UNIVERSITY OF NORTH CAROLINA
Department of Statistics
Chapel Hill, N. C.

Mathematical Sciences Directorate
Air Force Office of Scientific Research
Washington 25, D. C.

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# ON THE RECURRENCE OF PATTERNS

by

Marcel Paul Schützenberger

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Simple formula are derived for the generating function of the recurrence event defined by a set of words, (or "patterns").



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## I. Introduction.

Let us consider a Bernoulli process in which the individual outcomes are the letters $x_i \in X$ and a set of words (or "patterns") $U = u_j$ in these letters. By definition, the event $\underline{U}$ occurs for the first time at the end of a sequence f of letters if the last letters of f form a word from U and if this occurence has not happened earlier on f.

This type of recurrent event has practical applications in engineering and in statistics and it has been considered at some length by W. Feller (1) especially in the case where U reduces to the word $u = x_1^n$ ( $= x_1 x_1 \ldots x_1$ ), i.e. in the case of the so called success run of length n. In this note we develop Feller's results in order to get a simple expression for the generating function of $\underline{U}$ when U is a finite set and with the help of some combinatorial properties of free monoids we discuss in more details the case where U reduces to a single word.

## II. Notations.

We denote by F the free monoid generated by X (Cf 2, chap. 1), that is the set of all finite words in the letters $x_i \in X$ with the product ff' defined for any $f, f' \in F$ as the word made up of the word f followed by the word f'. The empty word is always denoted by e; $/f/$ denotes the degree (or length) of f (i.e. $/e/ = 0$ and $/fx/ = /f/ + 1$ for any $f \in F$ and $x \in X$).

We consider a mapping $\lambda_t$ which sends every $x \in X$ onto $t\Pr(x)$ where t is an ordinary variate and where $\Pr(x)$ is the probability of x; $\lambda_t$ extends in a natural fashion to an homomorphism of the free algebra generated by X into the ring of the polynomials in t. We further extend $\lambda_t$ to the module $\bar{F}$ of the infinite sums $\bar{f} = \sum_{f \in F} \langle \bar{f}, f \rangle f$ with real coefficients $\langle \bar{f}, f \rangle$ which have the property that the ordinary power series in t $\sum_{f \in F} |\langle \bar{f}, f \rangle| \lambda_t f$ is convergent for all probability distributions $\{\Pr(x)\}$ in some open domain around 0. Then, classically, $\bar{F}$ is a topological algebra and every $\bar{f} \in \bar{F}$ which is such that $\langle \bar{f}, e \rangle = 1$ admits a (continuous) inverse

$$\overline{\bar{f}}^{-1} = e + \sum_{n>0} (-\bar{f})^n .$$

We observe that for any subset F' of F the sum $\sum_{f \in F'} f$ ( = 0 , if F' is empty) belongs to $\bar{F}$ and we shall usually represent it by the same letter F'; then, $\lambda_t F'$ is just the ordinary generating function of the subset F'. With these notations, the intersection of two subsets F' and F" can be interpreted as a special case of the Hadamard product $\sum_{f \in F} \langle F', f \rangle \langle F'', f \rangle f$ of the two elements F' and F" of $\bar{F}$.

Let now $\underline{\underline{E}}$ be any recurrent event on F and A (resp. A*)be the set of the words at the end of which $\underline{\underline{E}}$ occurs (resp. occurs for the first time); we denote by S the complement in F of the right ideal A*F and, consequently, S is the set of the word on which $\underline{\underline{E}}$ has not yet occured; we recall that it is a natural convention to assume that $e \in A$, $e \in S$ but $e \notin A^*$.

Feller's original definition can be translated into the two following statements:

II.1. Every $f \in F$ can be factorized in one and only one manner as a product $f = as$ with $a \in A$ and $s \in S$.

II.1'. Every $a \in A^+$ ( $= A - \{e\}$ ) can be factorized in a unique manner as a product $a = a_{i_1} a_{i_2} \dots a_{i_m}$ where the words $a_{i_j}$ all belong to $A^*$.

Thus, after Feller, we can write the following identities in $\overline{F}$ :

$S = F - A^*F = (e - A^*)(e - X)^{-1}$ (since $F = (e - X)^{-1}$ );

$A = (e - A^*)^{-1}$ or, in equivalent fashion since $A = e + A^+$ is invertible,

$A^* = A^+ (e + A^+)^{-1}$ .

If $\underline{\underline{E}}$ is persistent (i.e., by definition, if $\lim_{t \to 1} \lambda_t A^* = 1$ ), the mean recurrence time $\tau$ of $\underline{\underline{E}}$ is equal to $\lim_{t \to} (1 - \lambda_t A^*)(1 - \lambda_t X)^{-1}$ and because of our previous formula we have also $\tau = \lim_{t \to 1} \lambda_t S$ .

III. <u>A general formula</u>.

We revert to the problem described in the introduction and we consider a fixed, not empty, set of words U; according to our definitions a word $a \in F$ belongs to $A^*$ if and only if it satisfies the two conditions:

i. the word a ends with some word $u \in U$, that is, a belongs to some left ideal Fu;

ii. it is not possible to factorise a in the form $a = fuf'$ with $f \in F$, $u \in U$ and $f' \in F^+$ ( $= F - \{e\}$ ).

It follows instantly that we still define the same recurrent event $\underline{\underline{U}}$ if we reduce U by eliminating from it all the words u' which are such that $u' \in FuF$, for some other word $u \in U$.

Let us introduce the following notations for each u belonging to the reduced set U:

$$A_u = A \cap Fu \quad ; \quad A^*_u = A^* \cap Fu.$$

In the algebra $\overline{F}$ we have the identities:

$$A = e + \sum_{u\in U} A_u \; ; \quad A_u^* \; ; \; A_u = A\, A_u^* \quad .$$

As observed by Feller the value of $\lambda_t\, A_u^*$ for $t = 1$ is exactly the probability $\pi_u$ that the word $u$ occurs before any other word $u'$ from $U$.

Multiplying on the left by $(e - X)$ the last identity above and using the fact that $e - X + \sum_{u\in U} (e - X)\, A_u$ is invertible we obtain

III.1. $$A_u^* = (e - X + \sum_{u\in U} (e - X)\, A_u)^{-1}\, (e - X)\, A_u .$$

Since, $\lambda_1 X = 1$ and, as well known, $U$ is persistent, we deduce

III.2. $$\pi_u = \lim_{t\to 1} \frac{\lambda_t (e-X)\, A_u}{\sum_{u\in U} \lambda_t (e-X) A_u}$$

In similar manner, we find that $S = (e+ \sum_{u\in U} A_u)^{-1} (e-X)^{-1} = ((e-X)(e+ \sum_{u\in U} A_u))^{-1}$ $= (e-X+ \sum_{u\in U} (e-X)A_u)^{-1}$ and finally,

III.2'. $$\tau = \lim_{t\to 1} \left( \sum_{u\in U} \lambda_t (e - X)\, A_u\right)^{-1} .$$

Thus the problem reduces to the computation of the sums $(e-X)A_u$ and we prove:

III.3. For any $u\in U$ one has the identity in $\overline{F}$

$$u = \sum_{u'\in U} (e-X)\, A_{u'}\, M_{u',u}$$

where the set corresponding to $M_{u',u}$ is defined as the set of the words $f$ of degree strictly less than $u$ that satisfy the relation $u'f \in Fu$.

Proof. Let us left-multiply the identity by $(e - X)^{-1}$; by definition, its left member $(e - X)^{-1}u$ is equal to Fu, the set of all words ending with u.

Because of II.1, any word fu $\epsilon$ Fu admits one and only one factorisation fu = as with a $\epsilon$ A and s $\epsilon$ S; by the very definition of U it is impossible that $/s/ \geq /u/$ because, then[1], s would have the form s = f'u and it would belong to A*; thus, $/s/ < /u/$ and consequently a $\neq$ e. Because of II.1'., this implies that a = a"a' with a"$\epsilon$A and a' contained in a certain $A^{*}_{u'}$ (u' $\epsilon$ U), that is a' = f'u'. Since U is assumed to be reduced, u' cannot be a proper factor of u and there exists a uniquely determined word f" satisfying the relations f = a"f'f" and f"u = u's. Thus we have proved that s belongs to $M_{u',u}$.

(1) Here as repeatedly in this paper we use the following theorem of F.W. Levy ( 3) which embodies the so called cancellation laws of the free monoid:

If $f_1, f_2, f_3, f_4 \in F$ are such that $f_1 f_2 = f_3 f_4$ and $/f_1/ \leq /f_3/$, there exists one and only one $f_5 \in F$ which satisfies the equations $f_3 = f_1 f_5$ and $f_2 = f_5 f_4$.

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Reciprocally, any word of the form f = as with a $\epsilon$ $A_{u'}$ and s $\epsilon$ $M_{u',u}$ belongs to Fu and because of the uniqueness of the factorisation described in the first part of the proof this remark concludes the verification of the identity.

Let us assume now that $U$ is a finite set and consider the UxU matrix $M$ whose entries are the sums $M_{u',u}$; since e belongs to $M_{u',u}$ if and only if $u = u'$, det $\lambda_t M$ is arbitrarily close to one when $t$ tends to zero. Consequently, the system formed by the linear equations

$$\lambda_t u = \sum_{u=\in U} (\lambda_t (e - X)A_{u'})(\lambda_t M_{u',u})$$

in the unknown generating functions $\lambda_t(e - X)A_{u'}$ can be solved by determinental expressions once the $\lambda_t M_{u',u}$'s are known and, using the formulas III.1, III.2 and III.2', this provides us with the desired result. For example, when $U = \{ u \}$, a single word, we have

$$\bar{\tau} = \lim_{t \to 1} \frac{\lambda_t M_{u,u}}{\lambda_t u}$$

It can be proved that the non commutative sums $A_u$ can also be obtained from the equations III.3 by successive elimination but, by necessity, the formulas are rather cumbersome. However, for $U = \{ u \}$ we have

III.4 $$A_u = (e - X)^{-1} u \; M_{u,u}^{-1}, \quad A^*_u = (e-X)^{-1} u((eX)M_{u,u}+u)^{-1}(e-X).$$

For $U = \{ u,u' \}$ we have for instance

III.4'. $$A_u = (e - X)^{-1} (u - u'M^{-1}_{u',u'})(M_{u,u} - M_{u,u'}M^{-1}_{u',u'} M_{u',u})^{-1}.$$

As a straightforward consequence of the general formulas we mention the following fact whose proof is left to the reader

III.5. If U is the direct union of sets $U_j$ which are such that $M_{u',u} = 0$ when $u' \in U_{j'}$, $u \in U_j$, $u \neq j'$, the generating function $A^+$ ($= A - \{e\}$)corresponding to the event U is the direct sum of the generating functions $A_j^+$ corresponding to each of the events $\underline{\underline{U}}_j$.

This explains the simplicity of the results when U is the union of runs $u_j = x_j^{n_j}$ of length $n_j$ in the letters $x_j$.

IV. <u>A symmetry property</u>.

Let us denote by $f \longrightarrow \tilde{f}$ the involution of F which sends every $f = x_{i_1} x_{i_2} \dots x_{i_m}$ onto $\tilde{f} = x_{i_m} \dots x_{i_2} x_{i_1}$ and consider the set $\tilde{\underline{\underline{U}}} = \{\tilde{u}_j\}$ to which we associate as above a recurrent event u and the set A'* of the words at the end of which $\tilde{\underline{\underline{U}}}$ occurs for the first time.

IV.1. There exists a bijection $\tilde{\beta}$ : A* $\longrightarrow$ A'* which commutes with $\lambda_t$.

<u>Proof</u>. It is more convenient to keep U unchanged and to define B* by the following conditions symmetric of those used in the definition of A*: the word b belongs to B* if and only if:

i'. it begins with some word $u \in U$, that is, it belongs to some right ideal uF;

ii'. it is not possible to factorise b in the form $b = f'uf$ with $f \in F$, $u \in U$ and $f' \in F^+$.

Thus, clearly, $A'^* = \{\tilde{b} : b \in B^*\}$ and we only have to prove the existence of a bijection $\beta$: A* $\longrightarrow$ B*.

Let us consider any word $a = fu$ from $A^*_u$ where $u = x_{i_1} x_{i_2} \dots x_{i_m}$.
If $f = e$, we define $\beta a$ as being a itself;

if $f \neq e$, we consider the words $a_{(k)} = x_{i_{m-k+1}} x_{i_{m-k+2}} \ldots x_{i_m} f x_{i_1} x_{i_2} \ldots x_{i_{m-k}}$ obtained by a cyclic permutation of the letters of $a$ and we define $\beta a$ as $a_{(k)}$ where $k$ is the smallest integer which is such that $a_{(k)}$ belongs to some right ideal $u'F$ ($u' \in U$).

We surely have $k \leq m$ ($= /u/$) since $a_{(m)} = uf$ belongs to $uF$; if $\beta a = a_{(k)}$, all the words $fx_{i_1} x_{i_2} \ldots x_{i_{m-k'}}$ with $k' < k$ belong to $S = F\text{-}A^*F$ and, by the very definition of $A^*$ they do not admit a factor from $U$; thus, by construction, $\beta a$ satisfies i' and according to this last remark it also satisfies ii'; this proves the existence of a mapping $\beta: A^* \longrightarrow B^*$.

Reciprocally, given any $b = uf$ in $B^*$, we can find a minimal cyclic permutation $\alpha b$ of its letters which belongs to some left ideal $Fu'$ ($u' \in U$) and, as above $\alpha b$ belongs to $A^*$. Since, trivially, $(\alpha \circ \beta) a = a$ and $(\beta \circ \alpha) b = b$ for any $a \in A^*$ and $b \in B^*$, the mapping $\beta$ is a bijection and the proof is ended.

Remark. A slightly less explicit result follows instantly from $S = F - A^*F$ and from the conditions i and ii used for defining $A^*$ which, together show that $S$ is the complement in $F$ of the two sided ideal $FUF$, or in our present notations, that $S$ can also be defined as $F - FB^*$.

V. The case of U consisting of a single word.

Although, as we shall see, $M_{u,u}$ has the simple form $(e - v^n)(e - v)^{-1}$ for "almost all" words $u$, its expression can happen to be more intricate (as, e.g., for $u = xyxzxyxyxzxyx$ ) and we shall discuss a recurrence formula for computing it.

These remarks have some relevance also for the general case since, if $u \neq u'$ and $M_{u',u} \neq 0$, we have $M_{u',u} = (M_{u'',u''})\, g$ where $g$ is the word of lowest degree in $M_{u',u}$ and where $u''$ is uniquely determined with the help of F. W. Levy's theorem by the relations $u = u''g$ and $u' \in Fu''$.

As usual, for any $f \in F$, $f^0$ is defined as e and we start by verifying two statements of independent interest which will be needed later on.

V.1. If $a,b,c, \in F$ are such that $ab = ca$ there exist $d,d' \in F$ and $m \geq 0$ which satisfy the relations $a = (dd')^m d$; $b = d'd$; $c = dd'$.

Proof. We use repeatedly F. W. Levy's theorem and we assume that the result is true for all similar equation with a strictly lower total degree $|ab|$.

If $|a| \leq |b|$, we have directly $b = d'a$ and $c = ad'$ (and $a = d$, i.e. $m = 0$).
If $|a| > |b|$, we can write two new relations $a = a'b$ and $a = ca''$ and, in fact, $a' = a''$ since we have $(ca'')b = c(a'b)$ by hypothesis; thus, we are back to an equation $a'b = ca'$ ( $= a$) of the same type as the original one and the result follows from the induction hypothesis.

V.2. If $a,b,c,d, \in F$ and $n,m \geq 0$ are such that

$$a^{2+n} = (bc)^{1+m} bd \text{ and } |a| \leq |(bc)^m b|$$

there exist $h \in F$ and $n_1,n_2,n_3 \geq 0$ which satisfy the relations:

$$a = h^{n_1};\ bc = h^{n_2};\ bd = h^{n_3}.$$

Proof. We distinguish three mutually exclusive cases:

$1^o$. $|a| \leq |b|$. Then, by considering the left factors of the given equation we can define in a unique manner the integers $p_i$ $(1 \leq i \leq 3)$ and the words $a_i$ $(1 \leq i \leq 6)$ satisfying the relations:

$$b = a^{1+p_1}a_1 \;;\; bc = a^{1+p_1+p_2}a_3 \;;\; bcb = a^{1+p_1+p_2+p_3}\, a_5;$$

$$a = a_1a_2 = a_3a_4 = a_5a_6.$$

From these we deduce the new equation $a^{1+p_1} a_1 = a_4\, a^{p_3}\, a_5$ ( = b) and, if $p_3 = 1 + p_3'$, we obtain $a_3a_4a^{p_1}a_1 = a_4a_3a_4a^{p_3'}\, a_5$, that is, finally $a_4a_3 = a_3a_4$ ( = a). Using V.1. and induction this shows that $a_3 = h'^{n_3'}$ and $a_4 = h^{n_4'}$ for some $h \in F$ and the full result follows easily. If $p_3 = 0$, we must also have $p_1 = 0$ and then $a_3a_4a_1 = a_4a_5$ ( = b); consequently, $/a_5/ \geq /a_3/$ and, taking into account the relation $a_3a_4 = a_5a_6$ (=a) we finally get $a_3a_4 = a_4a_3$ and the end of the proof is the same as previously.

$2^o$. $/b/ < /a/ \leq /bc/$. The hypothesis $/a/ \leq /(bc)^m b/$ implies now that $m = 1 + m'$, say. We write $b' = bc$; $c' = e$; $d' = (bc)^{m'}bd$ and the equation takes the form $a^{2+n} = b'c'b'd'$ with, now, $/a/ \leq /b'/$; thus we can apply $1^o$ and the result is proved in this case.

$3^o$. $/bc/ < /a/$. We define $n''$ $m'' \geq 0$ and $b''$, $c'' \in F$ by the relations $/a^{1+m''}/ \leq /(bc)^{2+n''}/ = /a^{1+m''}b''/$; $b''c'' = a$. Writing $a'' = bc$ we are back to an equation of the original type but in which $d'' = e$ and $/a''/$ $(=/bc/) <$ $/b''c''/$ $(=/a/)$. Thus we can apply $2^o$ and the result is proved in all cases.

We revert to our problem of studying $M_{u,u}$ for a given $u \in F^+ (= F - \{e\})$; from V.1 it instantly follows that $g \in M_{u,u}$ (i.e. $ug = g'u$ for some $g' \in F$ and $/g/ < /u/$) if and only if $u = vg = g'v$ for some $v \in F$. In order to obtain a more useful formula we define $\delta u$ for any $u \in F^+$ as the largest integer (possibly, zero) which is such that u can be written in the form $u = (ff')^{\delta u} f \ ( = f(f'f)^{\delta u})$ for some pair $(f \in F^+, f' \in F)$. It may happen (only if $\delta u = 1$ as will be seen later) that several pairs satisfy this relation; in this case we consider only the principal one which we characterise by the supplementary condition that its total degree $/ff'/$ is the lowest possible; in any case we define $\bar{u} = (ff')^{\delta u - 1} f$ if $\delta u > 0$ and $= e$ if $\delta u = 0$.

For instance, if $u = xyxzxyxyxzxyx$ we have $\delta u = 2$ since $u = (x)(yxzxy)(x)(yxzxy(x)$, $\bar{u} = xyxzxyx$ but, now, the principal pair of $u_1 = \bar{u}$ is $(xyx, z)$ and we have $\bar{u}_1 = xyx \ (= u_2)$; $\bar{u}_2 = x(=u_3)$; $\bar{u}_3 = e$. It is a natural convention to define $M_{e,e}$ as zero and we have:

V.3. For any $u \in F^+$, $M_{u,u} = e + (M_{\bar{u},\bar{u}})(f'f)$.

Proof. It follows immediately from V.1 that $M_{u,u} = e$ if and only if $\delta u = 0$ and our formula is valid in this special case. Thus we may assume from now on that $\delta u > 0$ and that $M_{u,u}$ contains a word $g \in F^+$, that is, a word which satisfies the conditions $0 < /g/ < /u/$, $ug = g'u$ for some $g' \in F$. We distinguish several (possibly overlapping) cases

1°. If $/g/ \leq /(ff')^{\delta u - 1} f/ \ (= /\bar{u}/)$ we have the equation $(ff')^{1+n} fg = g'f(f'f)^{1+n}$ (with $n + 1 = \delta u$); cancelling on the left a factor of degree $/g'f/$ we obtain a relation $f''(f'f)^{1+n'} g = (f'f)^{1+n}$ where $f''$ is a right factor of $f'f$, that is, where $f'''f'' = f'f$ for some $f'''$. We now compare the left factor of the last relation above and we obtain $f''f''' = f'''f''$; according to the remark made in the proof of V.2

this implies that $f'' = h^{m}$ and $f''' = h^{m'}$ for some $h \in F$; consequently $f = h^{m''}h'$ for some left factor $h'$ of $h$. Reverting to the expression of $u$ we now obtain $u = h^{n''}h'$ with $n'' = (m+m')(1+n)+m''$. Because of the maximal character of $1 + n = \delta u$, this is the only possible if $m+m' = 1$ and $m'' = 0$; thus, finally, $f'' = e$ (or $= f'f$) and $g$ is a power of $f'f$. Since, trivially, any such word belongs to $M_{u,u}$ we have proved that the set of the words of degree at most $/\bar{u}/$ from $M_{u,u}$ is just the set $\{ (f'f)^{m} \}$ $0 \leq m < \delta u$.

$2^{o}$. If $/g/ \geq /ff'/$ we can write $g = g''f'f$ and $g' = ff'g'''$ and after simplification we derive from the original equation the new relation $\bar{u}g'' = g'''\bar{u}$. Consequently the formula is also true in this case and $1^{o}$ and $2^{o}$ cover all the possibilities except when $\delta u = 1$.

$3^{o}$ $\delta u = 1$ and $/f < /g/ < /ff'/$. Applying V.1 to the equation $ug = g'u$ we find that $g = ba$, $g' = ab$ and $u = (ab)^{m}a$; we have $m = 1$ since $/g/ < /u/$ and $\delta u = 1$ and, thus, $u = ff'f = aba$ with $/a/ < /f/$ because $(f,f')$ is the principal pair; consequently, $f = aa'$, $f = a''a$ and $b = a'f'a''$ for some $a',a'' \in F^{+}$. Using once more V.1. it follows that $a = (cd)^{m'}c$, $a' = dc$, $a'' = cd$ and $f = (cd)^{m+1}c$ for some $c,d, \in F$. Finally, $g = dcf'cd\ (cd)^{m'}c = dcf'f$ with $dc \in M_{f,f}$ and this concludes this proof since $\bar{u} = f$ when $\delta u = 1$.

Let us say that a word $u \in F^{+}$ belongs to $F_{1}$ if $M_{u,u} = (e-v^{n})(e-v)^{-1}$ for some $v \in F^{+}$. It is an open question to find an exact expression for the probability $\sigma_{n}$ that a random word of degree $n$ belongs to $F_{1}$. However we have

V.4. The probability $\sigma_n$ tends to one when n tends to infinity.

Proof. The statement is a straightforward consequence of the remark to be proved below that every word not in $F_1$ admits at least one factorisation of the form abacaba with $a \in F^+$ and $b,c, \in F$.

Indeed, by definition, $u \in F_1$ if $\delta u = 0$ or if $\delta u = 1$ and $\delta f = 0$ since in those cases, $M_{u,u} = e$ or $= e+ f$; thus, when $\delta u = 1$, $u \in F_1$ only if $\delta f > 0$, i.e. if for some $a \in F^+$, $b \in F$ and $c$ $(= f') \in F$ we have $u =$ abacaba. Let us assume now that $\delta u > 1$. It is a direct consequence of V.2. that, then, the equation $u = (ff')^{\delta u} f$ uniquely determines f and f'; thus, by V.3., $u \in F_1$ if (f,f') is also the principal pair of ff'f and if $\delta f = 0$. The case where $\delta f > 0$ has already been discussed and we consider the case where the principal pair (g,g') of ff'f is not (f,f'); by hypothesis gg'g = ff'f and by definition $|f| > |f|$; thus, as in the proof of V.3. $f = (ab)^m a$ for some $a \in F^+$. Consequently, either $m > 0$ and then $\delta f > 0$ or, else, we have f = a, f' = bag'ab; and, finally, $u = (abag'ab)^{\delta u} a$ This concludes the proof.

Remark. A more accurate description can be given for the set $F_o = \{ u: \delta u=0 \}$. Indeed, if $u \in F^+$ is such that $\delta u > 0$ there exists at least one $f \in F_o^+$ which is such that u = ff'f for some $f' \in F$; as in the last proof above, it is easily checked that the word f defined in this manner is unique and consequently the proportion $\sigma'_n$ of the words u with $\delta u = 0$ among the words of degree n satisfies the recurrence relations $\sigma'_{2n+1} = \sigma'_{2n}$ ; $\sigma'_{2n} = \sigma'_{2n-2} - (\text{card } X)^{-n} \sigma'_n$. By standard techniques, it is easily deduced that $\lim_{n \to \infty} \sigma'_n > 0$.

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