UNIVERSITY OF NORTH CAROLINA
Department of Statistics
Chapel Hill, N. C.

ON THE ELEMENTARY RENEWAL THEOREM FOR NON-IDENTICALLY DISTRIBUTED VARIABLES

by

Walter L. Smith

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1. Introduction

Let $\{X_n\}$ be a sequence of independent, identically distributed, random variables with $0 < E\, X_n < \infty$; write $S_n = X_1 + X_2 + \dots + X_n$; let N_x be the number of partial sums which satisfy $S_n \leq x$; write $H(x) = \mathcal{E}\, N_x$. The elementary renewal theorem states that under certain conditions $H(x)/x \rightarrow \{ E\, X_n \}^{-1}$ as $x \rightarrow \infty$.

Kawata (1956) has proved a result which is equivalent to a generalization of the elementary renewal theorem to the case in which $\{X_n\}$ are non-identically distributed. Unfortunately, he found it necessary to impose quite heavy restrictions upon the distribution functions involved. In this note we prove a generalization of Kawata's result, but under rather weaker restrictions.

We write $F_n(x) \equiv P\{X_n \leq x\}$ and $G_n(x) \equiv P\{S_n \leq x\}$. Where necessary, we make use of the Heaviside unit function $U(x) = P\{0 \leq x\}$. We shall call the sequence of independent random variables $\{X_n\}$ a <u>W-sequence with average μ</u> if the following two conditions hold: --

(i) $\mathcal{E}\, X_n$ exists for all n and

$$\frac{1}{n} \sum_{r=1}^{n} \mathcal{E}\, X_r \rightarrow \mu \; , \text{ as } n \rightarrow \infty \; .$$

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(ii) S_n/n tends to μ in probability as $n \to \infty$.

It is assumed, of course, that μ is some finite constant. The requirement that $\{X_n\}$ be a W-sequence merely amounts to requiring that it satisfy a certain weak law of large numbers. The determination of necessary and sufficient conditions upon $\{F_n(x)\}$ for the satisfaction of such a law is a classical problem in probability theory and its solution is well-known (see, for example, Gnedenko and Kolmogorov (1954), p. 135). We shall also make use of functions of slow growth; the function $L(x)$, defined for all sufficiently large x, is said to be a function of slow growth if, for every $c > 0$,

$$\frac{L(cx)}{L(x)} \to 1 \quad \text{as} \quad x \to \infty .$$

Let k be a non-negative integer, α and γ be certain non-negative constants, and L a certain function of slow growth. We shall then say that a sequence $\{a_n\}$ of constants is an $\{L, k, \gamma, \alpha\}$ - sequence if (i) $a_n \geq 0$, all n; (ii) $a_n = O(n^k)$; (iii) as $x \to 1 - 0$,

$$(1.1) \qquad (1 - x)^{\gamma} \sum_{n=1}^{\infty} a_n x^n \sim \alpha L\left(\frac{1}{1-x}\right) .$$

We shall call α the limit-value of the sequence. For example, the Taylor expansion of

$$\frac{1}{(1-x)^{3/2}} \quad \log \left(\frac{1}{1 - x}\right)$$

generates a $\{\log, 1, \frac{3}{2}, 1\}$ -sequence (it can be shown without too much difficulty that the relevant coefficients a_n are $O(n^{1/2} \log n)$ in fact).

The main result of this paper is as follows.

Theorem. Suppose the following conditions hold.

(T1) For some integer $k \geq 0$, some constant $\gamma \geq 0$, some limit value $\alpha \geq 0$, and some function of slow growth $L(x)$, the sequence of constants $\{a_n\}$ is an $\{L, k, \gamma, \alpha\}$ -sequence.

(T2) The sequence of mutually independent random variables $\{X_n\}$ is a W-sequence with positive average μ (> 0).

(T3) There is a large positive constant A such that

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \int_{-\infty}^{A} \{U(x) - F_j(x)\} \, dx > 0 \tag{1.2}$$

(T4) There is a distribution function $K(x)$ of a negative-valued random variable with a finite moment of order $(k+2)$ such that $K(x) \geq F_n(x)$ for all n and x

Then it follows that, as $t \to \infty$,

$$\frac{\mu^{\gamma} \Gamma(1+\gamma)}{t^{\gamma} L(t)} \sum_{n=1}^{\infty} a_n G_n(t) \to \alpha . \tag{1.3}$$

Notice that, in view of the easily proved fact that $H(x) = \Sigma_1^{\infty} G_n(x)$, we easily have the following.

Corollary. If conditions (T2) and (T3) of the Theorem hold, and if condition (T4) holds with $k = 0$, then

$$\frac{H(x)}{x} \to \frac{1}{\mu}, \quad \text{as } x \to \infty . \tag{1.4}$$

The Corollary is, of course, a version of the elementary renewal theorem for non-identically distributed variables. From (1.6) we can infer that for any fixed $h > 0$

$$(1.5) \qquad \frac{1}{t} \int_{t}^{t+h} H(x)\, dx \longrightarrow \frac{h}{\mu}, \quad \text{as} \quad t \longrightarrow \infty ,$$

whence

$$\frac{1}{t} \int_{0}^{t} \{H(x+h) - H(x)\}\, dx \longrightarrow \frac{h}{\mu}, \quad \text{as} \quad t \longrightarrow \infty ,$$

or, in other words,

$$(1.6) \qquad \lim_{t \longrightarrow \infty} \frac{1}{t} \int_{0}^{t} \sum_{n=1}^{\infty} P\{ x < S_n \leq x + h \}\, dx = \frac{h}{\mu} .$$

This last limit (1.6) is Kawata's result (1956). We shall explain later in what way his conditions are more restrictive than ours. It is interesting to see, however, that one can quickly recover the simpler result (1.4) from (1.6); thus (1.6) seems an unduly complicated statement. For (1.6) is equivalent to (1.5), and from (1.5), since $H(x)$ is non-decreasing,

$$\limsup_{t \longrightarrow \infty} \frac{H(t)}{t} \leq \frac{1}{\mu}$$

and

$$\liminf_{t \longrightarrow \infty} \frac{H(t+h)}{t} \geq \frac{1}{\mu} .$$

A word about the conditions of the theorem seems appropriate. Condition (T1) is merely a regularity condition on the coefficients $\{a_n\}$, and therefore is irrelevant from the point of view of the elementary renewal theorem given in the Corollary. Condition (T2) seems a natural generalization of the conditions holding for the simpler case of identically distributed variables, and indeed the elementary renewal theorem can be proved by means of the weak law of large numbers in this case. Condition (T4) is concerned with ensuring that the renewal function $H(x)$, or the more general function $\Sigma\, a_n G_n(x)$, shall be finite;

we comment at more length on this type of condition elsewhere (Smith, 1962b), but some condition like it seems to be necessary. Of course, when the $\{X_n\}$ are non-negative the need for (T4) disappears. Condition (T3), on the other hand, does seem extraneous, and we suspect that a different attack on the present problem may be able to dispose of it. Nevertheless it should be noticed that there are sequences $\{X_n\}$ satisfying (T2) and (T4) which do not satisfy (T3). Such a sequence is obtained if independent random variables $\{X_n\}$ are chosen such that

$$P\left\{X_n = 0\right\} = 1 - \frac{1}{\sqrt{n}}$$

$$P\left\{X_n = \sqrt{n}\right\} = \frac{1}{\sqrt{n}} .$$

It can be shown that this particular sequence $\{X_n\}$ is a W-sequence with average 1, and it is trivial to verify (T4).; however, (T3) does not hold for this W-sequence. We do not know whether in this special case the elementary renewal theorem actually holds in spite of the failure of (T3).

2. Some Lemmas. In what follows we denote the familiar Stieltjes convolution of two distribution functions, say $A(x)$ and $B(x)$, by $A * B(x)$. We denote $A * A(x)$ by $A^{*2}(x)$, and, generally, $A * A^{*n}(x)$ by $A^{*(n+1)}(x)$, for $n = 1, 2, 3, \ldots$

Lemma 1. Under the conditions of the Theorem there is an $\eta > 0$ such that

$$\sum_{n=1}^{\infty} n^k G_n(n\eta) < \infty . \tag{2.1}$$

Proof. Truncate the $\{X_n\}$ at A, thus:--

$$\begin{aligned} X'_n &= X_n \quad \text{if } X_n \leq A , \\ &= A \quad \text{otherwise} . \end{aligned}$$

Let $\mu_n' = \mathcal{E} X_n'$. Then (1.2) states that

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \mu_j' > 0 \quad ,$$

so we can find $\delta > 0$ and $n_o(\delta)$ such that

$$\frac{1}{n} \sum_{j=1}^{n} \mu_j' > \delta > 0 \tag{2.2}$$

for all $n > n_o(\delta)$.

Since $K(x) \geq F_n(x)$ for all n and all x it is clear that the $\{\mu_n'\}$. are bounded; let c be an upper bound for these $\{\mu_n'\}$. Let us put $Z_n = X_n' - \mu_n'$ and $L_n(x) = P\{ Z_n \leq x \}$. Then it follows that for all n and all x

$$U(x - A) \leq L_n(x) \leq K(x + c) \quad .$$

This proves that $\{Z_n\}$ is a <u>stochastically stable sequence</u>, as defined by Smith (1962a), whose Theorem 7 allows us to draw the following conclusion.

For every integer p there is a distribution function $K_p(x)$ such that

$$P\{Z_n + Z_{n+1} + \dots + Z_{n+p-1} \leq p\,x\} \leq K_p(x) \tag{2.3}$$

for all n and all x, where

$$I_p \equiv \int_{-\infty}^{0} K_p(x)\, dx \tag{2.4}$$

is finite for all p, and $I_p \to 0$ as $p \to \infty$.

Given $\epsilon < \delta$, therefore, we can find $p_o(\epsilon)$ such that $I_{p_o} < \epsilon$. If Y is a random variable with distribution function K_{p_o} then it follows from (2.4) that $\mathcal{E}\, Y > -\epsilon$. Moreover, it is clear that

$$\mathcal{E}\{|\min(0,Y)|^{k+2}\} < \infty \quad .$$

Write $M(x)$ for the supremum of $P\{Z_1 + Z_2 + \dots + Z_r \leq p_o x\}$ for $r = 1, 2, \dots, p_o - 1$. Then if Y_o is a random variable with distribution function $M(x)$ it is also clear that $\mathcal{E}\{|\min(0,Y_o)|^{k+2}\} < \infty$.

Now choose and fix $r = 0$, or 1, or 2, ..., or $p_o - 1$. It follows from what we have established so far that

$$P\{\sum_{j=1}^{np_o+r} Z_j \leq p_o x\} \leq M * K_{p_o}^{*n}(x) \quad .$$

Thus, in view of (2.2), if we suppose $n > n_o(\delta)$,

$$P\{\sum_{j=1}^{np_o+r} X_j' \leq p_o x + (n p_o + r)\delta\} \leq M * K_{p_o}^{*n}(x)$$

Therefore, on putting $x = -n\epsilon$ and observing that $X_j \geq X_j'$, we deduce that

$$(2.5) \qquad P\{\sum_{j=1}^{np_o+r} X_j \leq n p_o(\delta - \epsilon)\} \leq M * K_{p_o}^{*n}(-n\epsilon) \;.$$

Let $Y_1, Y_2, \dots$, be a sequence of independent random variables, identically distributed, with distribution function $K_{p_o}(x)$; let $Y_1, Y_2, \dots$ be independent of Y_o. Then $\mathcal{E}(Y_j + \epsilon) > 0$, for $j = 1, 2, 3, \dots$ and $\mathcal{E}\{|\min(0, Y_j + \epsilon)|^{k+2} < \infty$ for $j = 0, 1, 2, \dots$ Thus it follows from Smith (1962b) that

$$\sum_{n=1}^{\infty} n^k \; P\,\{Y_o + \sum_{j=1}^{n} (Y_j + \epsilon) \leq 0\} < \infty \quad ;$$

that is,

$$(2.6) \qquad \sum_{n=1}^{\infty} n^k \, M * K_{p_o}^{*n} \, (-n\epsilon) < \infty \quad .$$

From (2.5) and (2.6) we may conclude that

$$(2.7) \qquad \sum_{n=1}^{\infty} n^k \; G_{np_o+r} \, (n \, p_o(\delta - \epsilon)) \; < \; \infty \quad .$$

The lemma follows from (2.7) by letting $r = 0, 1, 2, \ldots, p_o - 1$ in turn, and by putting $\eta = \frac{1}{2} (\delta - \epsilon)$ so that $n \, p_o(\delta - \epsilon) > (n \, p_o + r)\eta$ for all n and all r.

<u>Lemma 2</u>. <u>Under the conditions of the Theorem</u>

$$(2.8) \qquad \int_{\mu}^{\infty} \{1 - G_n(nx)\} \, dx \longrightarrow 0, \quad \text{as} \quad x \longrightarrow \infty \quad .$$

<u>Proof.</u> Let us write $n\gamma_n = \mathcal{E} S_n$; then

$$\gamma_n = \int_{-\infty}^{+\infty} \{U(x) - G_n(nx)\} \, dx \quad .$$

Thus, by property (i) of a W-sequence, we have that $\gamma_n \longrightarrow \mu$ as $n \longrightarrow \infty$. Let us next write

$$\gamma_n = \int_{\mu}^{\infty} \{1 - G_n(nx)\} \, dx \; + \int_{0}^{\mu} \{1 - G_n(nx)\} \, dx \; - \int_{-\infty}^{0} G_n(nx) \, dx \quad ,$$

$$(2.9) \qquad = A_n + B_n - C_n \; , \; \text{say} \quad .$$

The sequence $\{X_n\}$ is a W-sequence, so that $G_n(nx) = P\{S_n \leq nx\} \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x < \mu$. Thus, by bounded convergence, $B_n \longrightarrow \mu$ as

$n \to \infty$. But $\gamma_n \to \mu$ also, and therefore (2.9) shows that we will have proved that $A_n \to 0$ if we can show $C_n \to 0$.

Let us employ the notation of Lemma 1. Then (2.3) shows that for all x and all p

$$P\{ \sum_{j=1}^{p} X_j' \leq p\,x + \sum_{j=1}^{p} \mu_j' \} \leq K_p(x) ,$$

whence

$$G_p(px) = P\{ \sum_{j=1}^{p} X_j \leq px\} \leq K_p(x) .$$

Therefore $C_p \leq I_p$, and, as we know that $I_p \to 0$ when $p \to \infty$, the lemma is proved.

<u>Lemma 3</u>. <u>Under the conditions of the theorem, as $s \to 0+$,</u>

$$\sum a_n e^{-\mu s n} \sim \frac{\alpha}{\mu^\gamma s^\gamma} L\left(\frac{1}{s}\right) .$$

<u>Proof</u>. Plainly, $e^{-\mu s} \to 1 - 0$ as $s \to 0+$. Therefore, as $s \to 0+$,

$$\sum_{n=1}^{\infty} a_n e^{-\mu s n} \sim \frac{\alpha}{(1-e^{-\mu s})^\gamma} L\left(\frac{1}{1-e^{-\mu s}}\right)$$

$$\sim \frac{\alpha}{\mu^\gamma s^\gamma} L\left(\frac{1}{1-e^{-\mu s}}\right) .$$

Karamata (1930, p. 45) has shown that $L(rx)/L(x) \to 1$ as $x \to \infty$, uniformly for r in any interval not containing 0. Thus it follows that

$$L\left(\frac{1}{1-e^{-\mu s}}\right) \sim L\left(\frac{1}{s}\right)$$

as $s \to 0+$, since $1 - e^{-\mu s} \sim \mu s$ for small s. Thus the lemma is proved.

Lemma 4. Under the conditions of the theorem, as $s \to 0+$,

$$\frac{(\mu s)^{\gamma+1}}{L(\frac{1}{s})} \sum_{n=1}^{\infty} n\, a_n\, e^{-\mu s n} \sim \gamma\, \alpha$$

Proof. Choose η, $0 < \eta < 1$. Then

$$\mu\, n\, e^{-\mu n s} < \frac{e^{-\mu s n \eta} - e^{-\mu s n}}{(1-\eta)s} .$$

Thus

$$\mu \cdot \sum_{n=1}^{\infty} n\, a_n\, e^{-\mu s n} < \frac{\sum_{n=1}^{\infty} a_n\, e^{-\mu s n \eta} - \sum_{n=1}^{\infty} a_n\, e^{-\mu s n}}{(1-\eta)s} ,$$

and so

$$\frac{\mu s^{\gamma+1}}{L(\frac{1}{s})} \sum_{n=1}^{\infty} n\, a_n\, e^{-\mu s n} < \frac{s^{\gamma}}{(1-\eta)L(\frac{1}{s})} \left\{ \sum_{n=1}^{\infty} a_n\, e^{-\mu s n \eta} - \sum_{n=1}^{\infty} a_n\, e^{-\mu s n} \right\} .$$

It follows therefore, from Lemma 3, that

$$(2.10) \quad \limsup_{s \to 0+} \frac{\mu s^{\gamma+1}}{L(\frac{1}{s})} \sum_{n=1}^{\infty} n\, a_n\, e^{-\mu s n} \leq \frac{\alpha}{\mu^{\gamma}} \left\{ \frac{\eta^{-\gamma} - 1}{1 - \eta} \right\} .$$

If we let $\eta \to 1 - 0$ in (2.10) we obtain

$$(2.11) \quad \limsup_{s \to 0+} \frac{\mu s^{\gamma+1}}{L(\frac{1}{s})} \sum_{n=1}^{\infty} n\, a_n\, e^{-\mu s n} \leq \frac{\alpha\, \gamma}{\mu^{\gamma}} .$$

Similarly, by taking $\eta > 1$ and using the fact that

$$\mu\, n\, e^{-\mu n s} > \frac{e^{-\mu s n} - e^{-\mu s n \eta}}{\eta - 1}$$

we can show

$$(2.12) \qquad \liminf_{s \to 0+} \frac{\mu s^{\gamma+1}}{L(\frac{1}{s})} \sum_{n=1}^{\infty} n\, a_n\, e^{-\mu s n} \geq \frac{\alpha\, \gamma}{\mu^{\gamma}} \quad .$$

The lemma follows from (2.11) and (2.12) .

Proof of the Theorem. We shall write β for an upper bound to the numbers a_n/n^k , and we shall suppose that $\eta > 0$ is chosen in accordance with Lemma 1; we may suppose $\eta < \mu$.

Consider, to begin with,

$$(3.1) \qquad K_n = \int_{n\eta}^{n\mu} e^{-sx}\, G_n(x)\, dx \quad ,$$

$$= n \int_{\eta}^{\mu} e^{-nsx}\, G_n(nx)\, dx \quad .$$

Evidently,

$$0 \leq K_n \leq n\, e^{-n\eta s} \int_{\eta}^{\mu} G_n(nx)\, dx \ .$$

But $G_n(nx) \to 0$ as $n \to \infty$, for all $x < \mu$, so we can appeal to bounded convergence and write

$$(3.2) \qquad K_n = n\, e^{-n\eta s}\, \delta_n' \quad ,$$

where $\delta_n' \to 0$ as $n \to \infty$, uniformly in $s \geq 0$.

Next consider

$$(3.3) \qquad L_n = \int_{n\mu}^{\infty} e^{-sx}\, \{1 - G_n(x)\}\, dx$$

$$= n \int_{\mu}^{\infty} e^{-nsx} \{1 - G_n(nx)\} \, dx \quad .$$

In view of Lemma 2 and the assumption that $\eta < \mu$ we may thus conclude that

(3.4) $$L_n = n \, e^{-n\eta s} \, \delta_n'' \quad ,$$

where $\delta_n'' \longrightarrow 0$ as $n \longrightarrow \infty$, uniformly in $s \geq 0$.

Thus, if we write $\delta_n = \delta_n' - \delta_n''$,

(3.5) $$\sum_{n=1}^{\infty} a_n \, (L_n - K_n) = \sum_{n=1}^{\infty} n \, a_n \, \delta_n \, e^{-n\eta s} \quad .$$

Given an arbitrary $\epsilon > 0$, we can find $n_o(\epsilon)$ such that $|\delta_n| < \epsilon$ for all $n > n_o$. Moreover we can assume that

$$\frac{1}{s^{\gamma}} \, L(\tfrac{1}{s}) \longrightarrow \infty \, , \text{ as } s \longrightarrow 0+ \quad ,$$

since the alternative case is when $\Sigma \, a_n$ is convergent and our theorem is then trivial. Thus

(3.6) $$\left| \sum_{n=1}^{\infty} n \, a_n \, \delta_n \, e^{-n\eta s} \right| < \sum_{n=1}^{n_o} n \, a_n \, | \delta_n | \, e^{-n\eta s} + \epsilon \sum_{n=1}^{\infty} n \, a_n \, e^{-n\eta s} - \epsilon \sum_{n=1}^{n_o} n \, a_n \, e^{-n\eta s} \quad .$$

Therefore, by Lemma 4 ,

(3.7) $$\limsup_{s \longrightarrow 0+} \frac{(\mu s)^{\gamma+1}}{L(\frac{1}{s})} \left| \sum_{n=1}^{\infty} n \, a_n \, \delta_n \, e^{-n\eta s} \right| \leq \epsilon \, \gamma \, \alpha \quad .$$

But ϵ is arbitrary, and we can therefore deduce from (3.7) and (3.5) that, as $s \to 0+$,

$$(3.8) \qquad \frac{(\mu s)^{\gamma+1}}{L(\frac{1}{s})} \sum_{n=1}^{\infty} a_n(L_n - K_n) \to 0 .$$

Now consider the function

$$(3.9) \qquad \tilde{H}(x) = \sum_{n=1}^{\infty} a_n G_n(x) U(x - n\eta) .$$

Evidently $\tilde{H}(x)$ is non-decreasing, since each term in the summation is non-decreasing. We also note that

$$(3.10) \qquad \tilde{H}(x) = \sum_{n=1}^{\infty} a_n U(x - n\mu) U(x - n\eta)$$

$$- \sum_{n=1}^{\infty} a_n \{U(x - n\mu) - G_n(x)\} U(x - n\eta) .$$

Let us denote the Laplace transform of a function $A(x)$, say, thus: --

$$A^{o}(s) = \int_{0}^{\infty} e^{-sx} A(x)\, dx .$$

Then, from (3.10), we have

$$(3.11) \qquad \tilde{H}^{o}(s) = \frac{1}{s} \sum_{n=1}^{\infty} a_n e^{-n\mu s} + \sum_{n=1}^{\infty} a_n(L_n - K_n) ;$$

the term-by-term integration being justified by monotone convergence.

From (3.11), (3.8), and Lemma 3, it appears that

$$(3.12) \qquad \frac{(\mu s)^{\gamma+1}}{L(\frac{1}{s})} \tilde{H}^{o}(s) \to \alpha \mu , \text{ as } s \to 0+ .$$

An appeal to Doetsch (1950, p. 511) then allows the inference

(3.13) $$\frac{\mu^\gamma \Gamma(1+\gamma)}{t^\gamma L(t)} \tilde{H}(t) \to \alpha, \qquad \text{as } t \to \infty .$$

But, by (3.9) ,

(3.14) $$\sum_{n=1}^{\infty} a_n G_n(x) = \tilde{H}(x) + \psi(x), \text{ say} ,$$

where

(3.15) $$\psi(x) = \sum_{n=1}^{\infty} a_n G_n(x) \left\{ 1 - U(x - n\eta) \right\}$$

(3.16) $$\leq \beta \sum_{n=1}^{\infty} n^k G_n(n\eta) .$$

We have already explained that we are assuming the divergence of Σa_n , to avoid triviality. Thus, by (1.1), we may suppose $t^\gamma L(t)$ increases without bound as $t \to \infty$. Hence, by (3.16) and Lemma 1,

(3.17) $$\lim_{t \to \infty} \frac{\psi(t)}{t^\gamma L(t)} = 0 .$$

The theorem follows from (3.13), (3.14), and (3.17) .

. Some comments on the theorem of Kawata. We have already explained at the end of §1 that the conclusion of Kawata's theorem (Kawata, 1956) is equivalent to the simpler conclusion of the corollary to our theorem. It is of interest to see that Kawata's conditions are actually rather more restrictive than those needed by this corollary. In our notation, Kawata's conditions are as follows:

(K1) There is an $s_o > 0$ such that, for all $0 \leq s \leq s_o$,

$$\int_{-\infty}^{0} e^{-sx} \, d F_n(x) < \infty .$$

(K2) Uniformly in n,

$$\lim_{A \to \infty} \int_A^\infty x \, d\, F_n(x) = 0 \ .$$

(K3) Uniformly in both n and s, $0 \leq s \leq s_o$,

$$\lim_{A \to \infty} \int_{-\infty}^{-A} e^{-sx} \, d\, F_n(x) = 0 \ .$$

Kawata also requires, like us, that all the expectations $\{ \mathcal{E} X_n \}$ shall exist and that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \mathcal{E} X_r = \mu > 0 \ . \tag{4.1}$$

We first remark that (K3) implies the existence of a function $\lambda(-A)$ such that $\lambda(-A) \to 0$ as $A \to \infty$, and $F_n(-A) \leq \lambda(-A) \exp \{-s_o A\}$ for all n. Thus our condition (T4) is satisfied by a distribution function $K(x)$ which decreases exponentially rapidly as $x \to -\infty$. A consequence of this is that for any small $\epsilon > 0$ we can find a large $M_o(\epsilon)$ such that

$$\int_{-\infty}^{-M_o} |x| \, d\, F_n(x) < \epsilon \ , \tag{4.2}$$

uniformly in n.

From (4.1) we infer the existence of $n_o(\epsilon)$ such that for all $n > n_o(\epsilon)$

$$\frac{1}{n} \sum_{r=1}^{n} \mathcal{E} X_r > \mu - \epsilon \ . \tag{4.3}$$

Also, from (K2) , we can find $N_o(\epsilon)$ such that

$$(4.4) \qquad \int_{N_0}^{\infty} x \, d\, F_n(x) < \epsilon ,$$

uniformly in n. Thus we can infer from (4.3) and (4.4) that for all $n > n_0(\epsilon)$

$$(4.5) \qquad \frac{1}{n} \sum_{r=1}^{n} \int_{-\infty}^{N_0} x \, d\, F_n(x) > \mu - 2\epsilon .$$

It follows easily from (4.5) that condition (T3) is satisfied.

It remains to be proved that Kawata's sequence $\{X_n\}$ is a W-sequence. In view of (4.1), therefore, all we have to prove is that $S_n/n \to \mu$ in probability as $n \to \infty$. To do this we first observe that, by (4.2) and (4.4),

$$|\mathcal{E} X_n| < 2\epsilon + \max(M_0, N_0) .$$

Thus $\{|\mathcal{E} X_n|\}$ is a bounded sequence; we shall write C for any upper bound. We also remark that the uniformity of (4.2) and (4.4) imply that

$$\int_{|x|>n} d\, F_m(x) = o(\tfrac{1}{n})$$

uniformly in m. Thus

$$\sum_{m=1}^{n} \int_{|x|>n} d\, F_m(x + \mathcal{E} X_m) = n\, o(\frac{1}{n - C})$$

and it follows that

$$(4.6) \qquad \sum_{m=1}^{n} \int_{|x|>n} d\, F_m(x + \mathcal{E} X_m) \to 0, \text{ as } n \to \infty .$$

Furthermore, for $n > C$,

$$\frac{1}{n} \sum_{m=1}^{n} \int_{|x|>n} |x| \, d\, F_m(x + \mathcal{E} X_m)$$

$$\leq \frac{1}{n} \sum_{m=1}^{n} \int_{|x|>n-C} |x| d\, F_m(x) + \frac{C}{n} \int_{|x|>n-C} d\, F_m(x) \quad .$$

Thus, by (4.2), (4.4), and (4.6), we can deduce that for all sufficiently large n

$$\frac{1}{n} \sum_{m=1}^{n} \int_{|x|>n} |x| \, d\, F_m(x + \mathcal{E} X_m) < 3\epsilon \quad ;$$

therefore we may conclude that

$$(4.7) \qquad \frac{1}{n} \sum_{m=1}^{n} \int_{|x|<n} x \, d\, F_m(x + \mathcal{E} X_m) \longrightarrow 0, \text{ as } n \longrightarrow \infty \; .$$

Lastly we note that, for $n > 2C$,

$$\frac{1}{n^2} \sum_{m=1}^{n} \int_{|x|<n} x^2 \, d\, F_m(x + \mathcal{E} X_m)$$

$$< \frac{1}{n^2} \sum_{m=1}^{n} \int_{|x|<2C} + \frac{1}{n^2} \sum_{m=1}^{n} \int_{2C\leq|x|<n}$$

$$< \frac{4C^2}{n} + \frac{1}{n} \sum_{m=1}^{n} \int_{|x|>2C} |x| d\, F_m(x + \mathcal{E} X_m)$$

$$< \frac{4C^2}{n} + \frac{2}{n} \sum_{m=1}^{n} \int_{|y|>C} |y| \, d\, F_m(y)$$

$$< \frac{4C^2}{n} + 2\epsilon \text{ , by (4.2) and (4.4) .}$$

Thus

$$(4.8) \qquad \frac{1}{n^2} \sum_{m=1}^{n} \int_{|x|<n} x^2 \, d\, F_m(x + \mathcal{E} X_m) \longrightarrow 0, \text{ as } n \longrightarrow \infty \text{ .}$$

From (4.6), (4.7), and (4.8) it follows that Kawata's sequence $\{X_n\}$ satisfies the conditions of the classical weak law of large numbers, and our demonstration is complete.

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