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APPLICATION OF THE GEOMETRY OF QUADRICS  
FOR CONSTRUCTING PBIB DESIGNS

by

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In this paper a number of families of PBIB designs with two associate classes is constructed using the properties of quadrics in finite projective geometry.

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1. Introduction . Partially balanced incomplete block

(PBIB) designs with two associate classes were introduced by Bose and Nair [3]. Bose and Shimamoto [4] have rephrased the definition so as to stress the distinction between the association scheme and the design. Bose and Shimamoto definition for the PBIB design with  $m$  associate classes is substantially as follows.

A PBIB design with  $m$  associate classes is an arrangement of  $v$  treatments in  $b$  blocks such that

(1) Each of the  $v$  treatments is replicated  $r$  times in  $b$  blocks each of size  $k$  and no treatment occurs more than once in every block.

(2) There exists a relationship of association between every pair of the  $v$  treatments satisfying the following conditions:

(a) Any two treatments are either first associates or second associates ... or  $m$ -th associates.

(b) Each treatment has  $n_1$  first associates,  $n_2$  second associates, ...  $n_m$   $m$ -th associates.

(c) Given any two treatments which are  $i$ -th associates, the number  $p_{jk}^i$  of treatments which are  $j$ -th associates of the first and  $k$ -th associates of the second is independent of the pair of treatments with which we start. Furthermore  $p_{jk}^i = p_{kj}^i$ , for  $i, j, k = 1, 2, \dots, m$ .

(3) Any pair of treatments which are  $i$ -th associates occur together in exactly  $\lambda_i$  blocks for  $i=1, 2, \dots, m$ .

Bose and Clatworthy [2] have given a less demanding definition for the PBIB design with two associate classes which is as follows:

A PBIB design with two associate classes is an arrangement of  $v$  treatments in  $b$  blocks such that

(1) Each of the  $v$  treatments is replicated  $r$  times in  $b$  blocks each

of size  $k$  and no treatment occurs more than once in any block.

(2) There exists a relationship of association between the  $v$  treatments satisfying the following conditions:

(a) Any two treatments are either first associates or second associates.

(b) Each treatment has  $n_1$  first associates and  $n_2$  second associates.

(c) For any pair of the  $v$  treatments which are  $i$ -th associate the number  $p_{11}^i$  of treatments common to the first associates of the first and first associates of the second is independent of the pair of treatments with which we start,  $i = 1, 2$ .

(3) Any pair of treatments which are  $i$ -th associates occurs in exactly  $\lambda_i$  blocks,  $i=1, 2$ .

Since it is easier to check the conditions of Bose-Clatworthy definition, we shall use this definition. The following relations hold between the parameters of a PBIB design and are useful for computing some parameters when others are known.

$$\begin{aligned}
 vr &= bk, \\
 v &= n_1 + n_2 + \dots + n_m + 1, \\
 r(k-1) &= \lambda_1 n_1 + \lambda_2 n_2 + \dots + \lambda_m n_m, \\
 (1.1) \quad \sum_{k=1}^m p_{jk}^i &= \begin{cases} n_i - 1, & \text{for } i = j \\ n_j, & \text{for } i \neq j \end{cases} \quad i, j = 1, 2, \dots, m, \\
 n_i p_{jk}^i &= n_j p_{ik}^j, \quad i, j, k = 1, 2, \dots, m.
 \end{aligned}$$

The theory of linear spaces in finite projective geometry have been used very profitably by several authors in constructing BIB and PBIB designs. Bose [1] first used the properties of quadric surfaces in finite projective geometry of two and three dimensions for constructing experimental designs.

In [8] we have derived several properties of quadrics in finite projective spaces.

In this paper we have used the geometry of quadrics to construct several series of PBIB designs with two associate classes.

Let  $B$  denote the class consisting of the sets  $B_1, B_2, \dots, B_b$  where  $B_j, j=1, 2, \dots, b$ , is a set of points in  $PG(n, s)$ . Let  $V$  denote another class consisting of the sets  $V_1, V_2, \dots, V_v$  where  $V_i, i=1, 2, \dots, v$  is a set of points in  $PG(n, s)$ . These two classes generate a design  $D(B, V)$  with the following incidence matrix

$$N = \begin{pmatrix} (n_{ij}) \\ (v \times b) & v \times b \end{pmatrix}$$

where

$$n_{ij} = \begin{cases} 1, & \text{if the sets } V_i \text{ and } B_j \text{ intersect each other,} \\ 0, & \text{if the sets } V_i \text{ and } B_j \text{ do not intersect.} \end{cases}$$

Then we have the following.

$$r_i = \sum_{j=1}^b n_{ij} = \text{Number of sets of the class } B \text{ which intersect } V_i.$$

$$k_j = \sum_{i=1}^v n_{ij} = \text{Number of sets of the class } V \text{ which intersect } B_j.$$

$$\lambda_{ii'} = \sum_{j=1}^b n_{ij} n_{i'j} = \text{Number of sets of the class } B \text{ which intersect both } V_i \text{ and } V_{i'}, i \neq i' = 1, 2, \dots, v.$$

Let  $C_1, C_2, \dots, C_m$  be  $m$  classes of sets in  $PG(n, s)$  such that

$$V_i \cap V_{i'} \in C_j, \text{ for some } j, j=1, 2, \dots, m.$$

The sets  $V_i$  and  $V_{i'}$  are said to be  $j$ -th associates if

$$V_i \cap V_{i'} \in C_j, j=1, 2, \dots, m.$$

Let  $p_{uv}^t(V_i, V_{i'})$  denote the number of sets of the class  $V$  which are  $u$ -th associates of  $V_i$  and  $v$ -th associates of  $V_{i'}$ ,  $i = i', i, i' = 1, 2, \dots, v$ ,

$t, u, v = 1, 2, \dots, m$ . The following result, stated as a lemma for the sake of reference, follows easily.

Lemma 1. The design  $D(B, V)$  is a PBIB design with two associate classes if the following are true.

- (1) Any two sets are either first associates or second associates ... or  $m$ -th associates.
- (2) Each set  $V_i (i=1, 2, \dots, v)$  has  $n_1$  first associates,  $n_2$  second associates, ...  $n_m$   $m$ -th associates.
- (3)  $p_{uv}^t(V_i, V_{i'})$  is independent of the  $t$ -th associate pair of sets  $(V_i, V_{i'})$  and  $p_{uv}^t = p_{vu}^t, t, v, u = 1, 2, \dots, m$ .
- (4)  $r_i = r, i=1, 2, \dots, v$  and  $k_j = k, j=1, 2, \dots, b$ .
- (5) For any pair of  $t$ -th associate sets  $(V_i, V_{i'})$   $\lambda_{11'} = \lambda_t, t = 1, 2, \dots, m$ .

As a consequence of Bose-Clatworthy definition of PBIB designs with two associate classes we have the following result.

Lemma 2. The design  $D(B, V)$  is a PBIB design with two associate classes if the conditions (1), (2), (4) and (5) are satisfied for  $m=2$  and the following condition (3)' is satisfied instead of (3).

(3)'. The number  $p_{11}^t(V_i, V_{i'})$  is equal for every pair of  $t$ -th associate sets  $(V_i, V_{i'})$ ,  $t = 1, 2$ .

## 2. Preliminaries on quadrics in finite geometry.

Let  $\alpha_0, \alpha_1, \dots, \alpha_s$  be the  $s$  elements of a Galois field  $\mathbb{GF}(s)$  where  $s$  is a prime power. Any  $(n+1)$ -tuple  $\underline{x} = (x_0, x_1, \dots, x_n)$  can be considered as a point of  $\text{PG}(n, s)$ , the finite projective geometry of  $n$  dimensions, where each  $x_i$  is an element of the field and at least one  $x_i$  is a non null element of the field,  $i = 0, 1, 2, \dots, n$ . Two  $(n+1)$ -tuples  $\underline{x} = (x_0, x_1, \dots, x_n)$  and  $\underline{x}' = (x'_0, x'_1, \dots, x'_n)$  are assumed to represent the same point if for some non-

null element  $\rho$  of the field  $x_i' = \rho x_i, i=1,2,\dots,n$ .

A quadric  $Q$  in  $PG(n,s)$  is the set of all points  $\underline{x} = (x_0, x_1, \dots, x_n)$  which satisfies an equation of the form

$$(2.1) \quad \dots \quad \underline{x} A x' = 0$$

where  $A$  is a  $(n+1 \times n+1)$  triangular matrix with elements from the field  $GF(s)$  and  $\underline{x}'$  is the transpose of the row vector  $\underline{x}$ . The expression  $\underline{x} A \underline{x}'$  is called the form of the quadric  $Q$ . A quadric  $Q$  in  $PG(n,s)$  is said to be degenerate if there exists a nonsingular transformation  $\underline{x} = B \underline{y}$  which transforms the form of  $Q$  to

$$\sum_{\substack{j \geq i \\ i=0}}^r c_{ij} y_i y_j \quad .$$

A quadric which is not degenerate will be said to be nondegenerate. Two points  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_0, \beta_1, \dots, \beta_n)$  are said to be mutually conjugate with respect to (w.r.t.)  $Q$  if  $\underline{\alpha}(A + A') \underline{\beta}' = 0$ . The polar space  $\tau(\alpha)$  of a point  $\alpha$  is the set of all points  $\beta$  which are conjugate to  $\alpha$  w.r.t.  $Q$ . The polar of  $\alpha$  is the linear space determined by the equation

$$(2.2) \quad \dots \quad \underline{\alpha} (A + A') \underline{x}' = 0.$$

If  $\alpha$  is a point of  $Q$ ,  $\tau(\alpha)$  is called the tangent space of  $Q$  at the point  $\alpha$ . If  $\alpha$  and  $\beta$  are two points of  $Q$  which are mutually conjugate, then the line  $\alpha\beta$  determined by the two points  $\alpha$  and  $\beta$  is contained in  $Q$ . A line contained in  $Q$  is called a generator of  $Q$ . Every nondegenerate quadric in  $PG(2k,s)$  contains linear spaces of dimensionality  $(k-1)$  and does not contain any linear space of higher dimensionality.

If a nondegenerate quadric in  $PG(2k-1,s)$  contains  $(k-1)$ -flats (linear spaces of dimensionality  $k-1$ ), it is called a hyperbolic quadric in  $PG(2k-1,s)$ . If a nondegenerate quadric in  $PG(2k-1,s)$  contains  $(k-2)$ -flats and no flats of higher dimension, it is called an elliptic quadric in  $PG(2k-1,s)$ .  $N(p,n)$  will

denote the number of  $p$ -flats contained in a nondegenerate quadric in  $PG(n,s)$   $\phi(p,k)$ ,  $\phi_1(p,k)$  and  $\phi_2(p,k)$  respectively denote the number of  $p$ -flats contained in a nondegenerate quadric in  $PG(2k,s)$ , an elliptic quadric in  $PG(2k-1,s)$  and a hyperbolic quadric in  $PG(2k-1,s)$ . Formulas for these functions are obtained in theorem 3.2 of [8]. It is proved in theorem 4.1 of [8] that for every nondegenerate quadric in  $PG(2k,s)$  there is a point  $S$  such that every line through  $S$  intersects the quadric in a single point. This point  $S$  is called the nucleus of polarity of the quadric.

Now we prove a few lemmas which will be useful later. Lemma 3 gives simple equations for hyperbolic and elliptic quadrics in  $PG(2k-1,s)$ . These simple equations are useful for writing down the blocks of the designs which are obtained from the incidence properties of these quadrics.

Lemma 3. Let  $GF(s)$  be a Galois field with characteristic not equal to 2. Let  $\alpha$  be a non-zero element of  $GF(s)$  such that  $-\alpha$  is a square and  $\beta$  be a non-zero element such that  $-\beta$  is not a square. Let  $\lambda$  be an element of  $GF(2^m)$  such that  $\lambda(x_1^2 + x_2^2) + x_1x_2$  is irreducible in  $GF(2^m)$ . Then

(1) The quadric  $Q_{2k-1}$  in  $PG(2k-1,s)$ ,  $s \neq 2^m$ , with the equation

$$x_1^2 + \alpha x_2^2 + x_3^2 + \alpha x_4^2 + \dots + x_{2k-1}^2 + \alpha x_{2k}^2 = 0$$

is a hyperbolic nondegenerate quadric.

(2) The quadric  $Q_{2k-1}$  in  $PG(2k-1,s)$ ,  $s \neq 2^m$ , with the equation

$$x_1^2 + \beta x_2^2 + x_3^2 + \alpha x_4^2 + \dots + x_{2k-1}^2 + \alpha x_{2k}^2 = 0$$

is an elliptic nondegenerate quadric.

(3) The quadric  $Q_{2k-1}$  in  $PG(2k-1, 2^m)$  with the equation

$$x_1x_2 + x_3x_4 + \dots + x_{2k-1}x_{2k} = 0$$

is a hyperbolic nondegenerate quadric.

(4) The quadric  $Q_{2k-1}$  in  $PG(2k-1, 2^m)$  with the equation

$$\lambda(x_1^2 + x_2^2) + x_1x_2 + x_3x_4 + \dots + x_{2k-1}x_{2k} = 0$$

is an elliptic nondegenerate quadric.

Proof. (1) It is obvious that  $Q_{2k-1}$  is nondegenerate. We shall show that  $Q_{2k-1}$  is hyperbolic by finite induction on  $k$ . First we prove the result for  $k=1$ . Since  $-\alpha$  is a square element of  $GF(s)$ , there exists an element  $\lambda$  of  $GF(s)$  such that

$$-\alpha = \lambda^2 .$$

The equation of  $Q_1$  is

$$x_1^2 + \alpha x_2^2 = 0 .$$

It can be easily seen that  $Q_1$  contains the two points  $(\lambda, 1)$  and  $(-\lambda, 1)$ .

Since  $Q_1$  contains linear space of dimensionality  $0(=k-1)$  i.e. points,  $Q_1$  is hyperbolic. Assume that  $Q_{2k-3}$  is hyperbolic. Consider the nonsingular transformation

$$y_i = x_i , i=1,2,\dots,2k-2 .$$

$$y_{2k-1} = x_{2k-1} + \lambda x_{2k} .$$

$$y_{2k} = x_{2k-1} - \lambda x_{2k} .$$

It is easy to see that under this transformation  $Q_{2k-1}$  transforms to  $Q'_{2k-1}$  with the equation

$$y_1^2 + \alpha y_2^2 + y_3^2 + \alpha y_4^2 + \dots + y_{2k-3}^2 + \alpha y_{2k-2}^2 + y_{2k-1} y_{2k} = 0 .$$

Since the incidence properties in a projective geometry remain invariant over nonsingular transformations, it is sufficient to show that  $Q'_{2k-1}$  is hyperbolic.

Consider the point

$$P = (0 \ 0 \ \dots \ 0 \ 1 \ 0)$$

of  $Q'_{2k-1}$ . The equation of  $\tau(P)$  is obviously

$$y_{2k} = 0 .$$

Let  $\pi$  be the  $(n-1)$ -flat with the equation

$$y_{2k-1} = 0 .$$

Then  $Q'_{2k-1} \cap \tau \cap \pi$  has the equation



$$y_1^2 + \alpha y_2^2 + y_3^2 + \alpha y_4^2 + \dots + y_{2k-3}^2 + \alpha y_{2k-2}^2 = 0 .$$

By assumption  $Q_{2k-1}' \cap \tau \cap \pi$  is hyperbolic and hence contains a  $(k-2)$ -flat  $\Sigma_{k-2}$ . The point  $P$  and the  $(k-2)$ -flat  $\Sigma_{k-2}$  are both contained in the quadric  $Q_{2k-1}'$  and are mutually conjugate. Hence by Lemma 2.3 of [8] the  $(k-1)$ -flat determined by  $P$  and  $\Sigma_{k-2}$  is contained in  $Q_{2k-1}'$ . Hence  $Q_{2k-1}'$  is hyperbolic.

(2) We shall prove the result by induction of  $k$ . First we prove the result for  $k = 1$ . The quadric  $Q_1$  in  $PG(1, s)$  with the equation

$$x_1^2 + \beta x_2^2 = 0$$

will be elliptic if  $Q_1$  does not contain any point. If possible, suppose  $Q_1$  contains a point  $(x_1^i, x_2^i)$ . Then  $x_2^i \neq 0$ . We can easily get

$$\frac{x_1^i{}^2}{x_2^i{}^2} = -\beta$$

which contradicts the assumption that  $-\beta$  is a non-square element. Hence  $Q_1$  is elliptic.

Assume that the result is true for  $(k - 1)$  so that  $Q_{2k-3}$  is a nondegenerate elliptic quadric. Applying the nonsingular transformation used in part 1, we can transform  $Q_{2k-1}$  to  $Q_{2k-1}'$  with the equation

$$y_1^2 + \beta y_2^2 + y_3^2 + \alpha y_4^2 + \dots + y_{2k-3}^2 + \alpha y_{2k-2}^2 + y_{2k-1} y_{2k} = 0 .$$

As before it will be sufficient to show that  $Q_{2k-1}'$  is elliptic. If possible, suppose  $Q_{2k-1}'$  is hyperbolic. Then  $Q_{2k-1}'$  contains  $(k-1)$ -flats. Consider the point

$$P = (0 \ 0 \ \dots \ 0 \ 1 \ 0) .$$

The equation of  $\tau(P)$  is

$$y_{2k} = 0 .$$

Let  $\pi$  be the  $(n-1)$ -flat with the equation

$$y_{2k-1} = 0 .$$

Then the equation of  $Q'_{2k-1} \cap \tau \cap \pi$  is

$$y_1^2 + \beta y_2^2 + y_3^2 + \alpha y_4^2 + \dots + y_{2k-3}^2 + \alpha y_{2k-2}^2 = 0,$$

which is elliptic by induction assumption. By theorem 3.2 of [8] the number of  $p$ -flats passing through a point  $P$  of a nondegenerate quadric and contained in the quadric is equal for every point  $P$  of the quadric. Since  $Q'_{2k-1}$  contains  $(k-1)$ -flats, it follows that there exists a  $(k-1)$ -flat  $\Sigma_{k-1}$  contained in  $Q'_{2k-1}$  and passing through  $P$ . By theorem 2.2 of [8]  $\Sigma_{k-1}$  is contained in  $Q'_{2k-1} \cap \tau(P)$ . Hence  $\Sigma_{k-1}$  intersects  $Q'_{2k-1} \cap \tau(P) \cap \pi$  in a  $(k-2)$ -flat  $\Sigma_{k-2}$ . So  $Q'_{2k-1} \cap \tau(P) \cap \pi$  contains a  $(k-2)$ -flat  $\Sigma_{k-2}$ . This contradicts the assumption that  $Q'_{2k-1} \cap \tau(P) \cap \pi$  is elliptic. This completes the proof of part (2). Parts (3) and (4) of the theorem can be proved by arguments exactly similar to arguments used in parts (1) and (2) respectively.

### 3. PBIB designs from the configuration of generators for blocks and points for treatments of the quadric.

Definition. Generator. A line which is contained in the quadric is called a generator of the quadric.

Lemma 4. Let  $P_1$  and  $P_2$  be two points of a nondegenerate quadric  $Q_n$  in  $PG(n,s)$  such that the line  $P_1P_2$  is not a generator. The number of points  $P$  such that both the lines  $PP_1$  and  $PP_2$  are generators of the quadric is  $N(0,n-2)$  where  $N(p,n)$  denotes the number of  $p$ -flats contained in a non-degenerate quadric of the type of  $Q_n$  (elliptic hyperbolic) in  $PG(n,s)$ .

Proof. Since the line  $P_1P_2$  is not a generator, by lemma 2.3 of [8] the points  $P_1$  and  $P_2$  are not mutually conjugate. Let  $\tau_1$  and  $\tau_2$  denote the tangent spaces at  $P_1$  and  $P_2$  respectively. Let  $P$  be a point of  $Q_n$  such that both  $PP_1$  and  $PP_2$  are generators of  $Q_n$ . Since  $PP_1$  is a generator, by lemma 2.3 of [8]  $P$  must be conjugate to  $P_1$ . Hence  $P$  must be a point of  $\tau_1$ . Similarly,  $P$  must be a point of  $\tau_2$ . Hence  $P$  is a point of  $Q_n \cap \tau_1 \cap \tau_2$ .

Conversely if  $P$  is a point of  $Q_n \cap \tau_1 \cap \tau_2$ ,  $P$  is a conjugate to both  $P_1$  and  $P_2$  and hence both  $PP_1$  and  $PP_2$  are generators of  $Q_n$ . It follows from the above argument that the required number is equal to the number of points in  $Q_n \cap \tau_1 \cap \tau_2$ .

Since  $P_1$  and  $P_2$  are mutually not conjugate  $\tau_2$  does not pass through  $\tau_1$ . So  $\tau_1$  is a tangent space at  $P_1$  and  $\tau_2$  is an  $(n-1)$ -flat not passing through  $P_1$ . By theorem 2.1 of [87]  $Q_n \cap \tau_1 \cap \tau_2$  is a nondegenerate quadric  $Q_{n-2}$  in  $PG(n-2, s)$  which is elliptic or hyperbolic according  $Q_n$  is elliptic or hyperbolic. Hence the lemma follows.

Lemma 5. Let  $P_1$  and  $P_2$  be two points of  $Q_n$  in  $PG(n, s)$  such that  $P_1P_2$  is a generator of  $Q_n$ . Then the number of points  $P$  other than  $P_1$  and  $P_2$  such that both  $PP_1$  and  $PP_2$  are generators of  $Q_n$  is  $(s-1) + s^2 N(0, n-4)$ .

Proof. Let  $\tau_1$  and  $\tau_2$  be the tangent spaces at  $P_1$  and  $P_2$  respectively. Let  $\Sigma_{n-2}$  be an  $(n-2)$ -flat not intersecting the line  $\Sigma_1$  determined by  $P_1$  and  $P_2$ . Let  $P$  be a point of  $Q_n$  other than  $P_1$  and  $P_2$ . As in the proof of lemma 4, we can show that both  $PP_1$  and  $PP_2$  are generators of  $Q_n$  if and only if  $P$  is a point of  $Q_n \cap \tau_1 \cap \tau_2$ . Hence the required number of points is equal to the number of points of  $Q_n \cap \tau_1 \cap \tau_2$  other than  $P_1$  and  $P_2$ . By lemma 2.2 of [87],  $\tau_1 \cap \tau_2 = \tau(\Sigma_1)$ , the polar of  $\Sigma_1$ . By theorem 2.1 of [87],  $Q_n \cap \tau(\Sigma_1) \cap \Sigma_{n-2}$  is  $Q_{n-4}$ , a nondegenerate quadric in  $PG(n-4, s)$  and  $Q_n \cap \tau(\Sigma_1)$  is a cone with  $\Sigma_1$  as the vertex and  $Q_n \cap \tau(\Sigma_1) \cap \Sigma_{n-2}$  as the base.

To count the required number of points, we notice that for every point  $P$  of  $Q_n \cap \tau(\Sigma_1) \cap \Sigma_{n-2}$  the plane determined by  $P$  and  $\Sigma_1$  is contained in  $Q_n \cap \tau(\Sigma_1)$ . The number of points of such a plane which do not lie on  $\Sigma_1$  is  $s^2$ . Hence it follows easily that the required number of points is  $s(s-1) + s^2 N(0, n-4)$ .

Theorem 1. Let  $B$  be the class of generators of  $Q_n$ , a nondegenerate quadric in  $PG(n,s)$  and  $V$  be the class of points of  $Q_n$ , each point being regarded as a pointset. The  $D(B, V)$  is a PBIB design with two associate classes with the following parameters.  $v = N(0,n)$ ,  $b = N(1,n)$ ,  $k = s+1$ ,  $r = N(0,n-2)$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = 0$ ,  $n_1 = s N(0,n-2)$ ,  $p_{11}^1 = (s-1)+s^2 N(0,n-4)$  and  $p_{11}^2 = N(0,n-2)$ .

Proof. It will be sufficient to check the conditions of lemma 2. It follows easily that  $b =$  number of generators of  $Q_n = N(1,n)$ ,  $k =$  number of points on a generator  $= s+1$ ,  $v =$  number of points of  $Q_n = N(0,n)$  and  $r =$  number of generators passing through a point  $= N(0,n-2)$  (theorem 3.3 of [87]). The association scheme is defined as follows. Two points  $P_1$  and  $P_2$  of  $Q_n$  are first associates of each other if the line  $P_1P_2$  is a generator and are second associates of each other if the line  $P_1P_2$  is not a generator. Since there can be at most one generator passing through two points of  $Q_n$ , we have  $\lambda_1 = 1$  and  $\lambda_2 = 0$ . Let  $P_1$  be a particular point of  $Q_n$ . The number of points  $P$  which are first associates to  $P_1$  is equal to the number of points  $P$  such that  $PP_1$  is a generator and hence equal to the number of points lying on the generators passing through  $P_1$ . From the above argument, we have  $n_1 = sN(0,n-2)$ . Let  $P_1$  and  $P_2$  be two first associate points. Then  $P_1P_2$  is a generator of  $Q_n$ . The number of points  $P$  which are first associates of both  $P_1$  and  $P_2$  is equal to the number of points  $P$  other than  $P_1$  and  $P_2$  such that both  $PP_1$  and  $PP_2$  are generators. By lemma 5 this number is equal to  $(s-1)+s^2N(0,n-4)$ . From the above argument it follows that  $p_{11}^1 = (s-1) + s^2N(0,n-4)$ . Let  $P_1$  and  $P_2$  be two second associate points. Then the line  $P_1P_2$  is not a generator. The number of points  $P$  which are first associates to both  $P_1$  and  $P_2$  is equal to the number of points  $P$  which are such that both  $PP_1$  and  $PP_2$  are generators. By lemma 4 this number does not depend on the particular pair of second associate points and is equal to  $N(0,n-2)$ . From the above argument it follows that  $p_{11}^2 = N(0,n-2)$ . This completes the proof of the theorem.

Specializing the quadric  $Q_n$  of theorem 1, we shall get a number of series of PBIB designs. Taking  $n = 2t$ , we get the following series.

$$v = \frac{s^{2t} - 1}{s - 1}, \quad r = \frac{s^{2t-2} - 1}{s - 1}, \quad k = s+1,$$

$$b = \frac{(s^{2t} - 1)(s^{2t-2} - 1)}{(s - 1)^2 (s + 1)}, \quad \lambda_1 = 1, \quad \lambda_2 = 0,$$

$$n_1 = \frac{s(s^{2t-2} - 1)}{(s - 1)}, \quad p'_{11} = (s-1) + \frac{s^2(s^{2t-4} - 1)}{(s - 1)} \quad \text{and} \quad p''_{11} = \frac{s^{2t-2} - 1}{s - 1}.$$

Putting  $t = 2$ , we get the symmetric series with

$$v = b = s^3 + s^2 + s + 1, \quad r = k = s + 1, \quad \lambda_1 = 1, \quad \lambda_2 = 0$$

$$n_1 = s^2 + s, \quad p'_{11} = s - 1 \quad \text{and} \quad p''_{11} = s + 1. \quad \text{This series was obtained by}$$

Clatworthy [6] by a different method.

Taking  $n = 2t - 1$ ,  $t \geq 3$ , and  $Q_n$  elliptic, we get the series with

$$v = \frac{s^{2t-1} - s^t + s^{t-1} - 1}{s - 1}, \quad r = \frac{s^{2t-3} - s^{t-1} + s^{t-2} - 1}{s - 1},$$

$$k = s + 1, \quad b = \frac{(s^{2t-1} - s^t + s^{t-1} - 1)(s^{2t-3} - s^{t-1} + s^{t-2} - 1)}{(s - 1)^2 (s + 1)},$$

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad n_1 = \frac{s(s^{2t-3} - s^{t-1} + s^{t-2} - 1)}{s - 1},$$

$$p'_{11} = (s - 1) + \frac{s^2(s^{2t-5} - s^{t-2} + s^{t-3} - 1)}{s - 1} \quad \text{and}$$

$$p''_{11} = \frac{s^{2t-3} - s^{t-1} + s^{t-2} - 1}{s - 1}.$$

Taking  $n = 2t - 1$ ,  $t \geq 2$  and  $Q_n$  hyperbolic, we get the series with

$$v = \frac{s^{2t-1} + s^t - s^{t-1} - 1}{s-1}, \quad r = \frac{s^{t-3} + s^{t-1} - s^{t-2} - 1}{s-1},$$

$$k = s + 1, \quad b = \frac{(s^{2t-1} + s^t - s^{t-1} - 1)(s^{2t-3} + s^{t-1} - s^{t-2} - 1)}{(s-1)^2 (s+1)},$$

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad n_1 = \frac{s(s^{2t-3} + s^{t-1} - s^{t-2} - 1)}{s-1}$$

$$p_{11}^1 = (s-1) + \frac{s^2(s^{2t-5} + s^{t-2} - s^{t-3} - 1)}{s-1} \quad \text{and}$$

$$p_{11}^2 = \frac{s^{2t-3} + s^{t-1} - s^{t-2} - 1}{s-1}.$$

Putting  $t = 2$ , in the above series, we get the family of simple lattice designs.

4. PBIB designs from the configuration of points of a quadric for blocks and generators of a quadric for treatments.

In this section we consider a nondegenerate quadric  $Q_n$  in  $PG(n,s)$  which contains lines but does not contain planes. Therefore,  $Q_n$  can be a nondegenerate quadric in  $PG(4,s)$  or an elliptic quadric in  $PG(5,s)$  or a hyperbolic nondegenerate quadric in  $PG(3,s)$ .

Lemma 6. Let  $l_1$  and  $l_2$  be two intersecting generators of  $Q_n$ . Then the number of generators other than  $l_1$  and  $l_2$  which intersect both  $l_1$  and  $l_2$  is  $N(0,n-2) - 2$ .

Proof. Suppose the two generators in  $l_1$  and  $l_2$  intersect at the point  $P$ . Then there cannot be any generator which intersects both  $l_1$  and  $l_2$  at points other than  $P$ . If possible, suppose there exists a generator which intersects  $l_1$  at  $P_1$  and  $l_2$  at  $P_2$ . Then the three points  $P, P_1$  and  $P_2$  are mutually conjugate and are points of  $Q_n$ . By lemma 2.3 of [8], the

plane determined by them is contained in  $Q_n$ . This contradicts the assumption that  $Q_n$  does not contain any planes. So it follows that the required generators are those which intersect both  $l_1$  and  $l_2$  at the point  $P$ . By theorem 3.3 of [8] the number of generators passing through  $P$  is  $N(0, n-2)$  of which  $l_1$  and  $l_2$  are two generators. Hence the lemma follows.

Lemma 7. Let  $l_1$  and  $l_2$  be two nonintersecting generators of  $Q_n$ . Then the number of generators which intersect both  $l_1$  and  $l_2$  is  $(s+1)$ .

Proof. Let  $P$  be a point of the generator  $l_2$ . Consider the generators which intersect  $l_2$  at  $P$  and also intersect  $l_1$ . Any such generator will lie in the plane  $\Sigma_2$  determined by  $l_1$  and the point  $P$ . The plane  $\Sigma_2$  is not contained in  $Q_n$  and contains a generator  $l_1$  of  $Q_n$  and a point  $P$  of  $Q_n$  not lying on  $l_1$ . From these facts it follows easily that  $\Sigma_2$  intersects  $Q_n$  in a pair of lines. Hence there exists one and only one line passing through  $P$  and intersecting  $l_1$ . This is true for every point  $l_2$  and the number of points of  $l_2$  is  $(s+1)$ . Hence the lemma follows.

Theorem 2. Let  $B$  be the class of points of a nondegenerate quadric  $Q_n$  which does not contain planes and  $V$  be the class of generators of  $Q_n$ . Then  $D(B, V)$  is a PBIB design with two associate classes with the following parameters

$$v = N(1, n), \quad r = (s+1), \quad k = N(0, n-2), \quad b = N(0, n), \quad \lambda_1 = 1, \quad \lambda_2 = 0,$$

$$n_1 = (N(0, n-2) - 1)(s+1), \quad p'_{11} = N(0, n-2) - 2 \quad \text{and} \quad p^2_{11} = (s+1).$$

Proof. It is sufficient to check the conditions of lemma 2. It follows easily that  $v =$  number of generators of  $Q_n = N(1, n)$ ,  $r =$  number of points of  $Q_n$  intersecting a generator  $= s+1$ ,  $k =$  number of generators of  $Q_n$  intersecting a point  $= N(0, n-2)$  and  $b =$  number of points of  $Q_n = N(0, n)$ . The association scheme is defined as follows. Two generators  $l_1$  and  $l_2$  of  $Q_n$  are first associates if they intersect and second associates if they do not intersect. Since two generators can intersect in at most one point, we have

$\lambda_1 = 1$  and  $\lambda_2 = 0$ . Let  $l_1$  be a given generator. The number of generators which are first associates of  $l_1$  is equal to the number of generators which intersect  $l_1$ . Through every point of  $l_1$  there passes  $N(0, n-2)$  generators of which  $l_1$  is one. Hence it follows that  $n_1 = (N(0, n-2) - 1)(s+1)$ . Consider two generators  $l_1$  and  $l_2$  which are first associates. By definition  $l_1$  and  $l_2$  intersect each other. The number of generators which are first associates of both  $l_1$  and  $l_2$  is equal to the number of generators which intersect both  $l_1$  and  $l_2$ . By lemma 6, this number is independent of the particular pair of intersecting generators  $l_1$  and  $l_2$  and is equal to  $N(0, n-2) - 2$ . Hence  $p_{11}^1 = N(0, n-2) - 2$ . Similarly using lemma 7, we can show that  $p_{11}^2 = s+1$ . This completes the proof of the theorem.

Specializing  $Q_n$  in theorem 2, we shall get a number of series of PBIB designs with two associate classes. Taking  $n=5$  and  $Q_5$  an elliptic quadric, we get the series with

$$v = (s^3+1)(s^2+1), \quad r = s+1, \quad k = s^2+1, \quad b = (s^3+1)(s+1), \quad \lambda_1 = 1, \quad \lambda_2 = 0$$

$$n_1 = s^2(s+1), \quad p_{11}^1 = s^2 - 1 \quad \text{and} \quad p_{11}^2 = s+1.$$

Taking  $n = 3$  and  $Q_3$  a hyperbolic nondegenerate quadric in  $PG(3, s)$ , we get the series with

$$v = 2(s+1), \quad r = (s+1), \quad k = 2, \quad b = (s+1)^2, \quad \lambda_1 = 1, \quad \lambda_2 = 0, \quad n_1 = s+1,$$

$$p_{11}^1 = 0 \quad \text{and} \quad p_{11}^2 = s+1.$$

This series is also given by Clatworthy [7]. Taking  $n = 4$  and  $Q_4$  a nondegenerate quadric with

$$v = b = \frac{s^4 - 1}{s - 1}, \quad r = k = s + 1, \quad \lambda_1 = 1, \quad \lambda_2 = 0, \quad n_1 = s(s + 1),$$

$$p_{11}^1 = s - 1 \quad \text{and} \quad p_{11}^2 = s + 1.$$

##### 5. PBIB designs from the configuration of generators on generators.

Theorem 3. Let  $B$  be the class of generators of a nondegenerate quadric  $Q_n$



which contains lines but does not contain planes. If a generator is considered as non-intersecting with itself,  $D(B, V)$  is a PBIB design with two associate classes with the following parameters.

$$v = b = N(1, n), r = k = (N(0, n-2) - 1)(s+1), \lambda_1 = N(0, n-2) - 2, \lambda_2 = s+1,$$

$$n_1 = (N(0, n-2) - 1)(s+1), p_{11}^1 = N(0, n-2) - 2 \text{ and } p_{11}^2 = s+1.$$

Proof. Two generators are defined to be first associates if they intersect each other. If two generators do not intersect, they are defined to be second associates. With the association scheme so defined, the proof of the theorem follows easily from lemma 6 and 7.

Taking  $Q_n$ , a nondegenerate quadric in  $PG(4, s)$ , we get the following series

$$v = b = s^3 + s^2 + s + 1, r = k = s(s+1), \lambda_1 = s - 1, \lambda_2 = s + 1,$$

$$n_1 = s(s+1), p_{11}^1 = s - 1 \text{ and } p_{11}^2 = s + 1.$$

Taking  $n = 5$  and  $Q_5$  an elliptic nondegenerate quadric in  $PG(5, s)$ , we get the series with

$$v = b = (s^3 + 1)(s^2 + 1), r = k = s^2(s+1), \lambda_1 = s^2 - 1, \lambda_2 = s + 1$$

$$n_1 = s^2(s+1), p_{11}^1 = s^2 - 1 \text{ and } p_{11}^2 = s + 1.$$

Taking  $n = 3$  and  $Q_3$  a hyperbolic nongenerate quadric, we get the series with

$$v = b = 2(s+1), r = k = s+1, \lambda_1 = 0, \lambda_2 = s+1, n_1 = s+1,$$

$$p_{11}^1 = 0 \text{ and } p_{11}^2 = s+1.$$

Now we shall give an example to illustrate the actual method of constructing the blocks of the design. Consider a design of the series given in theorem 1. We take  $n = 4$  and  $Q_4$  a non-degenerate quadric and  $s = 2$ . The parameters are  $v = b = 15, r = k = 3, \lambda_1 = 1, \lambda_2 = 0, n_1 = 6, p_{11}^1 = 1$  and  $p_{11}^2 = 3$ . The design is obtained by taking generators of  $Q_4$  as blocks and points of  $Q_4$  as treatments. The equation of  $Q_4$  can be taken as

$$x_0^2 + x_1x_2 + x_3x_4 = 0.$$

The 15 points of  $Q_4$  are

$$\begin{array}{ll} P_1 = (00001), & P_9 = (11101), \\ P_2 = (00010), & P_{10} = (00110), \\ P_3 = (10011), & P_{11} = (01010), \\ P_4 = (00100), & P_{12} = (11110), \\ P_5 = (01000), & P_{13} = (10111), \\ P_6 = (11100), & P_{14} = (11011), \\ P_7 = (00101), & P_{15} = (01111), \\ P_8 = (01001), & \end{array}$$

To write down the blocks systematically we can proceed as follows. Consider the treatment 1. The blocks which contain treatment 1 correspond to the generators containing the point  $P_1$ . To find out the generators passing through  $P_1$ , we find out  $Q_4 \cap \tau(P_1)$  where  $\tau(P_1)$  is the tangent space at  $P_1$ . For any point  $P$  of  $Q_4 \cap \tau(P_1)$ ,  $PP_1$  is a generator. In this way we can exhaust all the blocks containing treatment 1. Next we proceed to treatment 2 and by a similar procedure can find out the blocks containing treatment 2 which are not already included. We continue in this manner until all the blocks are obtained.

In our example the 15 generators are  $P_1P_4, P_1P_5, P_1P_6, P_2P_4, P_2P_5, P_2P_6, P_3P_4, P_3P_5, P_3P_6, P_7P_{11}, P_7P_{12}, P_8P_{10}, P_8P_{12}, P_9P_{10},$  and  $P_9P_{11}$ . So the blocks of the designs are

$$\begin{array}{lll} (1,4,7), & (1,5,8), & (1,6,9), \\ (2,4,10), & (2,5,11), & (2,6,12), \\ (3,4,13), & (3,5,14), & (3,6,15), \\ (7,11,15), & (7,12,14), & (8,10,15), \\ (8,12,13), & (9,10,14), & \text{and } ((9,11,13)). \end{array}$$

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