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A CODING PROBLEM ARISING IN THE TRANSMISSION OF NUMERICAL DATA

by

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A biunique coding from the set of the 2^n integers 1, 2, ..., 2^n onto the set of the 2^n n-place binary sequences is shown. It has some minimal properties.

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A CODING PROBLEM ARISING IN THE TRANSMISSION OF NUMERICAL DATA*

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1. Introduction and Summary

Let I_n denote the set of integers (i), $i = 1, 2, \ldots, 2^n$ and B_n the set of 2^n n-place binary sequences (α). Let f_n be a (1,1) encoding of I_n into B_n such that the binary sequence $\alpha(i)$ corresponds to the integer i and the integer $a(\alpha)$ corresponds to the sequence α. Let $\alpha_j(i)$ denote the binary sequence which differs from $\alpha(i)$ only in the j-th position. To transmit the integer i over a symmetric binary channel, the n digits of the corresponding binary sequence $\alpha(i)$ are presented one by one as inputs to the channel. If the corresponding output is the sequence α' then the transmitted integer is interpreted as $a(\alpha')$. The quantity $|i-a(\alpha')|$ is then the error in the interpreted value. In case there is no error in transmission the distortion is zero. If, however, there is a single error in the j-th place the distortion is $|i-\alpha_j(i)|$.

If the probability of a single digit error in a received binary sequence is p, then the expected error in the interpreted value is pE where

$$E = \frac{1}{n2^n} \sum_{i,j} \left| i-a\left\{\alpha_j(i)\right\} \right|$$

provided that

(i) Each integer i is equally likely to occur

(ii) Errors are equally likely in any digit position

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(iii) Only single digit errors occur in any sequence (or we decide to neglect cases where more than one digit is in error, since when p is small the probability of such an occurrence is very small).

It is known from a theorem of H.H.B. Martens [1], of the Bell Telephone Laboratories, that E is minimum for the natural encoding scheme for which the integer

(1.1) $$i = 2^{n-1}a_1 + 2^{n-2}a_2 + \dots + 2a_{n-1} + a_n$$

corresponds to the binary sequence

(1.2) $$\alpha = (a_1, a_2, \dots, a_n), \quad a_i = 0 \text{ or } 1, \; i = 1, 2, \dots, n.$$

For any encoding f_n let

(1.3) $$d_i(f_n) = \sum_{j=1}^{n} \left| i - a\left\{\alpha_j(i)\right\} \right|$$

and

(1.4) $$s(f_n) = \sum_{i=1}^{n} d_i(f_n)$$

Then it follows from Marten's theorem that

(1.5) $$\min_{f_n} s(f_n) = 2^n(2^n - 1)$$

There are other encodings besides the natural encoding (1.1, 1.2) for which $s(f_n)$ has the minimum value $2^n(2^n - 1)$ given by (1.5). Let $\bar{\mathfrak{J}}_n$ denote the set of all such encodings i.e.

(1.6) $$\bar{\mathfrak{J}}_n = \left\{ f_n : s(f_n) = 2^n(2^n - 1) \right\}$$

The mean error for a particular integer i is $p\, d_i(f_n)/n$ where

(1.7) $$d_i(f_n) = \sum_{j=1}^{n} \left| i - a\left\{\alpha_j(i)\right\} \right|$$

Thus for a given f_n the smallest expected error for a fixed integer is

(1.8) $$u(f_n) = \min_i \; d_i(f_n)$$

Then it is of interest to find an $f_n^* \in \bar{\bar{\mathcal{J}}}_n$ such that

$$(1.9) \qquad u(f_n^*) = \min_{f_n \in \bar{\bar{\mathcal{J}}}_n} u(f_n) = u(\bar{\bar{\mathcal{J}}}_n) \quad \text{say.}$$

In this paper we give a method of construction of a set of encodings $\mathcal{J}_n$ such that for every f_n in $\mathcal{J}_n$, $s(f_n) = 2^n(2^n - 1)$. Hence $\mathcal{J}_n \subseteq \bar{\bar{\mathcal{J}}}_n$. We also show by actual construction that there exists an f_n^* in $\mathcal{J}_n$, such that

$$(1.10) \qquad u(f_n^*) = \frac{2^{n+1} + 3n - \delta}{6}$$

where $\delta = 1$ or 2 according as n is odd or even. Hence

$$(1.11) \qquad u(\mathcal{J}_n) \leq \frac{2^{n+1} + 3n - \delta}{6}$$

For $n = 2$ or 3, one may verify by actual enumeration that the equality holds in (1.11).

2. Method of construction of $\mathcal{J}_n$.

Consider the n dimensional unit hypercube Σ_n the coordinates of whose vertices are the n-place binary sequences 2^n in number. The hypercube Σ_n has $n.2^{n-1}$ edges and every vertex is connected to n different vertices by n edges. Let f_n be a way of assigning the integers of I to the 2^n different vertices, so that each integer corresponds to a unique vertex. Then f_n is a (1,1) encoding of (i) to (α). If we define the weight of an edge to be the absolute difference of the two integers assigned to its two vertices, then it is easily seen that

$$\tfrac{1}{2}s(f_n) = \text{sum of the weights of the } n.2^{n-1} \text{ edges (as determined by the encoding } f_n)$$

$d_i(f_n)$ = sum of the weights of the n edges meeting at the vertex which has the integer i assigned to it.

Hence it is easy to recognize that $\bar{\bar{\mathcal{F}}}_n$ is the set of all ways of assigning the integers (i) in a (1,1) manner to the vertices of the unit hypercube Σ_n, such that the sum of the weights of all the edges is $2^{n-1}(2^n - 1)$.

Let T be any rigid motion (resultant of an orthogonal transformation and a translation) which carries Σ_n into itself. The set of all such rigid motions forms a group $\mathcal{G}_n$. Let T transform the vertex α to $T(\alpha)$. Given any encoding f_n in which the integer i corresponds to the vertex α, we get a new encoding $T(f_n)$ in which i corresponds to $T(\alpha)$. $T(f_n)$ may be called equivalent to f_n. It is clear that the set of encodings is divided in this way into equivalence classes such that if f_n and f_n' are equivalent encodings there exists a $T \epsilon \mathcal{G}_n$ such that $f'_n = T(f_n)$. It is also clear that is $f_n \epsilon \bar{\bar{\mathcal{F}}}$, then any encoding equivalent to f_n also belongs to $\bar{\bar{\mathcal{F}}}$ since the sum of the weights of the edges of Σ_n is invariant under the transformations of $\mathcal{G}$. Also if f_n and f_n' are equivalent then

$$(2.1) \qquad d_i(f_n) = d_i(f_n'), \qquad i = 1, 2, \ldots, 2^n$$

since the sum of the weights of the edges meeting at a given vertex remains invariant.

We shall now describe a method of constructing an encoding $f^*_{n+1} \epsilon \bar{\bar{\mathcal{F}}}_{n+1}$ given an encoding $f_n \epsilon \bar{\bar{\mathcal{F}}}_n$. In the given encoding f_n let $\alpha(i)$ be the sequence corresponding to the integer i $(i = 1,2,\ldots,2^n)$.

Thus i is the integer assigned to the vertex $\alpha(i)$ of the unit hypercube Σ_n.

In f^*_{n+1} the sequence corresponding to i, $(i = 1,2,\ldots,2^n)$ is then defined to be

(2.2) $$\alpha^*(i) = (\alpha(i), 0)$$

i.e. $\alpha^*(i)$ is the (n+1)-place sequence obtained by adding the coordinate 0 to $\alpha(i)$ in the (n+1)-th place.

Let $f'_n = T(f_n)$ be another encoding equivalent to f_n. Then the sequence corresponding to the integer i in f'_n is $\alpha'(i) = T[\alpha(i)]$. We now define the sequence corresponding to the integer $i + 2^n$, $(i = 1, 2, \ldots, 2^n)$ as

(2.3) $$\alpha^*(i + 2^n) = (\alpha'(i), 1)$$

i.e. $\alpha^*(i + 2^n)$ is the sequence obtained from $\alpha'(i)$ by adding the coordinate 1 in the (n+1)-th place.

Thus if we consider the unit hypercube Σ_{n+1} of n+1 dimensions, then f^*_{n+1} establishes a correspondence between the integers $1,2,\ldots,2^{n+1}$ and the vertices of Σ_{n+1}, such that the integers $1,2,\ldots,2^n$ correspond to the vertices of the n-dimensional hypercube $\Sigma_{n,0}$ which is formed by those vertices of Σ_{n+1} for which the (n+1)-th coordinate is zero, and the integers $2^n+1, 2^n+2,\ldots,2^{n+1}$ correspond to the vertices of the n-dimensional hypercube $\Sigma'_{n,1}$ which is formed by those vertices of Σ_{n+1} for which the (n+1)-th coordinate is unity.

The encoding f^*_{n+1} defined above is completely determined by the equivalent encodings f_n and f'_n, and may be denoted by the notation

$f^*_{n+1} = (f_n, f'_n)$. We shall say that f^*_{n+1} is derived from f_n by extension.

We shall now show that if $f_n \in \overline{\overline{\mathcal{J}}}_n$, then $f^*_{n+1} \in \overline{\overline{\mathcal{J}}}_{n+1}$. i.e. if $s(f_n) = 2^n(2^n - 1)$, then $s(f^*_{n+1}) = 2^{n+1}(2^{n+1} - 1)$. If $f_n \in \overline{\overline{\mathcal{J}}}_{n+1}$ then the sum of the weights of the edges of Σ_{n+1} (under f^*_{n+1}) = sum of the weights of the edges of $\Sigma_{n,0}$ + sum of the weights of the edges of the $\Sigma'_{n,1}$ + sum of the weights of the edges of Σ_{n+1} for which one vertex belongs to $\Sigma'_{n,1}$ and one vertex belongs to $\Sigma_{n,0} = 2^{n-1}(2^n - 1) + 2^{n-1}(2^n - 1) + \sum_{i=1}^{n} [(2^n + i) - i] = 2^n(2^{n+1} - 1)$. This shows that $f^*_{n+1} \in \overline{\overline{\mathcal{J}}}_{n+1}$.

<u>Example 1.</u> If $n = 1$, the set $I = 1,2$ and the set B_n of the binary sequences consists of the two sequences (0), (1). Hence there are only two encodings

$$f_1 : (0) \longleftrightarrow 1, \quad (1) \longleftrightarrow 2$$

$$f_1 : (0) \longleftrightarrow 2, \quad (1) \longleftrightarrow 1$$

which are equivalent and both belong to $\overline{\overline{\mathcal{J}}}_1$.

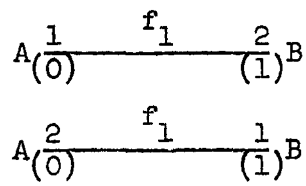

Fig. 1.

If $n = 2$, the set $I = 1, 2, 3, 4$ and the set B_n of two place binary sequences is (0,0), (1,0), (1,1), (0,1) which we may consider as the vertices A, B, C, D of the unit square shown in Fig.2.

Let M_1, M_2, N_1, N_2 be the mid-points AB, CD, BC and AD respectively. Let ℓ be the line perpendicular to the plane of the square through O. Then the group $\mathcal{G}_n$ consists of eight rigid motions described in the following table:

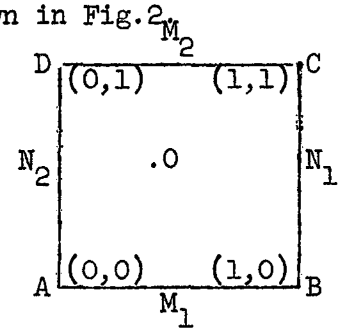

Fig. 2

Element of	Geometrical description	Analytical equation
I	Identity	$x_1' = x_1$
		$x_2' = x_2$
R	Rotation through a right	$x_1' = -x_2 + 1$
	angles about ℓ	$x_2' = x_1$
R^2	Rotation through two right	$x_1' = -x_1 + 1$
	angles about ℓ	$x_2' = -x_2 + 1$
R^3	Rotation through three right	$x_1' = x_2$
	angles about ℓ	$x_2' = -x_1 + 1$
P_1	Rotation through two right	$x_1' = -x_1 + 1$
	angles about M_1M_2	$x_2' = x_2$
P_2	Rotation through two right	$x_1' = x_1$
	angles about N_1N_2	$x_2' = -x_2 + 1$
D_1	Rotation through two right	$x_1' = x_2$
	angles about AC	$x_2' = x_1$
D_2	Rotation through two right	$x_1' = -x_2 + 1$
	angles about BD	$x_2' = -x_1 + 1$

Now there are 4! or 24 different encodings, divisible into 3 distinct equivalence classes each consisting of 8 members.

From f_1 we obtain by extension the two non-equivalent encodings (f_1, f_1) and (f_1, f_1') both belonging to $\bar{\bar{\mathfrak{J}}}_2$. These with their equivalents account for 16 of the encodings. They are diagramatically exhibited in Figs. 3(a) and 3(b).

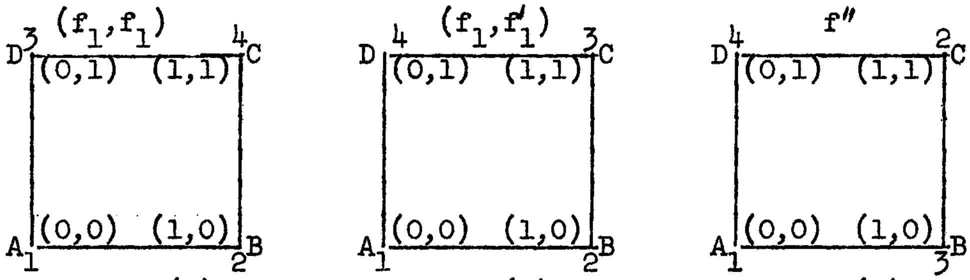

Fig.3(a) Fig.3(b) Fig.3(c)

By taking the weights of the edges we see that

$$s(f_1,\ f_1) = s(f_1,\ f_1') = 12$$

A third encoding f''_2 not equivalent to either $(f_1,\ f_1)$ or $(f_1,\ f_1')$ is exhibited in Fig. 3(c). The encodings equivalent to this account for the remaining 8 encodings.

$$s(f'''_2) = 16.$$

The encodings equivalent to $f^*_2 = (f_1,\ f_1')$ are exhibited in the following table, which gives in the column below each encoding the value of i corresponding to binary sequence α shown in the initial column.

α	f_2^*	$R(f_2^*)$	$R^2(f_2^*)$	$R^3(f_2^*)$	$P_1(f_2^*)$	$P_2(f_2^*)$	$D_1(f_2^*)$	$D_2(f_2^*)$
(00)	1	4	3	2	2	4	1	3
(10)	2	1	4	3	1	3	4	2
(11)	3	2	1	4	4	2	3	1
(01)	4	3	2	1	3	1	2	4

Each of the above encodings is characterized by

$$d_1(f_2^*) = 4, \quad d_2(f_2^*) = 2, \quad d_3(f_2^*) = 2, \quad d_4(f_2^*) = 4\ .$$

We now define a subset $\mathfrak{F}_n$ of $\overline{\mathfrak{F}}_n$ in the following manner. Let $\mathfrak{F}_1$ be the set of encodings f_1 and f_1' defined in Example 1. We then obtain the set of all encodings for the case n = 2 derivable from these by extension and equivalence. This is the set 16 of encodings $\mathcal{F}_2$, which are equivalent to (f_1, f_1) or (f_1, f_1'). Starting from $\mathcal{F}_2$ we obtain for the case n = 3 all possible encodings derivable from any encoding of $\mathcal{F}_2$ by extension and equivalence, and denote this set of encodings by $\mathcal{F}_3$.* Clearly $\mathcal{F}_n \subseteq \overline{\mathcal{F}}_n$ and the equality holds for n = 1 or 2. Whether $\mathcal{F}_n = \overline{\mathcal{F}}_n$ in general is not known, but it may be conjectured that the two sets are identical.

3. Existence of $f_n^* \in \mathcal{F}_n$ such that $u(f_n^*) \leqq \frac{2^{n+1}+3n - \delta n}{6}$ where $\delta_n = 1$ or 2 according as n is odd or even.

Let us define a sequence $\{r_i\}$ by the recurrence formulae

$$r_{2m} = r_{2m-1},\ r_{2m+1} = r_{2m} + 2^m,\ r_1 = 2;\ m = 1,2,3,\ \ldots \tag{3.1}$$

It follows that

$$r_{2m} = r_{2m-1} = \frac{2}{3}(1 + 2^{2m-1}), \qquad m = 1,2,3,\ldots \tag{3.2}$$

In particular

$$r_1 = r_2 = 2, \quad r_3 = r_4 = 6, \quad r_5 = r_6 = 22. \tag{3.3}$$

Suppose there exists an n-place binary encoding $f_n \in \mathcal{F}_n$, such that

$$d_{r_n}(f_n) = \frac{2^{n+1} + 3n - \delta_n}{6} \quad , \quad \delta_n = 1 \text{ or } 2 \text{ according} \tag{3.4}$$

as n is odd or even. Since $f_n \in \mathcal{F}_n$, it also has the minimal property that $s(f_n) = 2^n(2^n - 1)$.

We shall derive from f_n by extension an encoding f_{n+1}^* such that

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* Proceeding in this manner we can always derive $_n$ from $_{n-1}$.

(3.5) $$d_{r_{n+1}}(f^*_{n+1}) = \frac{2^{n+2} + 3(n+1) - \delta_{n+1}}{6}$$

Then $f^*_{n+1} \in \mathcal{F}_{n+1}$ and has the minimal property that $s(f^*_{n+1}) = 2^{n+1}(2^{n+1} - 1)$.

We shall first prove some useful Lemmas.

Lemma I. If $\gamma = (c_1, c_2, \ldots, c_n)$ is a binary sequence, an addition of binary sequences is defined as addition of corresponding co-ordinates (mod 2), then there exists a rigid motion T_γ which carries the n-dimensional unit hypercube Σ_n into itself and any vertex α into the vertex $\alpha + \gamma$.

The required transformation is

(3.6) $$x'_i = \pm x_i + c_i \qquad i = 1, 2, \ldots, n$$

where the upper or the lower sign is taken in $\pm x_i$ according as c_i is 0 or 1.

Lemma II. Let f_n be the encoding in which the integer i corresponds to the binary sequence $\alpha(i)$, $i = 1, 2, \ldots, 2^n$. Let $f'_n = T_\gamma(f_n)$ where T_γ is defined as in Lemma I, and let $f^*_{n+1} = (f_n, f'_n)$ be derived from f_n by extension, then for $1 \leqq i \leqq 2^n$

(3.7) $$\text{(i)} \quad d_i(f^*_{n+1}) = d_i(f_n) + \left|2^n - i + a\{\gamma + \alpha(i)\}\right|$$

(3.8) $$\text{(ii)} \quad d_{i+2^n}(f^*_{n+1}) = d_i(f_n) + \left|2^n + i - a\{\gamma + \alpha(i)\}\right|$$

The encoding f_n is equivalent to assigning the integer i to the corresponding vertex $\alpha(i)$ of the unit hypercube Σ_n. The equivalent encoding $f'_n = T_\gamma(f_n)$ is equivalent to assigning the integer i to the vertex $\alpha(i) + \gamma$ of Σ_n. If β is any arbitrary vertex of Σ_n then in f'_n the integer corresponding to β is $a(\gamma + \beta)$. The encoding f^*_{n+1} is equivalent to assigning the integer i to the vertex $\alpha(i)$, 0 or Σ_{n+1} and the integer $2^n + a\{\alpha(i) + \gamma\}$ to the vertex $\alpha(i)$, 1 of Σ_{n+1}.

$d_i(f^*_{n+1})$ = sum of the weights of the edges of Σ_{n+1} meeting at $\alpha(i)$, 0

= sum of the weights (under f_n) of the edges of Σ_n meeting a $\alpha(i)$ + weight of the edge joining $\alpha(i)$, 0 and $\alpha(i)$, 1.

The integer corresponding to $\alpha(i)$, 0 in f^*_{n+1} is i, and the integer corresponding to $\alpha(i)$, 1 is $2^n + a\{\alpha(i) + \gamma\}$. Hence

$$d_i(f^*_{n+1}) = d_i(f_n) + |2^n + a\{\alpha(i) + \gamma\} - i|,\ 1 \leqq i \leqq 2^n$$

Again

$d_{i+2^n}(f^*_{n+1})$ = sum of the weights of the edges of Σ_{n+1} meeting at $(\alpha(i) + \gamma, 1)$

= sum of the weights (under f'_n) of the edges of Σ_n meeting at $(\alpha(i) + \gamma)$ + weight of the edge joining $(\alpha(i) + \gamma, 1)$ and $(\alpha(i) + \gamma, 0)$.

The integer corresponding to $(\alpha(i) + \gamma, 1)$ in f^*_{n+1} is $2^n + i$ and the integer corresponding to $(\alpha(i) + \gamma, 0)$ is $a\{\alpha(i) + \gamma\}$. Hence

$$d_{i+2^n}(f^*_{n+1}) = d_i(f'_n) + |2^n + i - a\{\alpha(i) + \gamma\}|$$

$$= d_i(f_n) + |2^n + i - a\{\alpha(i) + \gamma\}|,\quad 1 \leqq i \leqq 2^n$$

This completes the proof of Lemma II.

Case I. Let n be odd, say n = 2m - 1. By hypothesis there exists an encoding $f_{2m-1} \epsilon \mathcal{F}_{2m-1}$ such that

$$d_{r_{2m=1}}(f_{2m-1}) = \frac{2^{2m} + 3(2m-1) - \delta_{2m-1}}{6}, \qquad \delta_{2m-1} = 1 .$$

Let the rigid motion T_γ be defined as in Lemma I, and let $f^*_{n+1} = (f_n, f'_n)$ be derived by extension from f_n, where $f'_n = T_\gamma(f_n)$. Putting $i = r_{2m} = r_{2m-1}$ in Lemma II, part (i) we have

$$(3.9) \qquad d_{r_{2m}}(f^*_{2m}) = d_{r_{2m-1}}(f_{2m-1}) + |2^{2m-1} - r_{2m-1} + a\{\gamma + \alpha(r_{2m-1})\}|$$

Let us choose γ such that

$$(3.10) \qquad \gamma + \alpha(r_{2m-1}) = \alpha(1)$$

i.e. in f' the integer 1 is assigned to the vertex $\gamma + \alpha(r_{2m-1})$. Then from (3.9)

$$d_{r_{2m}}(f^*_{2m}) = d_{r_{2m-1}}(f_{2m-1}) + 2^{2m-1} - r_{2m-1} + 1$$

$$= \frac{2^{2m} + 3(2m-1) - 1}{6} + \frac{1}{3}(2^{2m-1} + 1)$$

$$= \frac{2^{2m+1} + 6m - \delta_{2m}}{6}$$

i.e.

$$d_{n+1}(f^*_{n+1}) = \frac{2^{n+2} + 3(n+1) - \delta_{n+1}}{6}$$

Case II. Let n be even, say n = 2m. By hypothesis there exists an encoding $f_{2m} \in \mathcal{F}_{2m}$ such that

$$d_{r_{2m}}(f_{2m}) = \frac{2^{2m+1} + 6m - \delta_{2m}}{6}, \qquad \delta_{2m} = 2 .$$

Let the rigid motion T_γ be defined as in Lemma I, and let $f^*_{n+1} = (f_n, f'_n)$ where $f'_n = T_\gamma(f_n)$. Putting $i = r_{2m}$ in Lemma II, part (ii) and noting that $r_{2m} + 2^m = r_{2m+1}$ we have

$$(3.11) \qquad d_{r_{2m+1}}(f^*_{2m+1}) = d_{r_{2m}}(f_{2m}) + |2^{2m} + r_{2m} - a\{\gamma + \alpha(r_{2m})\}|$$

Let us choose γ such that

(3.12) $$\gamma + \alpha(r_{2m}) = \alpha(2^{2m})$$

i.e. in f' the integer 2^{2m} is assigned to the vertex $\gamma + \alpha(r_{2m})$. Then from (3.11),

$$d_{r_{2m+1}}(f^*_{2m+1}) = d_{r_{2m}}(f_{2m}) + r_{2m}$$

$$= \frac{2^{2m+1} + 6m - 2}{6} + \frac{2}{3}\ (1 + 2^{m-1})$$

$$= \frac{2^{2m+2} + 3(2m+1) - \delta_{2m+1}}{6}$$

i.e.

$$d_{r_{n+1}}(f^*_{n+1}) = \frac{2^{n+2} + 3(n+1) - \delta_{n+1}}{6}$$

To complete the induction we note that for the trivial encoding f_1 in Example 1

$$f_1: \qquad (0) \longleftrightarrow 1, \qquad (1) \longleftrightarrow 2$$

$$s(f_1) = 1 = 2^{n-1}(2^n - 1), \text{ and}$$

$$d_{r_1}(f_1) = 1 = \frac{2^{n+1} + 3n - \delta_1}{6}$$

since $n = 1$, $r_1 = 2$ and $\delta_1 = 1$.

Example 2. (i) Starting from f_1, we get $f_1 = T_\gamma(f_1)$ by choosing γ to satisfy (3.10) i.e.

$$\gamma + \alpha(2) = \alpha(1)$$

Hence $\gamma = (1)$ and

$$f_1: \quad (0) \longrightarrow 2, \qquad (1) \longrightarrow 1$$

Hence

$$f_2^* = (f_1, f'_1): \quad (0,0) \longleftrightarrow 1, \quad (1,0) \longleftrightarrow 2,$$
$$(0,1) \longleftrightarrow 4, \quad (1,1) \longleftrightarrow 3.$$

This encoding is geometrically exhibited in Fig. 3(b). Clearly

$$d_{r_2}(f_2^*) = \frac{2^{n+1} + 3n - \delta_n}{6}$$

since $d_2\,(f_2^*) = 2, \quad n = 2, \quad \delta_2 = 2$

(ii) Changing terminology we can now define f_2 by

$$f_2 : (0,0) \longleftrightarrow 1, \quad (1,0) \longleftrightarrow 2, \ (0,1) \longleftrightarrow 4,$$
$$(1,1) \longleftrightarrow 3.$$

We get $f'_2 = T_\gamma(f_2)$ by choosing γ to satisfy (3.12), i.e.

$$\gamma + \alpha(2) = \alpha(4)$$

Hence

$$\gamma = \alpha(4) - \alpha(2) = (0,1) - (1,0) = (1,1)$$

$$f'_2: \quad (1,1) \longleftrightarrow 5, \quad (0,1) \longleftrightarrow 6, \quad (1,0) \longleftrightarrow 8,$$
$$(0,0) \longleftrightarrow 7$$

$$f_3^* = (f_2, f'_2): (0,0,0) = 1, \ (1,0,0) = 2, \ (0,1,0) = 4,$$
$$(1,1,0) = 3, \ (0,0,1) = 7, \ (1,0,1) = 8,$$
$$(0,1,1) = 6, \ (1,1,1) = 5$$

f_3^* is geometrically exhibited in Fig. 4.

$$d_{r_3}(f_3^*) = \frac{2^{n+1} + 3n - \delta_n}{6}$$

Since $d_6(f_3^*) = 4$, $n = 3$, $\delta_3 = 1$

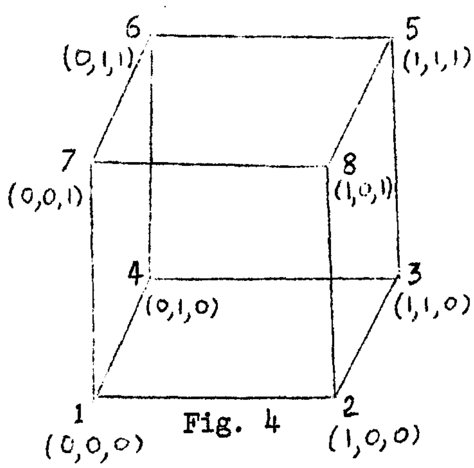

Fig. 4

(iii) Again changing the terminology and taking the encoding exhibited in Fig. 4 to be f_3, we get $f'_2 = T(f_2)$ by choosing γ to satisfy (3.10), i.e.

$$\gamma + \alpha(6) = \alpha(1)$$

Hence $\gamma = \alpha(1) - \alpha(6) = (0,0,0) - (0,1,1) = (0,1,1)$, and

$$
\begin{array}{llll}
f_4^* : & (0,0,0,0) \longleftrightarrow 1\ , & (1,0,0,0) \longleftrightarrow 2\ , \\
& (0,1,0,0) \longleftrightarrow 4\ , & (1,1,0,0) \longleftrightarrow 3\ , \\
& (0,0,1,0) \longleftrightarrow 7\ , & (1,0,1,0) \longleftrightarrow 8\ , \\
& (0,1,1,0) \longleftrightarrow 6\ , & (1,1,1,0) \longleftrightarrow 5\ , \\
& (0,0,0,1) \longleftrightarrow 14, & (1,0,0,1) \longleftrightarrow 13, \\
& (0,1,0,1) \longleftrightarrow 15, & (1,1,0,1) \longleftrightarrow 16, \\
& (0,0,1,1) \longleftrightarrow 12, & (1,0,1,1) \longleftrightarrow 11, \\
& (0,1,1,1) \longleftrightarrow 9\ , & (1,1,1,1) \longleftrightarrow 10\ .
\end{array}
$$

$$d_{r_4}(f_4^*) = \frac{2^{n+1} + 3n - \delta_n}{6}$$

since $\delta_6(f_4^*) = 7$, $n = 4$, $\delta_4 = 2$.

* * * * * * * *

Reference

[1] H.H.B. Martens, "A remark on error minimizing codes," Unpublished manuscript, with the Bell Telephone Laboratories, Murray Hill, New Jersey.