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Air Force Office of Scientific Research
Washington 25, D. C.

AFOSR Report No. 1090

SOME TERNARY ERROR CORRECTING CODES AND FRACTIONALLY REPLICATED DESIGNS

by

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May, 1961

Contract No. AF 49(638)-213

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Institute of Statistics
Mimeograph Series No. 295

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1. Summary

The connection between the theory of error correcting codes and the theory of fractionally replicated designs was pointed out by the author in [3]. It was shown that the packing problem plays a fundamental role in both theories. This is the problem of finding the maximum possible number of distinct points in the finite projective space PG(r-1, p), where p is a prime or a prime power, so that no d of the points are dependent. Here we find a configuration of 12 points in PG(5, 3) so that no 6 are dependent. Ternary error correcting codes and some fractional factorial designs (with or without blocking) are deduced.

2. A set of 12 points in PG(5, 3), no five of which are dependent.

Consider the 6 x 12 matrix

$$(2.1)\quad C = [A, I] = \left[\begin{array}{cccccc:cccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

This research was supported by the United States Air Force through the Air Force Office of Scientific Research and Development Command, under Contract No. AF (49) 638-213. Reproduction in whole or in part is permitted for any purpose of the United States Government.

whose elements belong to the Galois field GF(3), and where A is the matrix formed by the first six columns of [A, I], and I is the matrix formed by the last six columns of [A, I]. Let $\alpha_1, \alpha_2, \ldots, \alpha_6$ be the column vectors of A, and $\epsilon_1, \epsilon_2, \ldots, \epsilon_6$ the column vectors of I. Let P_i and Q_i be the points in PG(3, 5) with coordinates α_i and ϵ_i respectively ($i = 1,2,3,4,5,6$). We shall prove

Theorem 1. If Σ is the set of 12 points P_i, Q_i ($i = 1,2,3,4,5,6$) defined above, then any 5 points of Σ are independent.

(a) We note that A is a symmetric matrix and that

$$AA' = A^2 = 2I \tag{2.2}$$

The following relations follow at once

$$A^{-1} = 2A$$
$$A\alpha_i = 2\epsilon_i$$
$$A\epsilon_i = \alpha_i$$

It follows that the transformation

$$\rho x^* = Ax \quad , \qquad \rho \neq 0$$

transforms the point P_i to Q_i and vice versa ($i = 1,2,3,4,5,6$).

(b) The points Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 are independent. It follows from (a) that the points P_1, P_2, P_3, P_4, P_5, P_6 are independent.

(c) Since P_i has five non-zero coordinates it is clear that P_i and any four points chosen out of Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 are independent. It follows from (a) that Q_i and any four points chosen out of P_1, P_2, P_3, P_4, P_5, P_6 are independent.

(d) We note that if we choose any three rowed submatrix from A, any two columns of this submatrix are independent i.e. any 3 x 2 submatrix of A has rank 2.

(e) We shall now show that any three points Q_i, Q_j, Q_k chosen out of $Q_1, Q_2, \ldots, Q_6$, together with any two points P_u, P_v chosen out of $P_1, P_2, \ldots, P_6$ are independent. Let p, q, r be the rows of (A, I) other than the rows i, j, k. We can suppose without loss of generality that $i < j < k$. Then after suitable row interchanges the coordinates of P_u, P_v, Q_i, Q_j, Q_k are given by the columns of the matrix

$$(2.3) \qquad \begin{bmatrix} a_{iu} & a_{iv} & 1 & 0 & 0 \\ a_{ju} & a_{jv} & 0 & 1 & 0 \\ a_{ku} & a_{kv} & 0 & 0 & 1 \\ a_{pu} & a_{pv} & 0 & 0 & 0 \\ a_{qu} & a_{qv} & 0 & 0 & 0 \\ a_{ru} & a_{rv} & 0 & 0 & 0 \end{bmatrix} .$$

It follows from (d) that the rank of this matrix is 5. Hence P_u, P_v, Q_i, Q_j, Q_k are independent. It now follows from (a) that any three points P_i, P_j, P_k chosen out of $P_1, P_2, \ldots, P_6$, together with any two points Q_u, Q_v chosen out of $Q_1, Q_2, \ldots, Q_6$ are independent.

Corollary 1. The 12 x 6 matrix

$$(2.4) \qquad D = \begin{bmatrix} A \\ I \end{bmatrix}$$

has the property (P_5), i.e. no five rows of D are dependent.

Corollary 2. The 11 x 5 matrix D_1 obtained from D by omitting the last row and last column of D has the property (P_4), i.e. no four rows of D_1 are dependent.

3. Some ternary error-correcting systematic codes.

Consider a channel which is capable of transmitting any one of p distinct symbols. Such a channel is called a p-nary channel. We shall assume that p is a prime or a prime power and identify each symbol with a corresponding element of the field GF(p). For $p = 3$, the channel is called a ternary channel. Due to the presence of noise a transmitted symbol may be received as one of the other p-1 symbols. When this happens we say that there is an error in transmitting the symbol. The symbols successively presented to the channel of transmission constitute the input and the symbols received constitute the output .

Consider n-vectors whose coordinates are elements of GF(p). The totality of these vectors forms a vector space V_n. Consider a subspace V_k of V_n, generated by k independent vectors. V_k consists of p^k vectors which may be put in correspondence with a set of p^k messages. To transmit any message we use the symbols of the corresponding vector γ as input . The output is then a vector γ^* of V_k. If

$$\gamma^* = v + \epsilon \tag{3.1}$$

then ϵ is the error-vector . If we define $w(\epsilon)$ the weight of ϵ, as the number of non-zero elements in ϵ, then $w(\epsilon)$ is the number of errors committed in transmitting γ. If $r = n-k$, then there exists a vector space V_r of rank r, orthogonal to V_k which consists of all vectors orthogonal to each vector of V_k. Let D be an n x r matrix whose column vectors are independent and belong to V_k. Then the necessary and sufficient condition for the row vector γ to belong to V_k is

(3.2) $$\gamma D = 0$$

The matrix D is called the parity check matrix. Now

(3.3) $$\gamma^{*} D = (\gamma + \epsilon) D = \epsilon D$$

All error vectors which give rise to the same ϵD may be said to constitute an alias set. Each alias set consists of V^k error-vectors. If ϵ belongs to an alias set so does $\epsilon + \gamma_1$ where γ_1 is any element of V_k. The difference of any two error vectors belonging to the same alias set is an element of V_k and thus corresponds to a message.

To reconstruct the transmitted vector γ from the received vector γ^{*} we must have a rule which enables us to pick a unique ϵ given ϵ D. This vector may be called the leader of the alias set corresponding to ϵ D. Then our decoding rule consists of taking the transmitted message to be $\gamma = \gamma^{*} - \epsilon$. The set of vectors of V_k together with this decoding rule will be called an n-place systematic p-nary code with k information places. It is clear that a message will be correctly interpreted if and only if the true error vectors belongs to set of alias leaders. When errors occur independently and with equal probability, errors vectors with smaller weights have greater chance of occurring. Hence we should follow Slepian's [9] rule of choosing the leader of each alias set to be the vector with the minimum weight (in case of tie, one of the vectors having the smallest weight).

Let $\epsilon = (e_1, e_2, \ldots, e_n)$ and let the row vectors of the parity check matrix D be $\delta_1, \delta_2, \ldots, \delta_n$. Then

$$(3.4) \qquad \gamma^{*}D = \epsilon D = e_1\delta_1 + e_2\delta_2 + \dots + e_n\delta_n = \delta \text{ (say)}$$

Thus ϵD is a linear function of $\delta_1, \delta_2, \dots, \delta_n$, the number of non-zero coefficients in this linear function being $w(\epsilon)$, the weight of ϵ.

Now suppose that the parity check matrix D has the property (P_{2t}), so that no set of 2t vectors from among $\delta_1, \delta_2, \dots, \delta_n$ are dependent. If in transmitting γ, $t_o \leq t$ errors have been committed then $w(\epsilon) = t_o$ and there are exactly t_o non-zero coefficients in $e_1\delta_1 + e_2\delta_2 + \dots + e_n\delta_n$ in (3.4). If $\epsilon^{*} = (e_1^{*}, e_2^{*}, \dots, e_n^{*})$ is any other error vector belonging to the same alias set as ϵ^{*} then

$$(3.5) \qquad e_1\delta_1 + e_2\delta_2 + \dots + e_n\delta_n = e_1^{*}\delta_1 + e_2^{*}\delta_2 + \dots + e_n^{*}\delta_n = \delta$$

Now the weight $w(\epsilon^{*})$ must exceed t_o, otherwise (3.5) would give a linear relation between $2t_o$ or a lesser number of vectors from $\delta_1, \delta_2, \dots, \delta_n$ and this would contradict the fact that D has the property (P_{2t}). Thus ϵ is the leader of the coset corresponding to ϵD. There will exist exactly one linear function of $\delta_1, \delta_2, \dots, \delta_n$ with t or less non-zero coefficients equal to $\gamma^{*}D$, and the coefficients of this linear function determine ϵ and hence $\gamma = \gamma^{*} - \epsilon$. Thus the code is t error correcting, i.e. will correct t or a lesser number of errors.

Again suppose that the parity check matrix D has the property (P_{2t+1}) so that no set of 2t+1 vectors from among $\delta_1, \delta_2, \dots, \delta_n$ are dependent. We can show as before that if $t_o \leq t$ errors are committed, then the transmitted message γ will be correctly interpreted. Suppose now that t+1 errors occur in transmitting γ. Then $\gamma^{*}D$ cannot vanish. Also there will exist no linear function of $\delta_1, \delta_2, \dots, \delta_n$ with $t_o \leq t$

non-zero coefficients which will equal $\gamma^* D$. These two facts taken together will indicate to us that there have been at least t+1 errors. However it will be impossible to obtain the exact value of ϵ since there could exist more than one linear function of $\delta_1, \delta_2, \ldots, \delta_n$ with t+1 non-zero coefficients, producing the same value $\delta = \gamma^* D$.

Now consider the special case p = 3. Defining the matrices A and I as in section 2, we have proved in corollary 1 to Theorem 1, that

$$D = \begin{bmatrix} A \\ I \end{bmatrix}$$

possesses the property (P_5). Let

$$C = [A, I]$$

be given by (2.1). It follows from (2.2) that

$$CD = 0$$

Let V_6 be the vector space generated by the row vectors of C, and let our set of messages correspond to the vectors of V_6. Then we can take D as the corresponding parity check matrix. We therefore have the following theorem.

Theorem 2. The 12 place ternary code generated by the vectors

$\gamma_1 = (0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)$

$\gamma_2 = (1, 0, 1, 1, 2, 2, 0, 1, 0, 0, 0, 0)$

$\gamma_3 = (1, 1, 0, 2, 2, 1, 0, 0, 1, 0, 0, 0)$

$\gamma_4 = (1, 1, 2, 0, 1, 2, 0, 0, 0, 1, 0, 0)$

$\gamma_5 = (1, 2, 2, 1, 0, 1, 0, 0, 0, 0, 1, 0)$

$\gamma_6 = (1, 2, 1, 2, 1, 2, 0, 0, 0, 0, 0, 1)$

is a two error correcting and 3 error detecting systematic code with 6 information places.

For decoding we calculate $\gamma^* D$, where γ^* is the transmitted vector and D is given by (2.4). If $\gamma^* D$ is the null vector then there is no error. If $\gamma^* D = e_i \delta_i$ then ϵ is the vector which has e_i in the i-th place and zero in the other places. If $\gamma^* D = e_i \delta_i + e_j \delta_j$ then ϵ is the vector which has e_i and e_j and in the i-th and j-th positions and zero in the other places. In each case $\gamma = \gamma^* - \epsilon$. If $\gamma^* D$ is neither null nor of the form $e_i \delta_i$ or $e_i \delta_i + e_j \delta_j$ we conclude that there have been at least three errors.

Again let D_1 be the matrix considered in corollary 2. Let γ_{i1} be the vector obtained by dropping the last coordinate from γ_i in theorem 2, and let C_1 be the 6x11 matrix whose row vectors are γ_{11}, γ_{21}, γ_{31}, γ_{41}, γ_{51} and γ_{61}. Then

$$C_1 \mathcal{D}_1 = 0$$

Let $V_{6,1}$ be the vector space generated by $\gamma_{11}, \gamma_{21}, \ldots, \gamma_{61}$ and let our set of messages correspond to the vectors of $V_{6,1}$. Then we can take D_1 as the corresponding parity check matrix. Since D_1 has the property (P_4) we have the following theorem.

Theorem 3. The 11 place ternary code generated by the vectors γ_{11}, γ_{21}, γ_{31}, γ_{41}, γ_{51}, γ_{61} where γ_{i1} is obtained from γ_i in Theorem 2A, by dropping the last coordinate is a 2 error correcting systematic code with 6 information places.

For decoding we calculate $\gamma_1^* D_1$ where γ_1^* is transmitted vector and proceed as in Theorem 2A.

Corollary 3. Each non-null vector of the vector space V_6 generated by the vectors $\gamma_1, \gamma_2, \ldots, \gamma_6$ in Theorem 2, has weight 6 or more.

If γ is a vector of V_6 then $\gamma D = 0$. The corollary follows from the fact that D has the property (P_5), since if γ is of weight w there is a linear relation among w row vectors of D.

4. Some fractional factorial designs of the type $\frac{1}{3^k} \times 3^n$.

Let p be any prime number. Consider the group $\mathcal{G}_n$ generated by the elements

$$F_1, F_2, \ldots, F_n \tag{4.1}$$

satisfying the relations

$$F_1^p = F_2^p = \ldots = F_n^p = I \tag{4.2}$$

where I is the unit element of the group. If G is any element of the group $\mathcal{G}_n$, then by using (4.2), we can express G in the form

$$G = F_1^{a_1} F_2^{a_2} \ldots F_n^{a_n} \tag{4.3}$$

where

$$0 \leq a_i < p \tag{4.4}$$

$F_1, F_2, \ldots, F_n$ may be identified with the n factors of a factorial experiment in which each factor can be chosen at any one of the p distinct levels. The treatment in which the factor F_i occurs at the level x_i $(i = 0, 1, \ldots, n)$ may be written as

$$f_1^{x_1} f_2^{x_2} \ldots f_n^{x_n} \tag{4.5}$$

For any given G other than the unit element I the totality of the p^n treatments can be partitioned into p disjoint sets wuch that

(4.6) $$a_1x_1 + a_2x_2 + \ldots + a_nx_n$$

has the same value (mod p). The contrasts between these sets carry p-1 degrees of freedom which may be considered to belong to the interaction G. All the elements (other than the identity)

$$G, G^2, \ldots, G^{p-1}$$

of the sub-group generated by G define the same interaction, since it is readily seen that any of them generates the same partition of the set of treatments. Thus there are $(p^n - 1)/(p-1)$ interactions each carrying p-1 degrees of freedom, which account for the $p^n - 1$ independent contrasts between the treatments.

Any k independent elements $G_1, G_2, \ldots, G_k$ of $\mathcal{G}_n$ generate a subgroup $\mathcal{G}_k$ of order p^k. Let

(4.7) $$G_i = F_1^{a_{i1}} F_2^{a_{i2}} \ldots F_n^{a_{in}} \qquad i = 1, 2, \ldots, k$$

Consider the set S_{n-k} of treatments $f_1^{x_1} f_2^{x_2} \ldots f_n^{x_n}$ for which

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x^n = 0 \quad (\text{mod } p)$$

for i = ,. 2, ..., k. We then have p^{n-k} treatments in the set. Hence they form a $1/p^k$-th fraction of the totality of all possible treatments. Let L be any interaction not belonging to $\mathcal{G}_k$. Then the interactions

(4.8) $$L\, G_1^{n_1} G_2^{n_2} \ldots G_k^{n_k} \qquad 0 \leq n_i \leq p-1,\ i=1,2,\ldots,k$$

are called aliases of L. The set of interactions (4.8) is said to be an alias set. If in a factorial experiment the responses corresponding to the treatments of the set S_{n-k} are observed, then we can estimate the sum of all the aliases of L, though L individually cannot be estimated.

Since it is in general more important to estimate lower order interactions, it is of interest to choose the fundamental subgroup $\mathcal{G}_k$ in such a way that (for a specified t) no t-factor or lower order interaction is aliased with another t-factor or lower order interaction, i.e. any alias set should not contain more than one t-factor or lower order interaction. It is readily seen that for this it is necessary and sufficient that every interaction represented by an element of $\mathcal{G}_k$ should have 2t + 1 or more factors. In particular if each element of $\mathcal{G}_k$ has five or more factors then any main effect or two factor interaction will not be aliased with any other main effect or two factor interaction.

If we have s further elements $G_{k+1}, G_{k+2}, \ldots, G_{k+s}$ given by

$$G_i = F_1^{a_{i1}} F_2^{a_{i2}} \ldots F_n^{a_{in}}, \quad i = k+1, k+2, \ldots, k+s \tag{4.9}$$

such that $k+s < n$, and $G_1, G_2, \ldots, G_k, G_{k+1}, \ldots, G_{k+s}$ are independent, then the p^{n-k} treatments of the set S_{n-k} can be subdivided into subsets such that for a given subset

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = c_i \pmod p \tag{4.10}$$

$$i = k+1, \ldots, k+s$$

where $c_{k+1}, \ldots, c_{k+s}$ are fixed integers less than p. Since each c_i can be chosen in p different ways the number of subsets is p^s, and each subset contains p^{n-k-s} treatments. If each subset of treatments is assigned to a different block certain interactions will be confounded with block effects. These interactions are given by

$$G_1^{n_1} G_2^{n_2} \ldots G_k^{n_k} G_{k+1}^{n_{k+1}} \ldots G_{k+s}^{n_{k+s}} \tag{4.11}$$

$$0 \leq n_i \leq p-1, \quad i = 1, 2, \ldots, k+s$$

If we want that no t factor or lower order interaction is confounded with any block effect then each interaction in (4.11) must contain at least t+1 or more factors. In particular to leave main effects and two factor interactions unconfounded each interaction in (4.11) must be a 3-factor or higher order interaction. The subgroup generated by $G_1, G_2, \ldots, G_{k+s}$ may be called the block subgroup.

The theory of fractional replication briefly outlined above is due to Finney [5]. Kishen [8] using funite geometrical representation developed earlier by him and the present author [4] generalized this to the case when p is a prime power. We shall however not consider this as we are mainly interested in the case p = 3. For the theory of confounding reference may be made to [1], [2], [6] and [7].

If in a fractional factorial experiment no main effect or two factor interaction is confounded with a block effect or aliased with another main effect or two factor interaction we shall say that main effects and two factors interaction are measurable(neglecting three factor and higher order interactions).

Consider in particular the group $\mathcal{G}_{12}$ generated by $F_1, F_2, \ldots, F_{12}$ satisfying the relations

$$F_1^3 = F_2^3 = \ldots = F_{12}^3 = I$$

It follows from corollary 3, that the subgroup $\mathcal{G}_6$ generated by the elements $G_1, G_2, \ldots, G_6$ given below (4.12) is such that each element contains at least six or more of the letters F_i (with a non-zero power).

$$
(4.12) \qquad
\begin{array}{llllllll}
G_1 = & \cdot & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 \\
G_2 = & F_1 & \cdot & F_3 & F_4 & F_5^2 & F_6^2 & F_8 \\
G_3 = & F_1 & F_2 & \cdot & F_4^2 & F_5^2 & F_6 & F_9 \\
G_4 = & F_1 & F_2 & F_3^2 & \cdot & F_5 & F_6^2 & F_{10} \\
G_5 = & F_1 & F_2^2 & F_3^2 & F_4 & \cdot & F_6 & F_{11} \\
G_6 = & F_1 & F_2^2 & F_3 & F_4^2 & F_5 & F_6^2 & F_{12}
\end{array}
$$

We shall now use the above results to obtain some fractional factorial designs of the type $\frac{1}{3^k} \times 3^n$ where $n \leq 11$. The list of designs given is illustrative and not exhaustive.

(i) Suppose there are 11 factors $F_1, F_2, \ldots, F_{11}$. Let the fundamental subgroup be generated by $G_1, G_2, \ldots, G_6$ in (4.12) after deleting the letter F_{12}. Since only one letter has been deleted each interaction in the fundamental subgroup will contain five factors or more. Hence we have $\frac{1}{3^6} \times 3^{11}$ fractional experiment in which main effects and two factor interactions are measurable.

(ii) Suppose there are 10 factors $F_1, F_2, \ldots, F_{10}$. Drop the letters F_{11} and F_{12} from (4.12). Let the fundamental subgroup be generated by $G_1, G_2, \ldots, G_5$ and let the block subgroup have the additional generator F_6. Since only one letter viz F_{11} has been deleted from the generators of the fundamental subgroup, each interaction in this subgroup contains five or more letters. Similarly each interaction in the block subgroup contains four or more letters. Hence three factor or lower order interactions are unconfounded with block effects. We thus have $\frac{1}{3^5} \times 3^{10}$ fractional experiment with the treatments divided into 3 blocks, in which main effects and two factor interactions are measurable.

(iii) If there are 9 factors $F_1, F_2, \ldots, F_9$ then we can get a $\frac{1}{3^4} \times 3^9$ fractional experiment, with the treatments divided into 9 blocks, by dropping F_{10}, F_{11}, F_{12} from (4.12) and generating the fundamental subgroup from G_1, G_2, G_3, G_4 and the block subgroup from $G_1, G_2, G_3, G_4, G_5, G_6$. Main effects and two factor interactions are measurable.

(iv) Suppose there are 8 factors $F_1, F_2, \ldots, F_8$. If we drop the letters $F_9, F_{10}, F_{11}, F_{12}$ from (4.12) and generate the fundamental subgroup from G_1, G_2, G_3 and the block subgroup from $G_1, G_2, G_3, G_4, G_5, G_6$ then we get a $\frac{1}{3^3} \times 3^8$ fractional experiment in 27 blocks in which all the main effects and two factor interactions are measurable with the exception of the two factor interactions $F_1F_8^2$, $F_2F_7^2$, $F_3F_4^2$ and F_5F_6 which are confounded with block effects. If however we generate the block subgroup from G_1, G_2, G_3, G_4 and G_5 then we get a $\frac{1}{3^3} \times 3^8$ fractional experiment in 9 blocks in which all the main effects and two factor interactions are measurable.

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