses.

(4.1) $(S_4, \mathcal{H}_4)$: (i) $[\tau_j - \tau_{j'} = 0; j \neq j' = 1,2,\ldots,5] \cap [\tau_{j} - \tau_{j'} = 0; j \neq j' = 6,\ldots,9]$,

or

(ii) $[\bigcap_{\text{all } c} \sum_{j=1}^{5} c_j \tau_j = 0] [\bigcap_{\text{all } c} \sum_{j=6}^{9} c_j \tau_j = 0]$ with $\sum_{j=1}^{5} c_j = \sum_{j=6}^{9} c_j = 0$ and

$$\sum_{j=1}^{5} c_j^2 = \sum_{j=6}^{9} c_j^2 = 1;$$

(4.2) $(S_5, \mathcal{H}_5)$: $\tau_j - \tau_{j'} = 0$; $j=1,2,\ldots,5$; $j' = 6,\ldots,9$;

(4.3) $(S_6, \mathcal{H}_6)$: (i) (4.2) $\cap$ (4.1)(i)
or (ii) (4.2) $\cap$ (4.1)(ii) ;

(4.4) $(S_7, \mathcal{H}_7)$: $[\tau_j - \tau_5 = 0;\ j=1,2,3,4] \cap [\tau_j - \tau_9 = 0;\ j=6,7,8]$;

(4.5) $(S_8, \mathcal{H}_8)$: $\tau_5 - \tau_9 = 0$.

Finally, with these nine treatments against a different background (and without the division into two groups considered above) let us consider a cyclically structured hypothesis

(4.6) $(S_9, \mathcal{H}_9)$: $\tau_j - \tau_{j+1} = 0$; $j=1,2,\ldots,v$,

with the convention that $\tau_{v+1} = \tau_1$.

Before we discuss the associated test procedures, some remarks on the above structures might be helpful. It is easy to check that any one of $\mathcal{H}_5, \mathcal{H}_6$ or $\mathcal{H}_9$ constitutes the total hypothesis $\mathcal{H}$ (of equality of the nine treatment effects), but this is not true of $\mathcal{H}_4$ or $\mathcal{H}_7$ or $\mathcal{H}_8$, which suggests that to each of these we have to adjoin something else (depedning upon the nature of our interest) so that the resulting structure constitutesthe total hypothesis. This means, for example, under $\mathcal{H}_8$, that although we are more interested in $\tau_5 - \tau_9$, we shall keep open the possibility of studying other contrasts as well. This also means that by adjunction we shall be changing the nature of the original structure which, nevertheless, will still occupy a privileged position in the altered structure. This adjunction we shall make at the proper points. We next note that $\mathcal{H}_4$ is related to "within contrasts" of the two groups, expressed either in the form of all contrasts within one group plus all contrasts within another group or in the form of pairwise contrasts within one plus within another, depending upon the nature of our interest. Likewise, $\mathcal{H}_5$ is related to a modified version of "between contrasts," $\mathcal{H}_6$ is self-explained, $\mathcal{H}_7$ is related to contrasts within each group against two respective standards, $\mathcal{H}_8$ to a contrast between just these two standards. Finally, $\mathcal{H}_9$ needs no further comments.

For the above structures we offer test procedures with the following acceptance regions

$$(4.7)\quad \mathcal{D}_{41}: \Big[\bigcap_{j\neq j'=1}^{5}\Big\{(\hat{\tau}_j - \hat{\tau}_{j'})^2/\hat{\sigma}^2_{nu} = t_{jj'}^2 \leq \mu_\alpha^{(41)}\Big\}\Big]$$

$$\cap\ \Big[\bigcap_{j\neq j'=6}^{9}\Big\{(\hat{\tau}_j - \hat{\tau}_{j'})^2/\hat{\sigma}^2_{nu} = t_{jj'}^2 \leq \mu_\alpha^{(41)}\Big\}\Big]$$

$$\cap\ \Big[\Big(\frac{1}{5}\sum_{j=1}^{5}\hat{\tau}_j - \frac{1}{4}\sum_{j=6}^{9}\hat{\tau}_j\Big)^2/\hat{\sigma}^2_{nu} = t^2 \leq \mu_\alpha^{*(41)}\Big];$$

(4.8) $\mathcal{D}_{42}$: $\left[\bigcap_{\text{all normalized contrasts}} \left\{ \left(\sum_{j=1}^{5} c_j \hat{\tau}_j\right)^2 / \hat{\sigma}^2_{nu} \leq \mu^{(42)}_{\alpha 1} \right\} \text{ or } \left\{ F_1 \leq \mu^{(42)}_{\alpha} \right\} \right]$

$\bigcap \left[\bigcap_{\text{all normalized contrasts}} \left\{ \left(\sum_{j=6}^{9} c_j \hat{\tau}_j\right)^2 / \hat{\sigma}^2_{nu} \leq \mu^{(42)}_{\alpha 2} \right\} \text{ or } \left\{ F_2 \leq \mu^{(42)}_{\alpha} \right\} \right]$

$\bigcap \left[\left(\frac{1}{5} \sum_{j=1}^{5} \hat{\tau}_j - \frac{1}{4} \sum_{j=6}^{9} \hat{\tau}_j\right)^2 / \hat{\sigma}^2_{nu} \equiv t^2 \leq \mu^{*(42)}_{\alpha} \right]$;

(4.9) $\mathcal{D}_5$: $\bigcap_{\substack{j=1,\ldots,5 \\ j'=6,\ldots,9}} \left[(\hat{\tau}_j - \hat{\tau}_{j'})^2 / \hat{\sigma}^2_{nu} \equiv t^2_{jj'} \leq \mu^{(5)}_{\alpha} \right]$;

(4.10) $\mathcal{D}_{61}$: $\bigcap_{j \neq j'=1}^{9} \left[t^2_{jj'} \leq \mu^{(61)}_{\alpha} \right]$,

which is precisely the same as $\mathcal{D}_2$ of section 2;

(4.11) $\mathcal{D}_{62}$: $\left[F_1 \leq \mu^{(62)}_{\alpha}\right] \bigcap \left[F_2 \leq \mu^{(62)}_{\alpha}\right] \bigcap \left[t^2_{jj'} \leq \mu^{(62)}_{\alpha}\right]$,

where $t^2_{jj'}$ is the same as defined under (4.9);

(4.12) $\mathcal{D}_7$: $\left[\bigcap_{j=1}^{4} \left\{ (\hat{\tau}_j - \hat{\tau}_5)^2 / \hat{\sigma}^2_{nu} \equiv t^2_{j5} \leq \mu^{(7)}_{\alpha} \right\} \right]$

$\bigcap \left[\bigcap_{j=6}^{8} \left\{ (\hat{\tau}_j - \hat{\tau}_9)^2 / \hat{\sigma}^2_{nu} \equiv t^2_{j9} \leq \mu^{(7)}_{\alpha} \right\} \right]$

$\bigcap \left[\left(\frac{1}{5} \sum_{j=1}^{5} \hat{\tau}_j - \frac{1}{4} \sum_{j=6}^{9} \hat{\tau}_j\right)^2 / \hat{\sigma}^2_{nu} \equiv t^2 \leq \mu^{*(7)}_{\alpha} \right]$;

(4.13) $\mathcal{D}_8$: $\left[(\hat{\tau}_5 - \hat{\tau}_9)^2 / \hat{\sigma}^2_{nu} \equiv t^2 \leq \mu^{(8)}_{\alpha}\right] \bigcap \left[\bigcap_{j=1}^{4} \left\{ (\hat{\tau}_j - \hat{\tau}_5) / \hat{\sigma}^2_{nu} \equiv t^2_{j5} \leq \mu^{*(8)}_{\alpha} \right\} \right]$

$$\bigcap \left[\bigcap_{j=6}^{8} \left\{ (\hat{\tau}_j - \hat{\tau}_9)^2 / \hat{\sigma}^2_{nu} \equiv t^2_{j9} \leq \mu^{*(8)}_{\alpha} \right\} \right];$$

$$(4.14)\ \mathcal{D}_9: \bigcap_{j=1}^{9} \left[(\hat{\tau}_j - \hat{\tau}_{j+1})^2 / \hat{\sigma}^2_{nu} \equiv t^2_{j,j+1} \leq \mu^{(9)}_{\alpha} \right] .$$

Notice that in (4.7), (4.8) and (4.12) the adjunction is based on the hypothesis of zero contrast between the group means and is in terms of a corresponding t-region and in (4.13) is based on contrasts within the two groups against the respective standards. It may be remarked that in each case where adjunction was used it would be quite meaningful to use other kinds of adjunction. As in section 2, the μ_{α}'s in (4.9), (4.10), (4.11) and (4.14) are given by

$$(4.15)\quad P[\mathcal{D}_5 | \mathcal{H}] = P[\mathcal{D}_{61} | \mathcal{H}] = P[\mathcal{D}_{62} | \mathcal{H}] = P[\mathcal{D}_9 | \mathcal{H}] = 1-\alpha.$$

We next observe that in (4.7), (4.8) and (4.12), depending upon the nature of our interest, it is open to us to take $\mu^*_{\alpha} = \mu_{\alpha}$ or keep the two different in any prescribed manner that might be desired, but in any case with the general overall requirement that

$$(4.16)\quad P[\mathcal{D}_{41} | \mathcal{H}] = P[\mathcal{D}_{42} | \mathcal{H}] = P[\mathcal{D}_{62} | \mathcal{H}] = P[\mathcal{D}_7 | \mathcal{H}] = P[\mathcal{D}_8 | \mathcal{H}]$$

$$= 1 - \alpha .$$

However, in (4.13), the meaningful course seems to be to so choose μ^*_{α} as to make the discrimination with regard to (j5) and (j9) less sharp than with regard to (59).

Turning now to the power properties of test procedures offered for the different structured hypotheses we observe that for any design, in general, and for any particular structured hypothesis the suggested test procedure secures, in the sense indicated in section 3, better discrimination with regard to those particular deviations than the other test procedures. In other words, in this sense each procedure is appropriate to one structure. Furthermore, given one structure and the test procedure claimed to be appropriate to it (in that it is at least better than the others in the class considered), a design will be offered in the next section which would be appropriate to it, in that it would be at least better, in the sense indicated in section 3, than others in the class considered.

5. The problem of design of experiments when the given set of treatments has a structure.

We are now in a position to have a new look at the problem of experimental designs. Given a set of v treatments $\mathcal{T}$, we shall first suppose that the set $\mathcal{T}$ may have no structure, i.e., we may not possess any information regarding the nature of the treatments, or any relationship in which they stand with respect to each other. This will, of course,happen rarely. If $\mathcal{T}$ has no structure, we will proceed in the usual way with the design of our expreiment, remembering the three basic principles [1] (i) replication, (ii) randomization, and (iii) local control. The choice of the exact design to be used will be determined by these three principles together with the nature of our resources, especially in terms of the number of experimental units (n) available. While analyzing the data of the experiment so conducted, we will apply the usual over-all F-test for testing the customary hypothesis $\mathcal{H}$ of the equality of all treatment effects.

Next, let us suppose that the set of treatments $\mathcal{T}$ has a known structure S. In this case we can introduce a fourth principle to control the choice of the design viz. that the design, in a sense to be presently explained, should correspond to the structure. The class of designs $\{D\}$ which will be said to correspond to the structure S, is to be such that, if possible, it should control the relative variances of the treatment contrasts falling under S, in a predetermined way. After the class of design $\{D\}$ corresponding to S has been determined, an actual design will be selected from this class by use of the three principles referred to in the last paragraph, and the preassigned value of n. Examples of many types of structures are given in the earlier sections. These structures, though simple, are by no means the only ones to be found, nor are they necessarily the more general or more important ones. The discussion of other structures, because of their complexity, is being deferred to later communications.

As mentioned above, a class of designs $\{D\}$ which corresponds to a structure S controls the relative accuracies of the treatment contrasts contained in S in the desired way both from the point of view of ordinary point estimation and of ordinary testing. Also, in case we are interested in increasing from the testing point of view the accuracies of any or all of the treatments contrasts contained in S, we can apply the corresponding intersection test as exemplified in the earlier sections, instead of applying the over-all F-test.

The relationship in which the various treatments stand, each to the rest, was recognized in case of factorial experiments from the very beginning, and this information was utilized in a sense in the construction of confounded factorial designs. However, in this case also, the intersection tests were not used in any connection. In the case of experiments where the treatments are non-factorial in nature, as, for example, when the treatments represent several varieties of a crop, the information regarding structures, if any, was seldom utilized and of course the intersection tests, as far as we are sware, were never used.

We shall now consider classes of designs corresponding to the structures discussed in the earlier sections. S_o represents the unstructured case and the BIB design or the randomized block (RB) design can be utilized for this purpose. The principle of local control may sometimes necessitate the use of a PBIB design. However, it must be remembered that a PBIB design corresponds to certain other structures and decreases the precision of certain treatment contrasts while increasing the precision of certain other treatment contrasts.

For S_2, a BIB design appears to be most reasonable. In S_3 we want to increase the accuracy of the contrasts $(\tau_j - \tau_v)$ $(j=1,2,\ldots,v-1)$. The class of designs $\{D_3\}$ which are consistent with S_3 should obviously be such that λ_{jv} $(j=1,2,\ldots,v-1)$ are high compared to other $\lambda_{jj'}$ $(j,j'\neq v)$. The class of designs which are consistent with the structures of the types S_4, S_5 and S_6 are the Intra and Inter Group Balanced Incomplete Block Design (GBIB design, for short). These designs were first defined by K. R. Nair and C. R. Rao [2], and the analysis of the designs was also given for the case where the number of groups m = 2. Recently, the authors independently arrived at these designs (calling them generalized BIB or GBIB, for short) from the present viewpoint, and obtained the complete analysis when m is any positive integer. The authors also introduced the intersection tests for testing $\mathcal{H}$, which was not done earlier by Nair and Rao.

In the structures S_i $(i=4,5,6,7,8)$, considered for the purpose of a very simple illustration, the whole set of treatments has been divided into two groups which may be denoted by $G_1(\tau_1$ to $\tau_5)$ and G_2 $(\tau_6$ to $\tau_9)$. In S_4, we want to increase the accuracy of within group comparisons, and so the corresponding class $\{D_4\}$ is such that $\lambda_1 = \mu_{jj'}(j,j' \epsilon G_1)$ and $\lambda_2 = \lambda_{jj'}(j,j' \epsilon G_2)$ are high compared to all other λ's. In the class $\{D_5\}$ corresponding to S_5, λ_1 and λ_2 should be low compared to other λ's. The same considerations may eventually reduce the class D_6 corresponding to S_6 to the class of BIB designs.

Similarly, in $\{D_7\}$, $\lambda_1 = \lambda_{j5}$ $(j \in G_1,\ j \neq 5)$ and $\lambda_2 = \lambda_{j9}$ $(j \in G_2,\ j \neq 9)$ should be high compared to other λ's. The same considerations indicate that in $\{D_8\}$, λ_{59} is high compared to others, and in $\{D_9\}$, $\lambda = \lambda_{j,j+1}$, $j=1,2,\ldots,9$ is high.

For the sake of illustration one design D_i from each of the classes D_i for $i=4,5,7,8,9$ is exhibited below. The values of the parameters chosen are perfectly arbitrary, and thus the design may not be usable in a situation that involves another set of v, n and k. Furthermore even if usable, no optimality is claimed for any of these designs among the general class to which such a design belongs. However, it can be shown that the general pattern of any of these designs is such that, for the corresponding structure of the hypothesis, the design is better than any of the comparable customary designs. This, in fact, is the main point in exhibiting these designs. The vector $\underline{r} = (r_1,\ldots,r_9)$, where r_j = number of replications of j-th treatment, $j=1,2,\ldots,9$ has been indicated below each design beside the values of λ's.

D_4

$\lambda_1=4, \lambda_2=3$, other λ's$=1$,

$v=9$, $k=3$, $b=26$,

$\underline{r}=(10,10,10,10,10,6,6,6,6)$

DESIGN

123	186	321
456	429	215
789	753	154
147	678	124
258	789	235
369	896	134
159	967	135
726	543	245
483	432	

D_5

$\lambda_1=\lambda_2=0$, other λ's$= 1$

$v=9$, $k=2$, $b=20$

$\underline{r}' = (4,4,4,4,4,5,5,5,5)$.

DESIGN

16	36	56
17	37	57
18	38	58
19	39	59
26	46	
27	47	
28	48	
29	49	

D_7

$\lambda_{5j} = 1,\ j \in G_1,\ \lambda_{9j} = 1,\ j \in G_2$

$\lambda_{59} = 1$, other λ's $= 0$

$v=9,\ k=2,\ b=8$

$\underline{r}' = (1,1,1,1,5,1,1,1,4)$

DESIGN

15	69
25	79
35	89
45	59

D_8

$\lambda_{59} = 7,\ \lambda_{5j} = 3,\ j \in G_1$ or G_2

$\lambda_{9j} = 3,\ j \in G_1$ or G_2

$\underline{r}' = (3,3,3,3,7,3,3,3,7)$

$v=9,\ b=7,\ k=5$

DESIGN

1	2	3	5	9
1	4	5	8	9
1	5	6	7	9
2	4	5	6	9
2	5	7	8	9
3	4	5	7	9
3	5	6	8	9

D_9

$\lambda_{j,j+1 (\text{mod } 9)} = 2$

$\lambda_{j,j+2 (\text{mod } 9)} = 1$

other λ's $= 0$

$\underline{r}' = (3,3,3,3,3,3,3,3,3)$

DESIGN

1	2	3
2	3	4
3	4	5
4	5	6
5	6	7
6	7	8
7	8	9
8	9	1
9	1	2

6. Concluding remarks.

The inference procedures offered or discussed in this paper are based on a method extensively used by one of the authors and his collaborators, being a particular application of a general principle designated by them as the union-intersections principle [3,4,5]. The method used by them earlier and also in this paper consists essentially in expressing a complex (composite) hypothesis as the intersection of a number of more elementary (composite) components where each such elementary component has a test procedure that has an optimal property in a very natural and meaningful sense. For the complex hypothesis a test procedure is suggested that has an acceptance region which is the intersection of the acceptance regions of the test procedures for the elementary hypotheses. The chief motivation behind this method, only partially explained in previous publications, is the following. The expression of a complex hypothesis as the intersection of a number of components is by no means unique, and, when we express it as the intersection of a particular class of components, we are looking for a test procedure that should have a large power against each of the associated deviations, may be at the cost of being relatively poor against other directions of deviation. At this point of the process we merely hope that this is precisely what would be achieved by the procedure suggested. For a wide class of specific problems in the set up of univariate and multivariate "normal" responses, it so turns out that this is, in fact, achieved by the test procedures suggested, in the sense that, in each case, the suggested procedure does better, for its particular purpose, than the one ordinarily used or recommended. At this point the further question arises. Even assuming that, for its particular purpose, the suggested procedure is better than the one ordinarily recommended, is it in any sense optimal (again for particular purpose)? To answer this question we have to define carefully the criterion (or sense) under which we are looking for optimality, and, in test procedures involving several parametric functions, that go with the more complex problems (as opposed to those like F, r, the two kinds of multiple correlation and Hotelling's T^2, etc., that involve only one parametric function each, and go with much simpler problems) any such criterion that might be laid down would seem to be far less "natural" and convincing than the corresponding one for simpler types of problem. By and large, the situation is this. In each case the procedure we have offered can be shown to belong to a class that is good in a "natural" sense; and, among that class, the procedure can be shown to be optimal in a sense that is far less "natural" and convincing. Such results on the so-called "optimality" as are known to date on these specific problems, and any light in this sense that we may be able to throw on the general method itself will be offered later. But in these complex situations, we must caution once more against any hasty attempt to set up an optimality criterion, then obtain an optimal procedure under that criterion, and finally stay happy with that procedure.

Turning now to the design aspect of the total problem, we recall that, once we lay down the objective, it has been possible, at least for these problems, to suggest a test procedure that, under any design, in general, is good in the sense already explained (at this point no optimality is claimed). With this procedure at our disposal the further problem of choice of a good design, at least for these problems, would be governed by the possibility of (i) reducing the σ_k^2 (a widely occurring empirical phenomenon well known to the design of experiments and ANOVA people, but apparently almost totally unknown to other groups of mathematical statisticians) and (ii) further increasing the discrimination in the preferred directions (already favored under the test procedure) through the structure of the design sought for, if, necessary, by introducing some kind of asymmetry. The remarks made in the previous paragraph about "good" and "optimal" in relation to a test procedure would equally apply to the choice of a design. As, for example, in the above problems, it is possible to choose a design that would be good in the sense of being better than the customary ones, but optimality is another matter. For a general theoretical treatment of optimality, in addition to the difficulties already mentioned in relation to the test procedure, we encounter the further difficulty about defining a sufficiently wide and meaningful class of "good" designs among which the one sought for is to be "optimal". However, in the sense explained in section 5, we have a very restricted treatment of optimality. In other words, it is broadly indicated there how it is possible, on the basis of the criteria and techniques (i) and (ii) mentioned just now, to pick out an "optimal" design among a particular small class of designs each of which is "good". So far as the general treatment of optimality is concerned, the little that has been done, for whatever it is worth, will be discussed later. Finally, we would like to add the same note of caution as at the end of the last paragraph.

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