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ON A SPECIAL CLASS OF RECURRENT EVENTS

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I. Let F be the set of all finite sequences (words) in the symbols $x \in X$. A recurrent event (A,μ) on F is defined by the set A of all words at the end of which it occurs and by the probability measure μ; for any $f \in F$, μf is the measure of the set of all infinite sequences which begin by f. We call A the support of (A,μ) and we denote by $T(A,\mu)$ the mean recurrence time of (A,μ).

If (B,μ') is another recurrent event of F, $(A \cap B,\mu')$ is again a recurrent event and it results from the general theory of Feller (Cf 2. Chap. VIII) that when $T(B,\mu')$ is finite the ratio $\pi = T(B,\mu')/T(A \cap B,\mu')$ is, in a certain sense, the limit of the conditional probability that a random word $f \in F$ belongs to A when it is known to belong to B. For given A, it is in general possible to find infinitely many (B,μ') with finite $T(B,\mu')$ which are such that $\pi = 0$.

The main point of this note is to verify several statements which, together, imply the following property

If A is such that $T(A \cap B,\mu')$ is finite for every (B,μ') with finite $T(B,\mu')$, then, for every such (B,μ'), π^{-1} is an integer at most equal to a certain finite number δ^* which depends only upon A.

Classical examples of this occurrence are the return to the origin in random walks over a finite group (Cf 3) and in particular the recurrent event which occur at the end of every word whose length is an integral multiple of a fixed integer k.

In the section 2, we discuss some properties of a class of recurrent events which we call birecurrent; in section 3, we verify the statements mentioned above and in the section 4 we describe several examples of birecurrent supports.

II. We consider F as the free monoid (Cf 1. Chap. 1) generated by X; the empty word e is the neutral element of F and the product ff' of the words f and f' is the word f" made up of f followed by f'; f (f') is called a <u>left</u> (<u>right</u>) <u>factor</u> of f"; a word is <u>proper</u> if it is different from e.

Feller's condition (Cf 2. Chap. VIII) that the non empty subset A of F is the support of a recurrent event can be expressed as follows

U_r : if $a \in A$ and $f \in F$, then, $af \in A$ if and only if $f \in A$.

This condition implies that A is a submonoid of F (i.e. that $e \in A$ and $A^2 \subset A$). We shall say that A is <u>birecurrent</u> if it satisfies U_r and the symmetric condition U_l

U_l : if $a \in A$ and $f \in F$, then, $fa \in A$ if and only if $f \in A$.

It follows immediately that if $\{A_i\}$ is any collection of supports of recurrent (birecurrent) events the same is true of the intersection C of the sets A_i; indeed, C is a submonoid because every A_i is a submonoid and, if, e.g., $a, af \in C$, the word f belongs to all the sets A_i (because of U_r) and, consequently it belongs also to C.

In all this paper, A will denote a recurrent (or, eventually, birecurrent) support and we shall use the following standing notations:

A* = the set of all the proper words at the end of which the event whose support is A occurs for the first time (and, for any recurrent support B, B* is defined similarly).

S = F - A*F (= the complement in F of the right ideal A*F);

R = F - FA*.

We state explicitly the following well known facts:

II. 1. Every $f \in F$ admits one and only one factorization f = as with $a \in A$ and $s \in S$ and at least one factorization f = ra' with $a' \in A$ and $r \in R$; if and only if A is birecurrent the second factorization is unique for all $f \in F$.

II. 1'. Every proper a from A admits a unique factorization as a product of elements from A^*.

The two statements are quite intuitive but a formal proof of them has been given in (5); II.1' shows that any bijection (i.e. one to one mapping onto) of A^* onto a set Y can be extended to an isomorphism of A onto the free monoid generated by Y. The following remark will be used repeatedly in the course of this paper.

II.1''. When A is birecurrent, if $s, s' \in S$ $(r, r' \in R)$ are such that s is a right factor of s' (r is a left factor of r') and that $sf, s'f \in A$ $(fr, fr' \in A)$ for some $f \in F$,

then $s = s'$ $(r = r')$; if, furthermore, $f \in R$ $(f \in S)$, then $sf \in A^* \cup \{e\}$.

<u>Proof</u>. Because of the perfect symmetry of U_r and U_l we can limit ourself to the proof of the statement concerning s and s'.

By hypothesis, $s' = f's$ for some $f' \in S$ and $sf, f'sf \in A$; because of U_l, this implies $f' \in A$; because of $s' \in S = F - A^*F$ and II.1', this in turn implies $f' = e$ and we have proved that $s' = es = s$.

Let us assume now that $sr \in A$ with $s \in S$ and $r \in R$; if $sr = e$, the result is proved; if $sr \in A - \{e\}$, II.1' shows that $sr = aa'$ with $a \in A^*$ and $a' \in A$; as above, a cannot be a left factor of s and, consequently, a' is a right factor of r; but, by a symmetrical argument, this shows that $a' = e$ and that consequently $sr = a \in A^*$. This concludes the proof of II.1".

Let us assume now that A is birecurrent; we denote by ΔSf (ΔRf) the set of the right (left) factors of f that belong to S (R) and by Δf the set of the triples (r,a,s) which are such that $f = ras$ and that $r \in R$, $a \in A$, $s \in S$; such a triple will be called an <u>A-factorization</u> of f and δf will denote the number of distinct triples in the set Δf of the A-factorizations of f.

II.2. For any $f, f' \in F$, $\delta ff' \geq \max(\delta f, \delta f')$ and $\delta ff' = \delta f$ $(=\delta f')$ if and only if for every left (right) factor f'' of f' (of f) the product ff'' $(f''f')$ has a factorization $ff'' = sa$ $(f''f' = ar')$ where $a \in A$ and where f'' is a right (left) factor of a.

Proof. Let us consider any element $g \in F$ and prove that there exists a bijection $\sigma_g: \Delta Rg \rightarrow \Delta Sg$; indeed, by II.1 to any $r \in \Delta Rg$ (i.e., to any $r \in R$ which is such that $g = rg'$ for some $g' \in F$) it corresponds a unique $s \in \Delta Sg$ (determined by the conditions $g' = as$, $a \in A$, $s \in S$) which we call $\sigma_g r$; because of the symmetry implied by the hypothesis that A is birecurrent we can construct in a similar manner a mapping $\Delta Sg \rightarrow \Delta Rg$ which we call σ_g^{-1}; since, clearly, for any $r \in \Delta Rg$ we have $\sigma_g^{-1} \circ \sigma_g r = r$ this shows that σ_g is a bijection and also that the A-factorizations of g are in a one-to-one correspondence with the elements of ΔRg. We now revert to the proof of II.2. By the above construction we know that $\delta ff'$ is equal to δf (i.e. to the number of elements in ΔRf) plus the number of proper $r' \in \Delta Rf'$ which are such that $fr' \in R$; thus, $\delta ff' \geq \delta f$ with the equality if and only if we do not have $ff'' \in R - \Delta Rf$ for some left factor f'' of f', i.e. if and only if every such ff'' satisfies the condition stated in II.2.

Because of the symmetry this concludes the proof.

For any $f \in F$, let us denote by αf the smallest positive integer for which $f^{\alpha f} \in A$ with αf, infinite if the only finite power of f that belongs to A is f^0 (= e, by definition).

II.3. A sufficient condition that the recurrent support A is birecurrent is that αf is finite for all $f \in F$; reciprocally if A is a birecurrent support, then, for any $f \in F$, αf is at most equal to the supremum $\delta' f$ of δf^m over all the positive powers of f.

Proof. By hypothesis, A satisfies U_r and, in order to show that it is birecurrent, it will be enough to show that if a and fa belong to A

then f also belongs to A; let us assume that $(af)^m \epsilon A$ for some positive finite m; we have $(af)^m = a(fa)^{m-1} f \epsilon A$ and, because of a, $(fa)^{m-1} \epsilon A$ and U_r, this implies $f \epsilon A$. This proves the first part of II.3.

Let now A be birecurrent and f such that $\delta' f$ is finite; by II.1, any f^n $(0 \leq n \leq \delta' f)$ admits an A-factorization $f^n = (e, a_n, s_n)$ and, by II.2, to each such s_n it corresponds one A-factorization of $f^{\delta' f}$; since, by definition, $\delta f^{\delta' f} \leq \delta' f$, we must have $s_n = s_m$ (= s, say) with $0 \leq m,n \leq \delta' f$ and, e.g., $m < n$. Thus, $f^n = as$ and $f^m = a's$ with $a, a' \epsilon A$ and, after cancelling s, we obtain $f^{n-m} a' = a$. Because of U_1, this last relation shows that f^{n-m} belong to A and, since $0 < n-m \leq \delta' f$, by construction, the result is entirely proved.

Let us assume now that A is birecurrent and that f is such that $\delta f = \delta f^2 < \infty$. We consider the set K (containing at least f^2) defined by

$$K = \{f' \epsilon fFf : \delta f' = \delta f\}.$$

II.4. There exists a group G, a subgroup H of G and a mapping $\sigma : K \to G$ that have the following properties:

σ is an epimorphism (i.e. homomorphism onto) and G is finite;

$\sigma^{-1} H = K \cap A$ and the index of H in G is at most δf.

<u>Proof</u>. According to II.2, the hypothesis $\delta f = \delta f^2$ implies the existence of a bijection $\sigma^* : \Delta Sf \to \Delta Rf$ defined for each $s \epsilon \Delta Sf$ by $\sigma^* s$ = the unique $r \epsilon \Delta Rf$ which is such that $sr \epsilon A$; trivially, $\sigma^* e = e$. Also, by II.2 and the very definition of K, we have $\Delta Rk = \Delta Rf$ and $\Delta Sk = \Delta Sf$ for any $k \epsilon K$. Thus, recalling the definition of σ_f given in the proof of II.2, we can associate to any $k \epsilon K$ a bijection $\sigma^*_k : \Delta Rf \to \Delta Rf$ defined by $\sigma^*_k = \sigma^* \circ \sigma_k$.

Let us now verify that for any $k, k' \epsilon K$ we have $\sigma^*_{kk'} = \sigma^*_k \sigma^*_{k'}$; indeed, if $(r,a,s) \epsilon \Delta k$ and $(r',a',s') \epsilon \Delta k'$ we shall have $(r,a'',s') \epsilon \Delta kk'$

for some $a'' \in A$ if and only if $sr' \in A$ and the identity is verified.

Because of the hypothesis that δf is finite, this construction shows that the set $\{\sigma_k^*\}_{k \in K}$ is a group G and that the mapping σ which sends every $k \in K$ onto σ_k^* is an epimorphism.

Let us observe now that k belongs to A if and only if $(e,k,e) \in \Delta k$, that is, if and only if σ_k^* lets e invariant; again, because G is finite, the elements $k \in K$ which have this last property map onto a subgroup H of G and, clearly, $\sigma^{-1}H$ is contained in A. The fact that the index of H in G is at most equal to the number of elements in ΔRf (i.e., to δf) is a standard result from group theory.

As a corollary of II.4 we state

II.4'. If A is such that the supremum δ^* of $\delta f'$ over all $f' \in F$ is finite and if $\delta f = \delta^*$, then the representation σ_k^* is isomorphic to the representation of G over the cosets of H.

<u>Proof.</u> The property stated amounts to the statement that $\{\sigma_k^*\}$ is transitive or, in an equivalent fashion to the fact that for every $s \in \Delta Sf$ there exists at least one $k \in K$ which is such that $\sigma_k\, e = s$, i.e. which is such that $k = as$ with $a \in A$. For proving this, let $(r,a',s) \in \Delta f$; by II.3 we know that there exists finite positive integers m and m' which are such that $f^m \in A$ and $r^{m'} \in A$; thus the product $f^m r^{m'-1} f = f^m r^{m'} a's$ admits the factorization $a''s$ with $a'' = f^m r^{m'} a' \in A$ and it belongs to K since, under the hypothesis that δf is maximal, K is identical to fFf.

The next statement is not needed for the verification of the main property; its aim is to show that the representation described in section 4 below covers all the birecurrent supports with finite δ^* ($= \sup\ \delta f$, by definition).

II.5. If A is a birecurrent support with finite δ^* there exist a monoid M and an epimorphism $\gamma : F \rightarrow M$ which are such that

$\gamma^{-1}\gamma A = A$, and that M admits minimal ideals.

Proof. Let us consider any $f \epsilon F$ and denote by $\{\gamma f\}$ the set of all $f' \epsilon F$ which satisfy the following condition:

for any $f_1, f_2 \epsilon F$, $f_1 f f_2 \epsilon A$ if and only if $f_1 f' f_2 \epsilon A$.

The relation $f' \epsilon \{\gamma f\}$ is reflexive and transitive and it is well known that it is compatible with the multiplicative structure of F (i.e., it is a congruence); thus we can identify each set $\{\gamma f\}$ with an element γf of a certain quotient monoid M of F. Since $f \epsilon A$ if and only if $f_1 f f_2 \epsilon A$ with $f_1 = f_2 = e$, A is the union of the sets $\{\gamma a\}$, $a \epsilon A$ and trivially, $\gamma^{-1}\gamma A = A$.

Let us now take and element f which is such that $\delta f = \delta^*$, a finite quantity; according to II.2 the maximal character of δf implies that for every f_1 the product $f_1 f$ has a left factor $f_1 r \epsilon A$ for some $r \epsilon \Delta Rf$; thus, because of the symmetry, any relation $f_1 f f_2 \epsilon A$ implies $f_1 r, s f_2 \epsilon A$ with $(r,a,s) \epsilon \Delta f$.

It follows immediately that for any two $k, k' \epsilon K = fFf$, the relation $\gamma k = \gamma k'$ is equivalent to the relation $\sigma k = \sigma k'$ in the notations of II.4. Thus, γK is isomorphic to a group and since K is the intersection of a right and of a left ideal of F, this shows that M admits minimal ideals.

We now revert to the preparation of the proof of the main property and we consider A, a birecurrent support, B a recurrent support and $C = A \cap B$; we assume that C does not reduce to e and that consequently C^* (the set of the proper words at the end of which the events whose supports are A and B respectively occur together for the first time) is not empty.

II.6. Any element f from $F - C^*F$ has a unique factorization $f = f_1 f_2$ with $f_1 \epsilon B - C^*B$ and $f_2 \epsilon F - B^*F$; reciprocally any such product $f_1 f_2$ belongs to $F - C^*F$.

Proof. Because of II.1 any f has a unique factorization $f = f_1 f_2$ with $f_1 \in B$ and $f_2 \in F\text{-}B^*F$; since C is a recurrent support contained in B any product $= f_1' f_2'$ with $f_1' \in B$ and $f_2' \in F - B^*F$ belongs to $F\text{-}C^*F$ if and only if f_1' belongs to $B - C^*B$ and this concludes the proof.

As mentioned in II.1', there exists and isomorphism $\beta : B \rightarrow Q$ where Q is the free monoid generated by $Q^* = \beta B^*$ and it is easily verified that the image P of C by β satisfies U_r and U_l when, according to our hypothesis A is birecurrent; indeed, P is surely a submonoid of Q and it is enough to verify that the relations p,p', $pqp' \in P$ imply $q \in Q$ (because $\beta^{-1}p, \beta^{-1}p', \beta^{-1}pqp' \in A$ imply, e.g., $\beta^{-1}qp \in A$, by U_r, then $\beta^{-1}q \in A$, by U_l and, finally $q \in P = \beta(A \cap B)$.

As above we define a P-factorization of an element $q \in Q$ as a triple $(\bar{r}, p, \bar{s})$ which is such that $q = \bar{r}p\bar{s}$ and that $\bar{r} \in \bar{R} = Q - QP^*$, $p \in P$, $\bar{s} \in \bar{S} = Q\text{-}P^*Q$ with $P^* = \beta C^*$. All the remarks made in II.2 apply here since P is a birecurrent support in Q, and we define $\bar{\delta} q$ as the number of P-factorizations of q.

II.7. For any $b \in B$, $\bar{\delta}\,\beta b \leq \delta b$.

Proof. Let $\bar{r}$ be any element of $\bar{R}$ and define $\beta^* \bar{r}$ as the (uniquely determined) element $r \in R$ which is such that $(r,a,e) \in \Delta b$ for some $a \in A$. We show that the restriction of the mapping β^* to any set $\Delta\bar{R}q$ $(q \in Q)$ is an injection (i.e., is one to one into); indeed, if $\bar{r}, \bar{r}' \in \Delta\bar{R}q$ we have e.g. $r' = r\,q'$ for some $q' \in Q$; thus, if $\beta^*\bar{r} = \beta^*\bar{r}' = r$, say, we have the following relations $\beta^{-1}\bar{r} = ra \in B$ with $a \in A$; $\beta^{-1}\bar{r}' = r\,a' \in B$ with $a' \in A$; $ra' = rab'$ with $b' = \beta^{-1}q' \in B$; consequently, $a' = ab'$ and, because of U_r, $b' \in A$; this shows that $q' = \beta b'$ belongs to P and that finally $q' = e$ because of the relation $\bar{r}' = \bar{r}\,q' \in \bar{R}$.
Thus, $\bar{r}' = \bar{r}$ and our contention is proved.

The remark II.7 is also proved since we have shown that for any $b \in B$ there exists an injection $\Delta\bar{R}\,\beta b \rightarrow \Delta R b$.

II.8. If δ^* (= sup δf) is finite and if $\delta b = \delta^*$ for at least one $b \in B$, then $\bar{\delta}^*$ (= sup $\bar{\delta} q$) is a divisor of δ^*.

Proof. Under these hypothesis, we may assume without loss of generality that B contains an element f which is such that $\delta f = \delta^*$ and $\bar{\delta}\,\beta f = \bar{\delta}^*$; and we use the notations of II.4 and II.4'.

By construction the image G' by σ of $B \cap K$ is a subgroup of G and we have $B \cap \sigma^{-1}(H \quad G') = A \cap B \cap K$; thus, by a standard result of group theory the index δ'^* of $H \cap G'$ in G' is a divisor of that of H in G (i.e., of δ^*). We prove now that δ'^* is in fact equal to $\bar{\delta}^*$; for this we repeat the construction of II.4 and II.4' with $\beta(B \cap K)$ in the role of K and we obtain an epimorphism $\bar{\sigma} : \beta(B \cap K) \to \bar{G}$ which is such that $\bar{\delta}^*$ is the index of the subgroup $\bar{H}$ of $\bar{G}$. We recall the definition of the mapping β^* used in II.7 and we observe that we can define a bisection $\beta^{*-1} : \Delta Rf \cap \beta^*\Delta\bar{R}\beta f \to \Delta\bar{R}\beta f$ which is such that $\beta^{*-1} o \beta^*$ is the identity mapping of $\Delta\bar{R}\beta f$ onto itself; β^{*-1} induces in a natural fashion an epimorphism $\beta^{**} : G' \to \bar{G}$ and, trivially, $H \cap G'$ is the inverse image of $\bar{H}$ by β^{**}. Thus $\bar{\delta}^*$ is equal to δ'^* and II.8 is proved.

III. We keep the notations already introduced and we assume that (A,μ) is a recurrent event; according to Feller, μ satisfies the two conditions:

M_o: $\mu e = 1$ and for any $f \in F$, $\mu f = \Sigma(\mu fx: x \in X)$.

M_r: if $a \in A$ and $f \in F$ then, $\mu af = \mu a\ \mu f$.

We shall say that μ is a <u>positive product measure</u> if $\mu ff' = \mu f\ \mu f' > 0$ for any $f, f' \in F$, and, in this case, M_r is trivially satisfied.

We denote by $/f/$ the length of the element f and for any subset F' of F we use the following notations:

$$F'_n = \{f \in F' : /f/ \leq n\} \quad ; \quad \mu F' = \lim_{n \to \infty} \Sigma\,(\mu f : f \in F'_n).$$

It follows that $\mu F' \leq 1$ if F' is such that any $f \in F$ has at most one left factor which belongs to F'; this condition is satisfied in particular by any subset of A^* and, according to Feller's definition, we shall say that (A,μ) is <u>persistent</u> if and only if $\mu A^* = 1$. The next two statements are proved at the imitation of Feller.

III.1. For any recurrent event (A,μ) we have $T(A,\mu) = \mu S$.

<u>Proof.</u> Let us introduce for any $s \in S$ the notations $S(s) = S \cap sF$ and $A^*(s) = A^* \cap sF$. We verify the identities.

(III.1). for all $m \geq /s/$: $0 \leq \mu s - \mu A^*_{m+1}(s) = \mu S_{m+1}(s) - \mu S_m(s)$;

(III.1'). for all $m \geq 1$: $(1-\mu A^*) + (\mu A^* - \mu A^*_m) = \mu S_m - \mu S_{m-1}$.

Indeed, (III.1) is an immediate consequence of M_o and of the fact that the sets $\{s\} \cup S_m(s)\,X$ and $S_{m+1}(s) \cup A^*_{m+1}(s)$ are identical ; (III.1') is the special case of (III.1) for $s = e$.

From this second identity we deduce that if $\mu A^* = 1$ we have $\lim_{m \to \infty} (\mu S_m - \mu S_{m-1}) = 0$. Thus, a fortiori (from the first identity) that $\mu A^* = 1$ implies $\mu s = \mu A^*(s)$. We now sum the second identity from m=1 to m=n; after rearranging terms, we obtain:

(III.1"). $\mu S_n = (n+1)(1-\mu A^*_n) + \Sigma(\ /a/\ \mu a : a \in A^*_n)$

This shows that if (A^*,μ) is not persistent μS is infinite and we assume now that $\mu A^* = 1$. Under this hypothesis, $T(A,\mu)$ is defined as $\lim_{n \to \infty} \Sigma$ $(/a/\mu a : a \in A^*_n)$, and since $\mu A^* = 1$ implies that $(n+1)(1-A^*_n)$ $= \Sigma((n+1)\mu a : a \in A^*-A^*_n)$, we can write for all n

$$\Sigma(\ /a/\ \mu a : a \in A^*_n) \leq \mu S_n \leq \Sigma(\ /a/\ \mu a : a \in A^*-A^*_n) + \Sigma(/a/\mu a : a \in A^*_n)$$

This concludes the proof since it shows that $\mu S = T(A,\mu)$ when this last quantity is finite and that μS is infinite when $T(A,\mu)$ is so.

For any $s \in S$ let us define $R^*(s)$ as $\{e\}$ when $s = e$ and, when $s \neq e$ as the set of those $f \in F$ which are such that $sf \in A^*$.

III.2. If A is birecurrent, μ a product measure and (A,μ) persistent, we have $T(A,\mu) = \mu R$ and, for all $s \in S$, $1 = \mu R^*(s)$.

<u>Proof.</u> Under these hypothesis all the notions are perfectly symmetrical; thus, the identity (III.1") shows that $\mu R_n = \mu S_n$ and, as a special case, that $\mu R = T(A,\mu)$.

Since any $a \in A^*(s)$ has a unique factorization $a = sf$ with $f \in R^*(s)$ and since μ is a product measure, we have for all $m \geq /s/$ the identity

(III.2) $\mu A^*_m(s) = \mu s\ \mu R^*_{m-/s/}(s)$.

Thus, because of the formula (III.1) we have in any case $R(s) = \mu A^*(s)/\mu s$ ≤ 1 with the equality sign when (A,μ) is persistent since, then, μs $= \mu A^*(s)$.

III.3 If A is birecurrent and μ a product measure, $T(A,\mu) = \delta^*$.

<u>Proof.</u> We use the notations of the section II and we recall the following facts:

1) According to II.1", $R^*(s)$ is a subset of R;

2) for the same reason, if $s,s' \in \Delta Sf$ for some $f \in F$, the sets $R^*(s)$ and $R^*(s')$ are disjoints.

3) if δ^* is finite and $\delta f = \delta^*$ then, by II.2, to every $r \in R$ there corresponds one $s \in \Delta Sf$ which is such that $sr \in A^*$; thus, in this case, the union of the sets $R^*(s)$ over all $s \in \Delta Sf$ is equal to R.

Now to the proof! We shall show that if $\delta f = \delta^*$ we have the inequalities $\mu R \leq \delta f \leq \mu R$ and, trivially, the result will follow by III.2. The second inequality is vacuously true when (A,μ) is not persistent since, then, μR is infinite; when (A,μ) is persistent we have for any $f' \in F$ the inequality $\delta f' = \Sigma\, (\mu R^*(s) : s \in \Delta Sf') \leq \mu R$ since, then, $\mu R^*(s) = 1$ and since the sets $R^*(s)$ are pairwise disjoint. Thus the second inequality is always true.

If now $\delta f = \delta^*$, we know by 3 above that $\Sigma\, (\mu R^*(s) : s \in \Delta Sf) = \mu R$; since in any case as we have seen in the proof of III.2, we have $\mu R^*(s) \leq 1$, it follows that $\mu R \leq \delta^*$ and the result is proved.

III.4. If (B',μ) is a recurrent event and if A is birecurrent we have: $T(A \cap B',\mu) = \bar{\delta}^*\, T(B',\mu)$ where $\bar{\delta}^*$ is defined below.

<u>Proof.</u> Let $B = \{b \in B' : \mu b > 0\}$ and $C = A \cap B$; it is easily verified that (B,μ) is again a recurrent event and that according to III.1 we have

$$T(A \cap B',\mu) = T(A \cap B,\mu) = \mu(F - C^*F)$$

$$T(B',\mu) = T(B,\mu) = \mu(F - B^*F).$$

We keep the notation used in the proof of II.6 and II.7 and we observe that, by taking into account II.6 and the condition M_r on μ, the remark III.4 is equivalent to the relation $\mu(B - C^*B) = \bar{\delta}^*$.

For proving this identity we define a measure ν on Q by the relation $\nu\beta b = \mu b$, for all $b \in B$; because of M_r and of the definition of B, ν is a positive product measure and, since we know that $P = \beta C$ is birecurrent, (P,ν) is a recurrent event on Q. Because of III.1 and III.3 $T(P,\nu) = \nu(Q - P^*Q) = \bar{\delta}^*$.

But, by definition, $\nu(Q - P^*Q) = \nu\beta\,(B - C^*B) = \mu(B - C^*B)$ and the result is proved

III.5. If δ^* is finite and (B',μ) persistent for some measure μ which satisfies the condition that for every $f \in F$ at least one element from FfF has positive measure, then $\bar{\delta}^*$ is a divisor of δ^*.

Proof. Because of the conditions satisfied by μ and δ^* we can find an element f which is such that $\delta f = \delta^*$ and that $\mu f > 0$; we have $f = b's'$ with $b' \in B$ and $s' \in F - B^*F$. Because (B,μ) is persistent, it follows from III.1 that $\mu(B^* - s'F) = \mu s'$; since this last quantity is positive, there exists at least one element $b \in B^* \cap s'F$. Finally, because of II.2 we have $\delta b'b = \delta^*$ with $b'b \in B$. Thus, we can apply II.8 and the result is proved.

The next statement is intended to give a characterization of the birecurrent supports in terms of their intersection with other recurrent event; by E we mean any fixed birecurrent support which is such that $T(E,\mu^*)$ is finite for one positive product measure μ^*; E^* is defined as usual and we say that (E',μ) belongs to the family $((E))$ if the two following conditions are met:

(E',μ') is a recurrent event on F;

there exists a finite integer m which is such that any element from E'^* is the product of m words from E^*.

It is trivial that under these hypothesis E' is birecurrent. Since F itself is a birecurrent support (with $F^* = X$) a simple example of a family $((E))$ is the family of the birecurrent events $(F_{(m)}, \mu_m)$ where $F_{(m)}$ is the set of all words whose length is a multiple of m and where μ_m is a suitable measure.

III.6. If the recurrent support A is such that $(A \cap E',\mu')$ is persistent for every $(E',\mu') \in ((E))$, then, A is a birecurrent support.

Proof. This is a simple application of II.3 and we use the notations of this remark. If αf is finite for all f, then we know by II.3 that

A is birecurrent; thus we may suppose that A and f are such that αf is infinite and we show that $(A \cap E', \mu')$ is not persistent for some suitable (E', μ'). Indeed, by the second part of II.3 we know that $f^m \in E$ for some finite positive m; thus f^m admits a factorization as a product of m' elements from E*; we take E' defined by the condition $E'^* = E^{*m'}$ and μ' defined by the condition that $\mu' f^m = 1$ and that $\mu' f' = 0$ for any other $f' \in E'^*$. The conditions M_o and M_r recalled at the beginning of this section are obviously satisfied and $T(E', \mu')$ is finite. Finally, $(A \cap E', \mu')$ cannot be persistent since $A \cap E'$ reduces to $\{e\}$ and this ends the proof.

Clearly, the conditions of III.6 are satisfied if A is such that $T(A \cap B, \mu) < \infty$ that for any (B, μ) with finite $T(B, \mu)$.

(Other algebraic characterizations of the birecurrent supports have been discussed in "Publications scientifiques de l'Universite' d' Alger serie Mathematique, tome VI. 1959 p. 85-90").

The next statement is a simple application of II.2.

III.7. If A is birecurrent and if δ^* is finite, then, for any product measure, μ, the distribution of the recurrence time of (A, μ) has moments of every order.

Proof. Let $A' = \{a \in A : \mu a > 0\}$; trivially, A' is birecurrent and, by II.7 we know that every $f \in F$ has at most δ^* A'-factorizations; since the distribution of the recurrence times of (A, μ) and (A', μ) are the same, there is no loss of generality in assuming that $A = A'$, i.e., that μ is positive.

Since δ^* is finite there exists an element $f \in F$ which because of II.2 has the property that for any proper $s \in S$ the product sf has a factorization $sf = ar$ with $a \in A^*A$; thus, for any integer n the definition $S = F - A^*F$ allows us to write the inequality

$$\mu A^*_{(n+1)/f/} - \mu A^*_{n/f/} \leq (1 - \mu f)^{n+1} .$$

Consequently the distribution of the $/a/$ $(a \in A^*)$, i.e. of the recurrence time of A^*, is dominated by an exponential distribution and this proves the result.