EQUALITY OF TWO DISPERSION MATRICES AGAINST

ALTERNATIVES OF INTERMEDIATE SPECIFICITY

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equal dispersion matrices is considered against several alterna-
tives of intermediate specificity, and a similar region test of
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## SUMMARY

For two multivariate nonsingular normal distributions, the familiar null hypothesis of equal dispersion matrices is considered against various alternatives stated in terms of certain characteristic roots and a physical interpretation is given for the alternatives considered. An inference procedure which depends on similar regions and is based on one independent random sample from each of the two distributions, is proposed for the null hypothesis against each of the alternative hypotheses. Also, for each case, conservative confidence bounds are obtained on one or more parametric functions which might be interpreted as measures of departure from the null hypothesis in the direction of the corresponding alternative.

## 1. INTRODUCTION

For two nonsingular p-variate normal distributions, $N[\underline{\mu}_1, \underline{\Sigma}_1]$ and $N[\underline{\mu}_2, \underline{\Sigma}_2]$, we start from the familiar null hypothesis $H_0: \underline{\Sigma}_1 = \underline{\Sigma}_2$. The characteristic roots, all positive,

of $\underline{\Sigma}_1 \underline{\Sigma}_2^{-1}$ no matter whether $H_0$ is true or not will be denoted by $\gamma_1, \gamma_2, \ldots, \gamma_p$. Most often the largest and smallest roots will be denoted respectively by $\gamma_M$ and $\gamma_m$. $H_0$ can now be stated in the form $H_0$:all $\gamma$'s $= 1$. As alternatives, however, the following are considered: (i) $H_1$:all $\gamma$'s $> 1$; (ii) $H_2$:all $\gamma$'s $< 1$; (iii) $H_3$:all $\gamma$'s $> 1$ or all $\gamma$'s $< 1$; (iv) $H_4$:at least one $\gamma > 1$; (v) $H_5$:at least one $\gamma < 1$; (vi) $H_6$:at least one $\gamma > 1$ and at least one $\gamma < 1$; (vii) $H_7$:at least one $\gamma > 1$ or $< 1$. It may be noted that (iii) is the union of (i) and (ii), (vi) is the intersection of (iv) and (v), (vii) is the union of (iv) and (v), and (vi), together with $H_0$, is the complement of (iii). Also, while each of the alternatives forms a mutually exclusive pair with $H_0$, yet only (vii) is the complement of $H_0$, and it is only (vii) that has attracted attention heretofore [2,5,10]. The relations in logical structure between the various alternatives may be useful in understanding the forms of the inference procedures proposed in Section 2 of this paper for $H_0$ against each of the alternatives. Section 3 discusses some conservative confidence bounds of varying degrees of appropriateness associated with the tests. Section 4 consists of some concluding remarks.

We consider one possible physical meaning of the alternatives considered in this paper. If $\underline{x}_1$(px1) is p-variate nonsingular $N[\underline{\mu}_1, \underline{\Sigma}_1]$ and $\underline{x}_2$(px1) is p-variate

nonsingular $N[\underline{\mu}_2, \underline{\Sigma}_2]$ and the elements (variates) of $\underline{x}_1$ are
physically of the same nature as those of $\underline{x}_2$ (i.e., for e.g.,
the first element in both is amount of steel produced, the
second element is total farm produce, etc.), then, if
$\underline{a}'(1 \times p) = (a_1, a_2, \ldots, a_p)$ is a vector of nonstochastic utili-
tarian "weights" that go with the p-variates, the linear
functions $\underline{a}'\underline{x}_1$, and $\underline{a}'\underline{x}_2$ are of utilitarian interest. It is
well known that $\underline{a}'\underline{x}_1$ is univariate $N(\underline{a}'\underline{\mu}_1, \underline{a}'\underline{\Sigma}_1\underline{a})$ and $\underline{a}'\underline{x}_2$ is
univariate $N(\underline{a}'\underline{\mu}_2, \underline{a}'\underline{\Sigma}_2\underline{a})$. If $\underline{a}'$ is known then a direct com-
parison of $\underline{a}'\underline{x}_1$ and $\underline{a}'\underline{x}_2$, for observed values of $\underline{x}_1$ and $\underline{x}_2$,
using the usual univariate techniques would be quite appropriate.
Thus, for instance, one may be interested in differences be-
tween the means $\underline{a}'\underline{\mu}_1$ and $\underline{a}'\underline{\mu}_2$, or in the ratio of the variances,
$\underline{a}'\underline{\Sigma}_1\underline{a}/\underline{a}'\underline{\Sigma}_2\underline{a}$. For a known system of utilitarian weights then,
one may, for instance, wish to test $H_o: \underline{a}'\underline{\Sigma}_1\underline{a}/\underline{a}'\underline{\Sigma}_2\underline{a} = 1$,
against $H_1: \underline{a}'\underline{\Sigma}_1\underline{a}/\underline{a}'\underline{\Sigma}_2\underline{a} > 1$. The test is the well known, one-
sided F-test. But now, if $\underline{a}'$ is not known or given, then one
may want to obtain a weight-free solution by protecting one-
self against the worst possible set of weights (in a sort of
minimax sense) and pose the question as a test of
$H_o: \underline{a}'\underline{\Sigma}_1\underline{a}/\underline{a}'\underline{\Sigma}_2\underline{a} = 1$ for all $\underline{a}$, against $H_1: \underline{a}'\underline{\Sigma}_1\underline{a}/\underline{a}'\underline{\Sigma}_2\underline{a} > 1$ for
all $\underline{a}$. This is exactly the null hypothesis of $H_o:$ all $\gamma$'s $= 1$,
against $H_1:$ all $\gamma$'s $> 1$. Of the other alternatives, $H_2$ and $H_3$
can be interpreted in exactly the same manner. According to
this interpretation $H_4$, $H_5$, $H_6$ and $H_7$ are much weaker alterna-
tives. $H_4$, for example, means $\underline{a}'\underline{\Sigma}_1\underline{a}/\underline{a}'\underline{\Sigma}_2\underline{a} > 1$ for at least

one $\underline{a}$, or in other words, that we are considering (in terms

of acceptance of $H_4$) the most favorable kind of weights (and

trying to reach in a sense a minimin solution). However, in

terms of acceptance of $H_0$, we stay with the same worst set of

weights. Similarly for $H_5$ to $H_7$. The main point in introduc-

ing $H_4$, $H_5$, $H_6$ and $H_7$ is to indicate how the customary $H_7$ shows

up according to our interpretation. Of Fisher's approach to

discriminant analysis and Hotelling's approach to canonical

correlations (in terms of taking a linear compound of the

variates and then maximizing certain quantities with respect

to these compounding coefficients) we have always preferred

this interpretation to the one that is more customary. But

this is a matter of opinion and we shall not press it here.

## 2. INFERENCE PROCEDURES FOR $H_0$ AGAINST EACH OF THE
## ALTERNATIVES OF SECTION 1

Let $\underline{S}_1$ and $\underline{S}_2$ be two (pxp) matrices based on

independent random samples of sizes $(n_1+1)$ and $(n_2+1)$ from

the two populations. Let these denote the maximum likelihood

estimators of $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ with the conventional bias correction.

We assume that p $\leq$ the smaller of $n_1$ and $n_2$, so that $\underline{S}_1$ and $\underline{S}_2$

are positive definite almost everywhere. Let $c_M$ and $c_m$ denote,

respectively, the largest and the smallest characteristic

roots of $\underline{S}_1\underline{S}_2^{-1}$. Also let $ch(\underline{A})$ denote the characteristic

root of any general (square) matrix $\underline{A}$ and $ch_m(\underline{A})$ and $ch_M(\underline{A})$

the smallest and largest roots. Then, using a heuristic

argument similar to that of [5], the following inference procedures, some of them three-decision procedures, are proposed, wherein $W(H)$ denotes the acceptance region for the hypothesis $H$, and $W(I)$, where it occurs, denotes the region of indecision or no choice between the two hypotheses in question:

$$\text{(i)} \quad W(H_o):c_M \leq \lambda_1; \; W(H_1):c_m > \lambda_1; \; W(I):c_m \leq \lambda_1 < c_M \; ,$$

$$\text{(ii)} \quad W(H_o):c_m \geq \lambda_2; \; W(H_2):c_M < \lambda_2; \; W(I):c_m < \lambda_2 \leq c_M \; ,$$

$$\text{(iii)} \quad W(H_o):\lambda_3 \leq c_m \leq c_M \leq \lambda_3'; \; W(H_3):c_m > \lambda_3' \text{ or } c_M < \lambda_3 \; ;$$

$$\text{(2.1)} \qquad W(I):c_m < \lambda_3 < c_M \quad \text{and/or} \quad c_m \leq \lambda_3' < c_M \; ,$$

$$\text{(iv)} \quad W(H_o):c_M \leq \lambda_4; \; W(H_4):c_M > \lambda_4 \; ,$$

$$\text{(v)} \quad W(H_o):c_m \geq \lambda_5; \; W(H_5):c_m < \lambda_5 \; ,$$

$$\text{(vi)} \quad W(H_o):\lambda_6 \leq c_m \leq c_M \leq \lambda_6'; \; W(H_6):c_m < \lambda_6 \text{ and } c_M > \lambda_6' \; ;$$

$$W(I):c_m < \lambda_6 \text{ and } c_M \leq \lambda_6' \text{ or } c_m \geq \lambda_6 \text{ and } c_M > \lambda_6' \; ,$$

$$\text{(vii)} \quad W(H_o):\lambda_7 \leq c_m \leq c_M \leq \lambda_7'; \; W(H_7):c_m < \lambda_7 \text{ and/or } c_M > \lambda_7' \; .$$

For Case (i), given $\lambda_1$ the probabilities assigned to the three regions, $W(H_o)$, $W(H_1)$ and $W(I)$, under $H_o$ can be determined. Likewise, given the probability assigned to the region $W(H_o)$ under $H_o$, $\lambda_1$ can be determined by the methods described in [3,4], and hence the probabilities assigned to $W(H_1)$ and $W(I)$ under $H_o$ may be determined. It should be noted that the method of evaluating the probability assigned to the region $W(I)$ under $H_o$, for a given $\lambda_1$, has not been explicitly

considered. The authors, however, feel that this will not present any essentially new difficulty and that the methods of [3,4] will be applicable to this problem also.

Similar remarks hold concerning the determination of the other $\lambda$'s, in Cases (ii)-(vii), under (2.1). For Cases (iii), (vi) and (vii), where we have two constants to determine since the inference procedures are two sided in each of these cases, in addition to the conditions of a given probability for $W(H_o)$ under $H_o$, we may impose the condition of local unbiasedness of each of these tests. These two conditions taken together will enable us to determine both constants involved uniquely. As discussed in [3,5,9], for Case (vii), the condition of local unbiasedness implies certain optimum power properties of the test for this case. For the other two cases, however, such implications of the condition of local unbiasedness are yet to be established. Further, regarding all the $\lambda$'s in (2.1), it should be noted that, in addition to depending on the conditions discussed above, they are also functions of p, $n_1$ and $n_2$.

Case (vii), as noted in Section 1, with the test given under Case (vii) of (2.1), is the one that has been considered in great detail elsewhere [5,6,7,8] and is included here merely for completeness.

Finally, it can be seen that all the probabilities (under $H_o$) associated with the procedures proposed under (2.1) are independent of nuisance parameters.

## 3. ASSOCIATED CONFIDENCE BOUNDS

Given a pair $(H_o, H)$ of composite hypothesis and
alternative, disjoint but not necessarily exhaustive, we
seek a parametric function that might be regarded as a
measure of departure from $H_o$ in the direction of $H$, or,
alternatively, some kind of a distance function between the
set $H_o$ and the set $H$. We next seek a confidence interval
for this parametric function, one sided (one way or the
other) or two sided, depending upon the nature of the
pair $(H_o, H)$. No claim is made at this stage that the parametric
function chosen or the confidence interval proposed for it is
in some sense optimal. As to the confidence coefficient, it
would be very desirable if given any permissible $1-\alpha$, the
interval could be defined such that this coefficient were
equal to $1-\alpha$. If it does not turn out that way, the next
best thing would be to have a confidence coefficient $\geq 1-\alpha$,
given any permissible $\alpha$, such that the equality is attained,
or, in other words, that the probability of the interval
covering the parametric function, for some value of this
function, is equal to $1-\alpha$. If this does not happen, the
next best thing would be, for any permissible $1-\alpha$, to have a
confidence coefficient whose greatest lower bound $\geq 1-\alpha$ (and
might, in fact, be greater than $1-\alpha$), provided that the
interval itself is not trivial, for example, $(0, \infty)$ or $(-\infty, \infty)$,
etc., but is, in fact, much better than these. We shall say
that such a confidence coefficient is a conservative one, or

alternatively, such a confidence region is a conservative one.
For really complex problems even this may be difficult to
obtain, to say nothing of intervals of the first or the
second kind, and we would consider even this quite worthwhile,
especially in view of the fact that we consider it more im-
portant to estimate this "distance function," pointwise or
intervalwise, than to test (and accept or reject) the usual
null hypothesis as such. All confidence intervals obtained
in this section are conservative. For Case (i), $(H_o, H_1)$, we
have a lower bound on $\gamma_m$, for Case (ii), $(H_o, H_2)$, an upper
bound on $\gamma_M$ and for Case (iii) a lower bound on $\gamma_m$ and/or an
upper bound on $\gamma_M$. For Cases (iv) - (vii), that is, for
$(H_o, H_i)$ (i $= 4, 5, 6, 7$) we have attempted but have failed so
far to obtain a lower bound on $\gamma_M$ for Case (iv), an upper
bound on $\gamma_m$ for Case (v), a lower bound on $\gamma_M$ and an upper
bound on $\gamma_m$ for Case (vi). The trouble seems to stem from
the difficulty in obtaining a lower bound on $\gamma_M$ that is not
also a lower bound on $\gamma_m$ or an upper bound on $\gamma_m$ that is not
also an upper bound on $\gamma_M$. However, we find that, if we re-
place $\gamma_m$ by $\gamma_m^* = ch_m(\underline{\Sigma}_1)/ch_M(\underline{\Sigma}_2)$ and $\gamma_M$ by $\gamma_M^* = ch_M(\underline{\Sigma}_1)/ch_m(\underline{\Sigma}_2)$,
then bounds similar to those we were seeking for
Cases (iv) - (vii) become feasible. The question now is, how
are these intervals related to $(H_o, H_i)$ (i $= 4, 5, 6, 7$)? For ex-
ample, how is $\left[ \gamma_M^* \geq \mu \right]$ related to $(H_o, H_4)$. In our sense it is
not a natural associate of $(H_o, H_4)$. If we consider $H_o^*: \underline{\Sigma}_1 = \underline{\Sigma}_2 = \delta \underline{I}$
(a diagonal matrix with all diagonal elements equal to $\delta$) and

$H_4^* : \gamma_M^* > 1$, we observe that $H_o^* \subset H$ and $H_4^* \supset H_4$, and $\left[ \gamma_M^* \geq \mu \right]$ is really a natural associate of $\left( H_o^*, H_4^* \right)$. The reader can interpret similar bounds for the Cases (v), (vi) and (vii). We discuss the Cases (v) - (vii) very briefly and Case (iv) in some detail to exhibit the kind of mathematics used here that might also be useful (together with some additional tools) in obtaining the kind of bounds we sought and have so far failed to obtain. The main purpose in presenting the results for Cases (iv) - (vii) is to help in possible further attempts at obtaining the more meaningful confidence intervals that we sought.

Case (i): Using the canonical form of the distribution of the observations and proceeding exactly as in Sections 5.1 and 5.2 of [6] and Sections 1 and 2 of [8], we can attach a preassigned probability 1-$\alpha$ to the region in the sample space defined by

$$(3.1) \qquad ch_M\left( \underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_1 \underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_2^{-1} \right) \leq \lambda_1 \, ,$$

where $\lambda_1$ is the constant under (2.1) such that

$P\left[ ch_M\left( \underline{S}_1 \underline{S}_2^{-1} \right) \leq \lambda_1 \mid H_o \right] = 1-\alpha$. Also, $\underline{D}_a$ denotes a diagonal matrix whose diagonal elements are $a_1, a_2, \ldots,$ and $\gamma_1, \ldots, \gamma_p$ have been already defined to be characteristic roots of $\underline{\Sigma}_1 \underline{\Sigma}_2^{-1}$.

Notice that the parameters also enter into the characterization of the region. Now (3.1) is equivalent to

$ch_M\left(\underline{D}_{1/\sqrt{\gamma}}\,\underline{S}_1\underline{D}_{1/\sqrt{\gamma}}\,\underline{S}_1^{-1}\underline{S}_1\underline{S}_2^{-1}\right) \leq \lambda_1$ which, in turn, is equivalent to the set of simultaneous confidence regions

$$(3.2) \qquad \frac{\underline{a}'\left(\underline{D}_{1/\sqrt{\gamma}}\,\underline{S}_1\underline{D}_{1/\sqrt{\gamma}}\,\underline{S}_1^{-1}\right)\underline{a}}{\underline{a}'\underline{S}_2\underline{S}_1^{-1}\underline{a}} \leq \lambda_1$$

for all nonnull vectors $\underline{a}(p\times1)$, with a joint confidence coefficient $1-\alpha$. Equation (3.2) may be rewritten as

$$(3.3) \qquad \frac{\underline{a}'\left(\underline{D}_{1/\sqrt{\gamma}}\,\underline{S}_1\underline{D}_{1/\sqrt{\gamma}}\,\underline{S}_1^{-1}\right)\underline{a}}{\underline{a}'\underline{a}} \leq \lambda_1\,\frac{\underline{a}'\underline{S}_2\underline{S}_1^{-1}\underline{a}}{\underline{a}'\underline{a}}\,,$$

for all nonnull vectors $\underline{a}$. Choosing $\underline{a}$ successively so as to maximize, one after the other, both sides of (3.3), it follows that (3.3) implies that

$$(3.4) \qquad ch_M\left(\underline{D}_{1/\sqrt{\gamma}}\,\underline{S}_1\underline{D}_{1/\sqrt{\gamma}}\,\underline{S}_1^{-1}\right) \leq \lambda_1\,ch_M\left(\underline{S}_2\underline{S}_1^{-1}\right).$$

We shall be using the phrase "choosing $\underline{a}$ successively,..."
repeatedly. What this precisely implies is the following.
Choose $\underline{a}$ so as to maximize the left side of (3.3) and denote
this value of $\underline{a}$ by $\underline{a}^*$. Then it follows that (3.3) implies

$$\text{ch}_M\left(\underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_1\underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_1^{-1}\right) \leq \lambda_1\, \frac{\underline{a}^{*\prime}\underline{S}_2\underline{S}_1^{-1}\underline{a}^*}{\underline{a}^{*\prime}\underline{a}^*}\ .$$

But the right side of the last inequality can be further in-
creased to $\lambda_1\text{ch}_M\left(\underline{S}_2\underline{S}_1^{-1}\right)$, whence (3.4) follows. This line of
reasoning has been repeatedly used in [8,9]. Returning to
(3.4) and writing $\underline{S}_1 = \underline{T}\,\underline{T}'$, where $\underline{T}$ is a triangular matrix
with zeros above the diagonal, and remembering that any non-
zero $\text{ch}(\underline{A}\,\underline{B})$ = a nonzero $\text{ch}(\underline{B}\,\underline{A})$, we obtain from (3.4),

$$(3.5) \qquad \text{ch}_M\left[\underline{T}^{-1}\underline{D}_{1/\sqrt{\gamma}}\, \underline{T}\,\underline{T}'\underline{D}_{1/\sqrt{\gamma}}\, (\underline{T}')^{-1}\right] \leq \lambda_1\text{ch}_M\left(\underline{S}_2\underline{S}_1^{-1}\right)\ .$$

But if $\underline{A}$ is a real matrix with real $\text{ch}(\underline{A})$, then it is known
that

$$(3.6) \qquad \text{ch}_m(\underline{A}\,\underline{A}') \leq [\text{ch}_m(\underline{A})]^2 \leq [\text{ch}_M(\underline{A})]^2 \leq \text{ch}_M(\underline{A}\,\underline{A}')\ .$$

Hence, (3.5) implies that

$$\left[ \text{ch}_M\!\left( \underline{T}^{-1}\underline{D}_{1/\sqrt{\gamma}}\ \underline{T} \right) \right]^2 = \left[ \text{ch}_M\!\left( \underline{D}_{1/\sqrt{\gamma}} \right) \right]^2 = \text{ch}_M\!\left( \underline{\Sigma}_2\underline{\Sigma}_1^{-1} \right) \leq \lambda_1 \text{ch}_M\!\left( \underline{S}_2\underline{S}_1^{-1} \right),$$

or, equivalently,

$$(3.7) \qquad\qquad \gamma_m \geq \mu_1 c_m\ ,$$

where $\mu_1 = 1/\lambda_1$. Equation (3.7) is thus a confidence interval with a conservative confidence coefficient $\geq 1-\alpha$.

Case (ii): Our starting point here is the region with a preassigned probability $1-\alpha$,

$$(3.8) \qquad\qquad \text{ch}_m\!\left( \underline{D}_{1/\sqrt{\gamma}}\ \underline{S}_1\underline{D}_{1/\sqrt{\gamma}}\ \underline{S}_2^{-1} \right) \geq \lambda_2\ ,$$

where $\lambda_2$ is the constant under (2.1) such that $P\left[ \text{ch}_m\!\left( \underline{S}_1\underline{S}_2^{-1} \right) \geq \lambda_2 \,\middle|\, H_o \right] = 1-\alpha$. Reasoning the same way as we did in obtaining (3.3) to (3.7), we show that (3.8) implies

$$(3.9) \qquad\qquad \gamma_M \leq \mu_2 c_M\ ,$$

where $\mu_2 = 1/\lambda_2$. Equation (3.9) is thus a confidence interval with a conservative confidence coefficient $\geq 1-\alpha$.

Case (iii): Our starting point here is the region with preassigned probability 1-α,

$$(3.10) \quad \mathrm{ch}_m\left(\underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_1 \underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_2^{-1}\right) \geq \lambda_3' \quad \text{or} \quad \mathrm{ch}_M\left(\underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_1 \underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_2^{-1}\right) \leq \lambda_3 \,,$$

where $\lambda_3 < \lambda_3'$ are constants such that

$$P\left[\mathrm{ch}_m\left(\underline{S}_1\underline{S}_2^{-1}\right) \geq \lambda_3' \quad \text{or} \quad \mathrm{ch}_M\left(\underline{S}_1\underline{S}_2^{-1}\right) \leq \lambda_3 \,\Big|\, H_o\right] = 1-\alpha, \text{ and}$$

further such that the test (iii) under (2.1) is locally unbiased. As before, we notice that (3.10) is equivalent to the set of simultaneous confidence regions

$$\frac{\underline{a}'\left(\underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_1 \underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_1^{-1}\right)\underline{a}}{\underline{a}'\underline{a}} \geq \frac{\lambda_3'\,\underline{a}'\underline{S}_2\underline{S}_1^{-1}\underline{a}}{\underline{a}'\underline{a}} \,,$$

(3.11)                    or

$$\frac{\underline{a}'\left(\underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_1 \underline{D}_{1/\sqrt{\gamma}}\, \underline{S}_1^{-1}\right)\underline{a}}{\underline{a}'\underline{a}} \leq \frac{\lambda_3\,\underline{a}'\underline{S}_2\underline{S}_1^{-1}\underline{a}}{\underline{a}'\underline{a}} \,,$$

for all nonnull vectors $\underline{a}$, with a joint confidence coefficient = 1-α. Proceeding now exactly as in Cases (i) and (ii), we find that (3.11) implies

(3.12)         $\gamma_m \geq \mu_3 c_m$   or   $\gamma_M \leq \mu_3' c_M$ ,

where $\mu_3 = 1/\lambda_3$ and $\mu_3' = 1/\lambda_3'$ so that $\mu_3 > \mu_3'$. Equation (3.12) is thus a confidence region with a conservative confidence coefficient $\geq 1-\alpha$.

In Case (i), if in addition to a lower bound, we are also interested in an upper bound on the $\gamma$'s, or in Case (ii), if in addition to an upper bound, we are also interested in a lower bound on the $\gamma$'s, we can find a confidence region

(3.13)         $\mu_3^* c_M \geq \gamma_M \geq \gamma_m \geq \mu_3^{**} c_m$ ,

with a conservative confidence coefficient $\geq 1-\alpha$, where $\mu_3^*$ and $\mu_3^{**}$ are given by

(3.14)         $P\left[\dfrac{1}{\mu_3^*} \leq c_m \leq c_M \leq \dfrac{1}{\mu_3^{**}} \,\middle|\, H_o\right] = 1-\alpha$ ,

and the condition of local unbiasedness. This is precisely the confidence statement that in [8] was associated with Case (vii). From our present viewpoint this association is inappropriate and the proper association of (3.13) is with the situation mentioned just before (3.13). It may be noticed, however, that in seeking also an upper bound

in Case (i) or also a lower bound in Case (ii), we are going beyond what is suggested by the pair $(H_o, H_1)$ or the pair $(H_o, H_2)$, and we are basing our additional interest on some additional consideration or requirement.

Case (iv): Taking the approach of [1], for this case, we write $\underline{S}_1^* = \underline{D}_1/\sqrt{\gamma_1}\,\underline{\Lambda}_1\underline{S}_1\underline{\Lambda}_1'\underline{D}_1/\sqrt{\gamma_1}$ and $\underline{S}_2^* = \underline{D}_1/\sqrt{\gamma_2}\,\underline{\Lambda}_2\underline{S}_2\underline{\Lambda}_2'\underline{D}_1/\sqrt{\gamma_2}$, where $\gamma_1$'s are $ch(\underline{\Sigma}_1)$, $\gamma_2$'s are $ch(\underline{\Sigma}_2)$ and $\underline{\Lambda}_1, \underline{\Lambda}_2$ are orthogonal matrices defined by the transformations $\underline{\Sigma}_i = \underline{\Lambda}_i'\underline{D}_{\gamma_i}\underline{\Lambda}_i$, $(i = 1,2)$. We take as our starting point the region

$$(3.15) \qquad D_4: \quad \frac{ch_m\left(\underline{S}_1^*\right)}{ch_M\left(\underline{S}_2^*\right)} \leq \lambda \ ,$$

where $\lambda$ is such that $P[D_4] = 1-\alpha$, no matter what $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ happen to be. It is known that if A is positive definite and B is at least positive semidefinite, then

$$(3.16) \qquad ch_m(\underline{A})ch_m(\underline{B}) \leq ch_m(\underline{A}\,\underline{B}) \leq ch_M(\underline{A}\,\underline{B}) \leq ch_M(\underline{A})ch_M(\underline{B}) \ .$$

Using (3.16), we have

$$ch_m\left(\underline{S}_1^*\right) \geq ch_m\left(\underline{D}_1/\gamma_1\right)ch_m\left(\underline{\Lambda}_1\underline{S}_1\underline{\Lambda}_1'\right) = ch_m\left(\underline{S}_1\right)/ch_M\left(\underline{\Sigma}_1\right) \ , \text{ and}$$

$$ch_M\left(\underline{S}_2^*\right) \leq ch_M\left(\underline{D}_1/\gamma_2\right)ch_M\left(\underline{\Lambda}_2\underline{S}_2\underline{\Lambda}_2'\right) = ch_M\left(\underline{S}_2\right)/ch_m\left(\underline{\Sigma}_2\right) \ . \quad \text{Hence,}$$

(3.15) implies

$$\frac{ch_m(\underline{S}_1)}{ch_M(\underline{\Sigma}_1)} \cdot \frac{ch_m(\underline{\Sigma}_2)}{ch_M(\underline{S}_2)} \leq \lambda \quad,$$

or, equivalently,

$$(3.17) \qquad \gamma_M^* \geq \nu \, \frac{ch_m(\underline{S}_1)}{ch_M(\underline{S}_2)} \quad,$$

where $\nu = 1/\lambda$. Equation (3.17) is thus a confidence interval with a conservative confidence coefficient $\geq 1-\alpha$.

<u>Case (v)</u>: Using the notation above for Case (iv), we take as our starting point the region

$$(3.18) \qquad D_5 : \; \frac{ch_M\left(\underline{S}_1^*\right)}{ch_m\left(\underline{S}_2^*\right)} \geq \lambda' \quad,$$

where $\lambda'$ is such that $P[D_5] = 1-\alpha$, no matter what $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ happen to be. Reasoning as in Case (iv) we end up with

$$(3.19) \qquad \gamma_m^* \leq \nu' \, \frac{ch_M(\underline{S}_1)}{ch_m(\underline{S}_2)} \quad,$$

where $\nu' = 1/\lambda'$.

<u>Case (vi)</u>: Our starting point here is the region

$$(3.20) \quad D_6: \quad \frac{ch_m\left(\underline{S}_1^*\right)}{ch_M\left(\underline{S}_2^*\right)} \leq \lambda^* \quad \text{and} \quad \frac{ch_M\left(\underline{S}_1^*\right)}{ch_m\left(\underline{S}_2^*\right)} \geq \lambda^{*\prime} \quad ,$$

where $\lambda^* < \lambda^{*\prime}$ are such that $P[D_6] = 1-\alpha$, no matter what $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ happen to be. Reasoning exactly as before, we end up with

$$(3.21) \quad \gamma_m^* \leq \nu^{*\prime} \, \frac{ch_M(\underline{S}_1)}{ch_m(\underline{S}_2)} \quad \text{and} \quad \gamma_M^* \geq \nu^* \, \frac{ch_m(\underline{S}_1)}{ch_M(\underline{S}_2)} \, ,$$

where $\nu^* = 1/\lambda^*$ and $\nu^{*\prime} = 1/\lambda^{*\prime}$ so that $\nu^* > \nu^{*\prime}$.

Case (vii): Proceeding as in the Cases (iv)-(vi), we have

$$(3.22) \quad \gamma_m^* \leq \nu_0^{\prime} \, \frac{ch_M(\underline{S}_1)}{ch_m(\underline{S}_2)} \quad \text{and/or} \quad \gamma_M^* \geq \nu_0 \, \frac{ch_m(\underline{S}_1)}{ch_M(\underline{S}_2)} \, ,$$

where $\nu_0 = 1/\lambda_0$, $\nu_0^{\prime} = 1/\lambda_0^{\prime}$ and $\lambda_0 < \lambda_0^{\prime}$ are constants such that $P\left[\frac{ch_m\left(\underline{S}_1^*\right)}{ch_M\left(\underline{S}_2^*\right)} \leq \lambda_0^{\prime} \quad \text{and/or} \quad \frac{ch_M\left(\underline{S}_1^*\right)}{ch_m\left(\underline{S}_2^*\right)} \geq \lambda_0^{\prime}\right] = 1-\alpha$, no matter what $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ happen to be.

## 4. CONCLUDING REMARKS

The procedures proposed here are heuristic, and investigations are underway as to the properties of these procedures, as, for example, unbiasedness, monotonicity and admissibility for the two-decision procedures and analogous properties of the three-decision procedures. Such properties have already been established for some of the two-decision procedures, including Case (vii) of (2.1) and (3.13) which we obtain by "inverting" the former. Also under consideration are the problem of partial statements in the sense of [8] and a generalization to the case of more than two dispersion matrices.

However, the more urgent and immediate problems are, if possible, to obtain (a) the meaningful bounds on $\gamma_m$ and $\gamma_M$ (for Cases (iv)-(vii)) that we sought but could not present in this paper and (b) the greatest lower bound on the conservative confidence coefficients obtained so far.

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