A SAMPLING STUDY OF ESTIMATORS OF THE NONLINEAR PARAMETER IN THE EXPONENTIAL MODEL

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INTRODUCTION

In recent years, nonlinear equations have been receiving close study. The estimation of the parameters α, β and ρ in the useful equation

$$y = \alpha - \beta\rho^x$$

has received much attention.

Stevens (1951) presented an iterative least squares solution for the estimation of the parameters. Patterson (1956, 1958, 1959) developed a class of linear and quadratic estimators, r, for ρ. He then estimated α and β from the linear regression of y on r^x. This class of estimators includes the ones derived by Hartley (1948) and Monroe (1949).

Approximate biases and variances for the class of quadratic estimators of r were presented by Patterson (1958, 1959). Portman (1961) derived different approximate bias and variance formulae for the Monroe estimator.

The purpose of this study is to determine empirically the small sample properties of these estimators of ρ and to contrast them with the approximate or asymptotic properties which have been derived.

METHODS OF ESTIMATING THE PARAMETERS

Least Squares Estimator

An observation, y, which comes from the single exponential non-linear equation, can be represented by

$$y = \alpha - \beta\rho^{x} + \epsilon \, . \tag{1}$$

The random errors, ϵ's, are assumed to be uncorrelated and to have a variance of σ^2. The estimators of the parameters, α, β, and ρ are a, b, and r respectively. The equation then becomes

$$y = a - br^{x} + e$$

where e is the residual.

In least squares we minimize N, the sum of the squares of these residuals. That is,

$$N = N(a, b, r) = \sum_{1}^{n} e^2 = \Sigma\,(y - a - br^{x})^2 \, .$$

The least squares equations are

$$N_a = \frac{\partial N}{\partial a} = - an - b\,\Sigma\, r^{x} + \Sigma\, y = 0$$

$$N_b = \frac{\partial N}{\partial b} = - a\,\Sigma r^{x} - b\,\Sigma\, r^{2x} + \Sigma\, r^{x} y = 0$$

$$N_r = \frac{\partial N}{\partial r} = - ab\,\Sigma\, xr^{x-1} - b^2\,\Sigma\, xr^{2x-1} + b\,\Sigma\, xr^{x-1}\, y = 0.$$

It is impossible to obtain explicit solutions for a, b, and r. Several schemes have been devised to iterate or otherwise obtain solutions.

Stevens (1951) used a method which he attributes to Fisher. By taking the total derivatives of the partials we obtain:

$$dN_a = \frac{\partial N_a}{\partial a}\Delta a + \frac{\partial N_a}{\partial b}\Delta b + \frac{\partial N_a}{\partial r}\Delta r = 0 = N_a$$

$$dN_b = \frac{\partial N_b}{\partial a}\Delta a + \frac{\partial N_b}{\partial b}\Delta b + \frac{\partial N_b}{\partial r}\Delta r = 0 = N_b$$

$$dN_r = \frac{\partial N_r}{\partial a}\Delta a + \frac{\partial N_r}{\partial b}\Delta b + \frac{\partial N_r}{\partial r}\Delta r = 0 = N_r$$

where $\frac{\partial N_a}{\partial a} = \frac{\partial^2 N}{\partial a^2}$, $\frac{\partial N_a}{\partial b} = \frac{\partial^2 N}{\partial a \partial b}$, etc.

Let
$$M = \begin{bmatrix} \frac{\partial^2 N}{\partial a^2} & \frac{\partial^2 N}{\partial a \partial b} & \frac{\partial^2 N}{\partial a \partial r} \\ \frac{\partial^2 N}{\partial a \partial b} & \frac{\partial^2 N}{\partial b^2} & \frac{\partial^2 N}{\partial b \partial r} \\ \frac{\partial^2 N}{\partial a \partial r} & \frac{\partial^2 N}{\partial b \partial r} & \frac{\partial^2 N}{\partial r^2} \end{bmatrix} .$$

Then
$$M \begin{bmatrix} \Delta a \\ \Delta b \\ \Delta r \end{bmatrix} = \begin{bmatrix} N_a \\ N_b \\ N_r \end{bmatrix} .$$

For given values of a, b, and r, the three equations become functions of Δa, Δb and Δr for which we can solve. That is, if M^{-1} exists, then:

$$\begin{bmatrix} \Delta a \\ \Delta b \\ \Delta r \end{bmatrix} = M^{-1} \begin{bmatrix} N_a \\ N_b \\ N_r \end{bmatrix} .$$

Stevens (1951) has shown that M can be altered so that it is only a function of r. Then, if we start with a value of r, say r_1, we can find a Δr_1 so that $r_2 = r_1 + \Delta r_1$. With r_2 we find an a_2 and b_2 so that $N(a_2, b_2, r_2)$ will be smaller than $N(a_1, b_1, r_1)$. This iteration process is then repeated until Δr is as small as desired.

The negative of the expected value of M is the information matrix, I. The inverse of I is the asymptotic variance covariance matrix, as is well known. Apart from a constant term, b, in some of the elements, M can be written as

$$M = \begin{bmatrix} n & \sum_{i=0}^{n-1} r^i & \sum_{i=0}^{n-1} ir^{i-1} \\ & \sum_{i=0}^{n-1} r^{2i} & \sum_{i=0}^{n-1} ir^{2i-1} \\ & & \sum_{i=0}^{n-1} i^2 r^{2i-2} \end{bmatrix}$$

which is symmetric. It is easy to see that the matrix is singular when r=0 and r=1. It follows that the matrix is ill-conditioned when r is near either of the two values. Stevens' method, then, will not necessarily iterate to a stable value at the extremes of ρ.

The least squares method of obtaining r requires a good deal of calculation. If there are less than thirteen equally spaced observations, then tables are available to aid in the calculations. This method also has the unfortunate property of using an ill-conditioned matrix for large or small values of ρ. To avoid this difficulty one can fit a quadratic polynomial as recommended by Stevens (1951).

Quadratic Estimators

Patterson (1956, 1958, 1959) presented some estimators of ρ which are simpler to calculate than the least squares estimator and are reasonably efficient and unbiassed. Estimates of α and β are then obtained by linear regression of y on r^x. The method requires that the intervals between the x's be equal. This is not required in the least squares procedure. However, Patterson (1956) showed that the least squares procedure presented by Stevens (1951) is a complicated quadratic method.

The estimators presented by Patterson are ratios of either linear or quadratic functions of the observations. The quadratic estimators are of most interest. They are of the form

$$r(\rho_0, k/\ell) = \frac{\underline{y}_1' D(k\underline{y}_0 + \ell\underline{y}_1)}{\underline{y}_0' D(k\underline{y}_0 + \ell\underline{y}_1)} \qquad (2)$$

where $\underline{y}_1' = (y_1, \ldots, y_{n-1}) \qquad \underline{y}_0' = (y_0, \ldots, y_{n-2})$.

ρ_0, k and ℓ are arbitrary constants and D is an n-1 by n-1 matrix whose elements we can choose.

The elements of D are chosen to minimize an approximate variance which is given by Patterson (1958). This variance is derived using the method of Finney (1958). The D matrix turns out to be a function of ρ, the parameter we wish to estimate, and so we must first specify that $\rho = \rho_0$ in order to have a determined matrix to calculate the estimator

$r(\rho_0, k/\ell)$. Thus the procedure of estimating ρ becomes most efficient when $\rho = \rho_0$.

Patterson (1959) asserts that Hartley (1948) derived an estimator of ρ which is the same as r (1, 1). Patterson believes that Monroe (1949) developed the estimator equivalent to r (1, 0). This assertion is shown to be correct in Appendix A. Patterson also suggests that r (1, 1.5) might be the best of the three estimators. In these three cases $\rho_0 = 1$.

When $\rho_0 = 1$, Patterson (1958) has shown that we can form the matrix D by first forming c_{ij} where

$$c_{ij} = \frac{i(n-j)}{n} \qquad i \leq j \quad .$$

Then

$$d_{ij} = c_{ij} - \frac{\sum_i c_{ij} \sum_j c_{ij}}{\sum_{i,j} c_{ij}}$$

or

$$d_{ij} = \begin{cases} \dfrac{i(n-j)}{n} - \dfrac{3ij(n-i)(n-j)}{n(n^2-1)} & i \leq j \\ \\ \dfrac{j(n-i)}{n} - \dfrac{3ij(n-i)(n-j)}{n(n^2-1)} & i > j \end{cases} \quad . \tag{3}$$

The three estimators are then

$$\begin{aligned} r\ (1,\ 1.5) &= \frac{3\underline{y}_1' D\underline{y}_0 + 2\underline{y}_1' D\underline{y}_1}{3\underline{y}_0' D\underline{y}_0 + 2\underline{y}_1' D\underline{y}_0} \\ r\ (1,\ 1) &= \frac{\underline{y}_1' D\underline{y}_0 + \underline{y}_1' D\underline{y}_1}{\underline{y}_0' D\underline{y}_0 + \underline{y}_1' D\underline{y}_0} \\ r\ (1,\ 0) &= \frac{\underline{y}_1' D\underline{y}_1}{\underline{y}_1' D\underline{y}_0} \quad . \end{aligned} \tag{4}$$

It should be noted that the last estimator, r (1, 0), does not contain the observation y_0 in the numerator. The first observations are undoubtedly the most important elements in determining the slope of the curve, particularly when ρ is small. For this reason, we might expect the r (1, 0) estimator to be less stable than the other two quadratic estimators.

BIAS AND VARIANCE OF THE ESTIMATORS

The least squares procedure gives an asymptotic variance of r which is obtained from I, the information matrix. For known values of ρ, the asymptotic variances of r have been calculated and are given in Table 7.

Patterson (1958) presented formulae for the approximate bias and variance of r. These formulae were developed using the method of Finney (1958).

Let A and B be functions of y and consistent estimators of two functions of ρ whose ratio is ρ. Then

$$r = \frac{A}{B}$$

where

$$E(A) = \xi \qquad E(B) = \eta \qquad \text{and} \qquad \rho = E(r) = \frac{\xi_0}{\eta_0} .$$

Now $$r = \frac{A}{B} = \xi_0 \left(\frac{A}{\xi_0}\right)\left(\frac{1}{\eta_0}\right)\left(\frac{B}{\eta_0}\right)^{-1} = \rho\left(1 + \frac{A - \xi_0}{\xi_0}\right)\left(1 + \frac{B - \eta_0}{\eta_0}\right)^{-1}$$

and by use of the expansion of $\left(1 + \frac{B - \eta_0}{\eta_0}\right)^{-1}$ we get

$$r = \rho + \frac{1}{\eta_0}\left[(A-\xi_0) - \rho(B-\eta_0)\right] - \frac{1}{\eta_0^2}\left[(A-\xi_0)(B-\eta_0) - \rho(B-\eta_0)^2\right] + \ldots .$$

Patterson (1958) used parts of the first three terms to obtain the bias of r and of the first two terms to obtain the variance of r. The parts used were the constants and terms involving σ^2. Terms of higher order in σ were ignored. Patterson showed that the approximations for the bias and variance of the three estimators are given by

$$\text{bias } r = \frac{\sigma^2}{\beta^2}\left[\frac{(\ell-\rho k)\text{ trace } D + k \text{ trace } DU - \ell\rho \text{ trace } DU' + (\ell\rho^2 - 2k\rho - \ell)F_1 - 2kF_2}{(k+\ell\rho)\, F_0}\right]$$

$$\text{var } r = \frac{\sigma^2}{\beta^2}\left[\frac{(1+\rho^2)\, F_1 - 2\rho F_2}{F_0^2}\right]$$

where D is as given in (3)

ℓ and k refer to the specific quadratic estimate

$F_0 = R'DR \qquad F_1 = R'DDR \qquad F_2 = R'D'UDR$

$$U = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 & 0 \\ 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & 0 & . & . & . & 0 & 0 \\ . & . & . & & & & . & . \\ . & . & . & & & & . & . \\ . & . & . & & & & . & . \\ . & . & . & & & & . & . \\ 0 & 0 & 0 & . & . & . & 1 & 0 \end{bmatrix} \qquad R' = (1, \rho, \rho^2, \ldots, \rho^{n-2})$$

and trace $D = \frac{n^2-4}{15}$ $\qquad$ trace $DU = \frac{2n^2-15n+22}{30}$.

The approximate variance of r is a function of ρ, β and the matrix D. It is by minimizing this variance that Patterson specifies the elements of D.

The calculations of the approximate bias and variance require formulae for F_0, F_1 and F_2. These are given in Appendix B for n=4 to n=10.

The coefficients of the bias and the variance are given in Tables 3 and 4. Multiplying the entries in the bias and variance tables by σ^2/β^2 gives the bias and the variance approximations.

Portman (1961) derived different approximations to the bias and variance for the estimator r (1, 0). She wrote

$$r = \frac{\underline{y}'(T'\rho T)\underline{y}}{\underline{y}'T'\rho T\underline{y} - \underline{y}'T'\rho\underline{y}} = \frac{A}{A-C} .$$

If we let A-C = B we could then expand the ratio of A to B in the same manner that Patterson and Finney did. However, Portman expanded the estimator in a Taylor's series and obtained

$$r = r_0 + r_A(A-A_0) = r_C(C-C_0) + \frac{r_{CC}(C-C_0)^2}{2!} + \frac{r_{AA}(A-A_0)^3}{2!}$$

$$+ r_{AC}\frac{(A-A_0)(C-C_0)}{2!} + \ldots .$$

The partial derivatives of r with respect to A and C evaluated at A_0 and C_0 are:

$$r_0 = \frac{A_0}{A_0-C_0} \qquad r_C = \frac{1}{(A_0-C_0)^2} \qquad r_A = \frac{-C_0}{A_0-C_0}$$

$$r_{CC} = \frac{2}{(A_0-C_0)^3} \qquad r_{AA} = \frac{-2}{(A_0-C_0)^3} \qquad r_{AC} = \frac{-2}{(A_0-C_0)^3} .$$

If $A_0 = E(A)$ and $C_0 = E(C)$, then

$$E(A-A_0) = E(C-C_0) = 0$$

$$E(A-A_0)^2 = \text{Var } A$$

$$E(C-C_0)^2 = \text{Var } C$$

$$E(A-A_0)(C-C_0) = \text{cov }(A, C) .$$

Using terms through the quadratic in the Taylor's expansion we find that

$$E(\hat{r}) = \frac{A_0}{(A_0-C_0)} [1 + \frac{C_0 \text{ Var } A}{A_0(A_0-C_0)} - \frac{(A_0+C_0) \text{ cov}(A,C)}{A_0(A_0-C_0)^2} + \frac{\text{Var } C}{(A_0-C_0)^2}] .$$

This gives the bias as:

$$\text{bias } (r) = E(\hat{r}) - \rho .$$

By a similar procedure, using only the linear terms, the approximation to the variance was found to be:

$$\text{Var}(r) = \frac{1}{(A_0-C_0)^4} [C_0^2 \text{ Var } A + A_0^2 \text{ Var } C - 2 A_0C_0 \text{ cov } (A,C)] .$$

Tables of the bias and variance for certain values of α, β and ρ were calculated by Portman (1961). They are reproduced as Tables 5 and 6.

It is worth noting that the two approximations to the bias and variance of r given by Portman and Patterson are structurally quite different. Patterson used the first terms of the series $(1 + \frac{\beta-\eta_0}{\eta_0})^{-1}$ and Portman used the first terms of the Taylor series expansion of A/A-C. Also, Portman used the expansion about E(A) and E(C) whereas Patterson expanded about ξ_0 and η_0 where $\rho = \xi_0/\eta_0$. Thus, although the ideas of the approximation are similar, the practical methods of obtaining these approximations are quite different.

THE SAMPLING STUDY

An empirical study of the four estimating procedures was carried out to see which might be the better of the four methods of estimating ρ in the equation

$$y = \alpha - \beta\rho^{x} + \epsilon \qquad x = 0,1,\ldots,n-1 \quad . \tag{5}$$

The four methods are:

(a) Stevens' iterative least squares estimate

and Patterson's quadratic estimates

(b) r (1, 1.5)

(c) r (1, 0)

(d) r (1, 1)

where the quadratic estimates are given in (4). ρ was taken to be .1, .3, .5, .7, and .9 with values of β of 5, 10 or 50 and n was taken as 4, 6, 8 or 10. The value of α is immaterial to all estimating procedures but it was generally taken to be 100. The ϵ's are uncorrelated and have a variance of one. Not all combinations of the β, ρ and n were taken. However, a wide enough selection of the combinations was used so that some conclusions could be drawn.

For a given set of α, β and ρ, a set of ϵ's was added to form the equation (5). These y's were then used to calculate a value of r for each of the procedures. For this given α, β and ρ, 100 sets of ϵ's were used to give us 100 values of r. The particular values of the parameters α, β and ρ were chosen to conform with the work Portman (1961) did on approximate bias and variance for the r (1, 0) procedure.

The sampling computations were done on an IBM 650. As was stated earlier, Stevens' iterative procedure for estimating r might not converge due to the properties of the matrix being inverted. The program stopped iterating after calculating nine new r's from the original one. The criterion for saying that a value of r was the correct solution to the normal equations was that Δr be less than or equal to 0.0005. For some samples, particularly when $\rho = .9$, Stevens' estimate was too unstable to calculate. Stevens' procedure was started using the r (1, 1.5) estimator.

The mean of the sample of the r's and the bias are shown in Table 1 along with the approximations to the bias. In Table 2 the sample variance and the error mean square of the sample of r's are given along with approximations to the variance and the efficiency relative to the least squares procedure.

For those samples where the least squares estimator was calculated the residual variance was also computed. This was formed by taking the sum of the squares of the observations subtracted from the predicted value and dividing this sum by n-3, where n is the number of observations. The means of these for each sample is shown in Table 8.

RESULTS AND DISCUSSION

Table 1 gives the information on the bias of the estimates. Where samples were drawn for each estimator, the least squares one has in general a smaller bias than the quadratic procedures but only slightly better than the r (1, 1.5) estimator. However, the least squares estimate is unstable for some points as shown by Table 1 and Table 8. The bias in the r (1, 0) estimator is much larger than any of the others.

The approximate bias derived by either Patterson or Portman does not appear to be too accurate.

The variance and the error mean square of r are given in Table 2. Here we see that the quadratic estimator r (1, 1.5) has a smaller variance than the other estimators. Again, the estimator r (1, 0) shows up as being unstable. This is partially due to wild values of r, one of which went as high as 359. The efficiencies of the quadratic estimators as compared to Stevens clearly contrasts the procedures.

The approximate variances are quite accurate. The agreement between the two approximations is surprising, particularly for larger β's. There appears to be no significant difference between Patterson's approximate variance and Stevens' asymptotic variance. It is the r (1, 0) estimator that is not approximated well. The wild values that come out of the r (1, 0) estimator make it hard to predict. If Portman's (1961) procedure of a Taylor's series expansion on r (1, 1.5) were used to obtain approximate variances, it could more fairly be contrasted with Patterson's procedure.

The coefficients of the approximate variance as derived by Patterson are strikingly similar to the asymptotic variance coefficients as derived by Stevens. See Tables 3 and 7.

Table 8 shows some of the difficulties given by Stevens' estimator and r (1, 0).

RECOMMENDATIONS

The sampling study suggests that the quadratic r (1, 1.5) estimator is the better of the four estimators of the parameter ρ in

$$y = \alpha - \beta\rho^{x} .$$

Its mean sample bias is lower than all but the least squares estimator. Its sample variance is quite a bit better than the least squares sample variance.

However, although the r (1, 1.5) estimator appears to be the best estimator of ρ, this does not say that the prediction equation

$$y = a - b\,[r\,(1, 1.5)]^{x}$$

is the best equation that can be fitted. It is certain that the least squares equation will give a better fit in the sense of minimizing the error sum of squares. This is in the process of being studied further.

Other least squares procedures might give interesting results. The procedure proposed by Stevens (1951) converges quickly for large β and $\rho < .9$. However at $\rho = .9$ and/or small β convergence is not certain. Possibly the procedure advocated by Pimental Gomes (1953) would not have these problems.

Besides estimating ρ, it is also necessary to estimate α, β and the residual variance. It would be interesting to see how the estimates for the quadratic procedure compare with the least squares method for these values.

Table 1. Sampling results, bias

	r(1,1.5)	r(1,0)	r(1,1)	L.S.[a]	r(1,1.5)	r(1,0)	r(1,1)	L.S.
	n=4 β=5 ρ=.1				n=4 β=5 ρ=.5			
$\bar{r}$	.2154	.2596	.2341	.2068	.6173	.7088	.6833	.6132
bias	.1154	.1596	.1341	.1068	.1175	.2088	.1833	.1132
P.bias[b]	.0138	.5016	.0276	-	.0535	.2940	.0803	-
M.bias[c]	-	≈ 37	-	-	-	1.2724	-	-
	n=4 β=10 ρ=.1				n=4 β=10 ρ=.3			
$\bar{r}$	.0866	.2009	.0897	.0870	.2824	.3741	.2872	.2828
bias	-.0134	.1009	-.0103	-.0130	-.0176	.0741	-.0128	-.0172
P.bias	.0032	.1254	.0069	-	.0058	.0632	.0102	-
M.bias	-	.7722	-	-	-	.1049	-	-
	n=4 β=50 ρ=.5				n=4 β=50 ρ=.7			
$\bar{r}$	.5059	.5077	.5061	.5061	.6970	.7013	.6976	.6975
bias	.0059	.0077	.0061	.0061	-.0030	.0013	-.0024	-.0025
P.bias	.0005	.0029	.0008	-	.0017	.0060	.0023	-
M.bias	-	.0030	-	-	-	.0061	-	-
	n=4 β=50 ρ=.9				n=6 β=5 ρ=.3			
$\bar{r}$	.9236	.9482	.9275	M.S.[d]	.3295	.1301	.3575	.3224
bias	.0236	.0482	.0275	M.S.	.0295	-.1699	.0575	.0224
P.bias	.0158	.0436	.0202	-	.0091	.3304	.0338	-
M.bias	-	.0493	-	-	-	.8119	-	-
	n=6 β=5 ρ=.7				n=6 β=10 ρ=.7			
$\bar{r}$	.7232	1.2115	.8614	.7300	.6707	.8036	.6811	.6744
bias	.0232	.5115	.1614	.0300	-.0293	.1036	-.0189	-.0256
P.bias	.0078	.2658	.0433	-	.0020	.0665	.0108	-
M.bias	-	.4419	-	-	-	.0774	-	-
	n=6 β=50 ρ=.9				n=8 β=5 ρ=.5			
$\bar{r}$	.9022	.9146	.9041	.9036	.5107	1.4322	.5378	.4896
bias	.0022	.0146	.0041	.0036	.0107	.9322	.0378	-.0104
P.bias	.0009	.0118	.0026	-	.0039	.2067	.0265	-
M.bias	-	.0120	-	-	-	.2544	-	-

[a] L.S. is the least squares estimator.

[b] P.bias is the approximate bias derived by Patterson, the coefficient of which comes from Table 4.

[c] M.bias is the approximate bias derived by Portman for r(1,0), the value of which comes from Table 5.

[d] M.S. is a missing sample.

Table 1 (continued)

	r(1,1.5)	r(1,0)	r(1,1)	L.S.	r(1,1.5)	r(1,0)	r(1,1)	L.S.
	n=8 β=5 ρ=.9				n=8 β=10 ρ=.1			
$\bar{r}$	.9105	4.7744	1.0029	M.S.	.1200	.4252	.1317	.0935
bias	.0105	3.8744	.1029	M.S.	.0200	.3252	.0317	-.0065
P.bias	.0067	.4869	.0712	-	.0218	.4392	.0345	-
M.bias	-	.7814	-	-	-	1.1292	-	-
	n=8 β=10 ρ=.9				n=8 β=50 ρ=.7			
$\bar{r}$	.8849	1.0064	.9024	M.S.	.7003	.7021	.7005	.7002
bias	-.0151	.1064	.0024	M.S.	.0003	.0021	.0005	.0002
P.bias	.0017	.1217	.0178	-	.0000	.0017	.0002	-
M.bias	-	.1464	-	-	-	.0017	-	-
	n=8 β=50 ρ=.9				n=10 β=5 ρ=.7			
$\bar{r}$	.9041	.9089	.9049	M.S.	.6824	1.0525	.7053	.6683
bias	.0041	.0089	.0049	M.S.	-.0176	.3525	.0053	-.0317
P.bias	.0001	.0049	.0007	-	-.0026	.1274	.0152	-
M.bias	-	.0049	-	-	-	-	-	-
	n=10 β=10 ρ=.5				n=10 β=10 ρ=.9			
$\bar{r}$	.5052	.5636	.5110	.5150	.8891	.9596	.8997	M.S.
bias	.0052	.0636	.0110	.0150	-.0109	.0596	-.0003	M.S.
P.bias	.0029	.0536	.0853	-	-.0033	.0635	.0073	-
M.bias	-	-	-	-	-	-	-	-
	n=10 β=50 ρ=.3							
$\bar{r}$	.3031	.3080	.3035	.3024				
bias	.0031	.0080	.0035	.0024				
P.bias	.0006	.0055	.0016	-				
M.bias	-	-	-	-				

Table 2. Sampling results, variance

	r(1,1.5)	r(1,0)	r(1,1)	L.S.	r(1,1.5)	r(1,0)	r(1,1)	L.S.
	n=4 β=5 ρ=.1				n=4 β=5 ρ=.5			
Var.a/	.0934	2.8871	.0947	.1056	.5459	2.0254	.7199	.5804
E.M.S.b/	.1058	2.884	.1117	.1159	.5542	2.0487	.7464	.5874
P.S., M.Var.c/	.0700	48.7576	-	.0689	.1736	.7288	-	.1736
Eff.d/	113%	5%	112%	100%	106%	29%	81%	100%
	n=4 β=10 ρ=.1				n=4 β=10 ρ=.3			
Var.	.0190	1.5175	.0191	.0199	.0228	.0963	.0228	.0238
E.M.S.	.0190	1.5125	.0190	.0198	.0229	.1008	.0227	.0239
P.S., M.Var.	.0175	.1684	-	.0172	.0246	.0238	-	.0246
Eff.	105%	1%	104%	100%	104%	25%	105%	100%
	n=4 β=50 ρ=.5				n=4 β=50 ρ=.7			
Var.	.0021	.0021	.0021	.0021	.0040	.0039	.0040	.0040
E.M.S.	.0021	.0021	.0021	.0021	.0039	.0039	.0039	.0039
P.S., M.Var.	.0017	.0017	-	.0017	.0046	.0046	-	.0046
Eff.	100%	100%	100%	100%	100%	100%	100%	100%
	n=4 β=50 ρ=.9				n=6 β=5 ρ=.3			
Var.	.0466	.0490	.0470	M.S.	.0625	55.1956	.0654	.0722
E.M.S.	.0467	.0508	.0473	M.S.	.0627	56.3475	.0681	.0720
P.S., M.Var.	.0401	.0433	-	.0401	.0553	.2491	-	.0534
Eff.	-	-	-	-	115%	.1%	110%	100%
	n=6 β=5 ρ=.7				n=6 β=10 ρ=.7			
Var.	.1529	11.3344	.9465	.1688	.0278	.2442	.0274	.0294
E.M.S.	.1519	11.4827	.9631	.1680	.0284	.2525	.0275	.0297
P.S., M.Var.	.0945	.1646	-	.0944	.0236	.0251	-	.0236
Eff.	110%	1%	18%	100%	106%	12%	107%	100%
	n=6 β=50 ρ=.9				n=8 β=5 ρ=.5			
Var.	.0054	.0055	.0054	.0055	.0391	22.104	.0380	.0550
E.M.S.	.0054	.0057	.0054	.0055	.0388	22.752	.0391	.0546
P.S., M.Var.	.0053	.0053	-	.0053	.0375	.0453	-	.0367
Eff.	102%	100%	101%	100%	141%	.2%	145%	100%

a/ Var $= \frac{\Sigma(r-\bar{r})^2}{99}$

b/ E.M.S. $= \frac{\Sigma(r-\rho)^2}{100}$

c/ (1) Under r(1,1.5) column is Patterson's approximation to the variance for r(1,1.5), r(1,0) and r(1,1).
(2) Under the r(1,0) column is Portman's approximation to the variance for r(1,0).
(3) Under the L.S. column is the asymptotic variance given by Stevens.

d/ Eff. is the ratio of the least squares variance/quadratic variance.

Table 2 (continued)

	r(1,1.5)	r(1,0)	r(1,1)	L.S.	r(1,1.5)	r(1,0)	r(1,1)	L.S.
	n=8 β=5 ρ=.9				n=8 β=10 ρ=.1			
Var.	.1163	1,246.5	.1446	M.S.	.0138	4.2046	.0138	.0101
E.M.S.	.1153	1,249.1	.1537	M.S.	.0140	4.2683	.0146	.0101
P.S. M.Var.	.1440	.3188	-	.1440	.0155	.3620	-	.0118
Eff.	-	-	-	-	73%	.2%	73%	100%
	n=8 β=10 ρ=.9				n=8 β=50 ρ=.7			
Var.	.0345	.0445	.0357	M.S.	.0004	.0004	.0004	.0004
E.M.S.	.0344	.0554	.0354	M.S.	.0004	.0004	.0004	.0004
P.S. M.Var.	.0360	.0406	-	.0360	.0004	.0004	-	.0004
Eff.	-	-	-	-	100%	101%	100%	100%
	n=8 β=50 ρ=.9				n=10 β=5 ρ=.7			
Var.	.0013	.0013	.0013	M.S.	.0210	3.5296	.0224	.0303
E.M.S.	.0013	.0014	.0013	M.S.	.0220	3.6186	.0222	.0310
P.S. M.Var.	.0014	.0010	-	.0014	.0231	-	-	.0230
Eff.	-	-	-	-	138%	1%	136%	100%
	n=10 β=10 ρ=.5				n=10 β=10 ρ=.9			
Var.	.0087	.0062	.0084	.0098	.0162	.0189	.0166	M.S.
E.M.S.	.0086	.0102	.0085	.0097	.0162	.0222	.0164	M.S.
P.S. M.Var.	.0076	-	-	.0072	.0140	-	-	.0094
Eff.	113%	159%	117%	100%	-	-	-	-
	n=10 β=50 ρ=.3							
Var.	.0005	.0004	.0005	.0004				
E.M.S.	.0005	.0005	.0005	.0004				
P.S. M.Var.	.0005	-	-	.0004				
Eff.	92%	101%	93%	100%				

Table 3. Coefficient of the approximate variance of r as derived by Patterson

ρ	n=4	n=5	n=6	n=7	n=8	n=9	n=10
0	1.5556	1.5000	1.5400	1.6178	1.7143	1.8214	1.9352
.1	1.7488	1.5276	1.4815	1.5006	1.5506	1.6177	1.6951
.2	2.0311	1.5773	1.4211	1.3709	1.3693	1.3935	1.4330
.3	2.4637	1.6790	1.3817	1.2506	1.1929	1.1731	1.1752
.4	3.1575	1.8758	1.3898	1.1614	1.0423	.9779	.9443
.5	4.3394	2.2465	1.4837	1.1280	.9372	.8259	.7576
.6	6.5548	2.9678	1.7392	1.1901	.9022	.7344	.6293
.7	11.3845	4.5397	2.3617	1.4468	.9904	.7344	.5782
.8	25.2457	8.9375	4.1441	2.2724	1.3986	.9369	.6695
.9	100.2214	31.7554	13.2004	6.5006	3.5998	2.1737	1.4030

$$\text{Var } r = \frac{\sigma^2}{\beta^2} \text{ (coefficient)}$$

Table 4. Coefficient of the approximate bias of r as derived by Patterson

Coefficient for bias r(1,1.5)

ρ	n=4	n=5	n=6	n=7	n=8	n=9	n=10
0	.2963	.5833	1.2600	2.2193	3.4286	4.8750	6.5525
.1	.3198	.3656	.7665	1.3857	2.1848	3.1494	4.2730
.2	.4100	.2430	.4339	.7995	1.2930	1.8980	2.6076
.3	.5783	.1969	.2284	.4141	.6925	1.0438	1.4605
.4	.8597	.2167	.1178	.1799	.3161	.5005	.7239
.5	1.3384	.3073	.0776	.0514	.0986	.1819	.2887
.6	2.2255	.5064	.0973	-.0071	-.0167	.0102	.0542
.7	4.1472	.9468	.1961	-.0167	-.0745	-.0797	-.0652
.8	9.6444	2.1874	.5049	.0318	-.1094	-.1447	-.1436
.9	39.4543	8.7550	2.1501	.3601	.1677	-.3107	-.3276

Coefficient for bias r(1,0)

ρ	n=4	n=5	n=6	n=7	n=8	n=9	n=10
0	-	-	-	-	-	-	-
.1	12.5393	19.7893	27.1012	35.0887	43.9195	53.6544	64.3198
.2	7.4871	10.1611	12.7376	15.5490	18.6672	22.1113	25.8865
.3	6.3154	7.3037	8.2593	9.3719	10.6599	12.1184	13.7401
.4	6.3697	6.2717	6.3344	6.5968	7.0179	7.5658	8.2199
.5	7.3497	6.1875	5.5518	5.2548	5.1667	5.2168	5.3650
.6	9.6705	7.0086	5.5771	4.7662	4.2899	4.0103	3.8555
.7	14.9698	9.4118	6.6459	5.1088	4.1796	3.5831	3.1839
.8	30.0253	16.5004	10.3583	7.1486	5.2898	4.1278	3.3577
.9	108.8989	52.6795	29.4729	18.2483	12.1725	8.5963	6.3506

Coefficient for bias r(1,1)

ρ	n=4	n=5	n=6	n=7	n=8	n=9	n=10
0	.6667	1.2500	2.2400	3.5467	5.1429	7.0179	9.1667
.1	.6900	.9542	1.5645	2.4070	3.4495	4.6799	6.0926
.2	.8032	.7940	1.1174	1.6189	2.2583	3.0210	3.9009
.3	1.0196	.7435	.8462	1.1032	1.4592	1.8957	2.4051
.4	1.3844	.7933	.7099	.7910	.9543	1.1734	1.4378
.5	2.0063	.9606	.6859	.6295	.6617	.7413	.8527
.6	3.1561	1.3191	.7823	.5896	.5216	.5102	.5293
.7	5.6327	2.1087	1.0814	.6868	.5094	.4231	.3807
.8	12.6638	4.3078	1.9647	1.0862	.6905	.4883	.3751
.9	50.4193	15.6905	6.4643	3.1846	1.7807	1.0957	.7269

$$\text{bias} = \frac{\sigma^2}{\beta^2}\,(\text{coefficient})$$

Table 5. Approximate bias of r(1,0) as given by Portman (1961)

n	β	ρ=.1	ρ=.3	ρ=.5	ρ=.7	ρ=.9
	5	37.6491	1.5225	1.2724	4.6115	24,742,973.1
4	10	.7722	.1049	.1072	.2584	14.7250
	50	.0058	.0026	.0030	.0061	.0493
	5	10.2634	.8119	.4489	.4419	3.5997
6	10	.9360	.1132	.0658	.0774	.4488
	50	.0120	.0033	.0022	.0028	.0120
	5	6.0148	.6568	.2544	.2262	.7814
8	10	1.1292	.1373	.0588	.0460	.1464
	50	.0192	.0044	.0021	.0017	.0049

Table 6. Approximate variance of r(1,0) as given by Portman (1961)

n	β	ρ=.1	ρ=.3	ρ=.5	ρ=.7	ρ=.9
	5	48.7576	.6748	.7288	5.3793	18,195,014,208.0
4	10	.1684	.0238	.0472	.1633	29.1107
	50	.0007	.0010	.0017	.0046	.0433
	5	13.6430	.2491	.0896	.1646	4.1872
6	10	.2651	.0146	.0148	.0251	.2225
	50	.0006	.0005	.0006	.0009	.0053
	5	6.6462	.1898	.0453	.0451	.3188
8	10	.3620	.0115	.0084	.0096	.0406
	50	.0006	.0005	.0004	.0004	.0010

Table 7. Coefficient of the asymptotic variance of the least squares estimator

ρ	n=4	n=5	n=6	n=7	n=8	n=9	n=10
.1	1.7216	1.4337	1.3038	1.2306	1.1838	1.1512	1.1273
.2	2.0189	1.5299	1.3246	1.2161	1.1501	1.1059	1.0743
.3	2.4587	1.6573	1.3354	1.1730	1.0786	1.0179	.9761
.4	3.1556	1.8668	1.3702	1.1278	.9914	.9067	.8503
.5	4.3396	2.2431	1.4763	1.1155	.9182	.7989	.7214
.6	6.5547	2.9666	1.7367	1.1861	.8963	.7262	.6183
.7	11.3845	4.5393	2.3609	1.4458	.9890	.7325	.5756
.8	25.2457	8.9374	4.1439	2.7216	1.3983	.9365	.6691
.9	100.2214	31.7553	13.2004	6.5005	3.5997	2.1736	.9443

$$\text{Asymptotic variance} = \frac{\sigma^2}{\beta^2}\,(\text{coefficient})$$

Table 8. Computational notes

n	β	ρ	d.n.c.[a]	R.V.[b]	r(1,0) >10[c]	Δr > 10[d]
4	5	.1	6/100	2.1756	0	0
4	5	.5	16/100	4.0824	1	2
4	5	.9	2/10	s.n.u.[e]	0	1
4	10	.1	0/100	.9011	0	0
4	10	.3	0/100	1.0516	0	0
4	10	.9	m.s.[f]	-	7	-[g]
4	50	.7	0/100	.7498	0	0
4	50	.7	0/100	.9753	0	0
4	50	.9	7/30	s.n.u.	0	0
4	50	.9	m.s.	-	0	-
6	5	.3	4/100	1.5617	2	0
6	5	.7	17/100	1.2699	2	1
6	10	.7	4/100	18.2580	0	1
6	50	.9	12/100	-[h]	0	0
8	5	.5	5/100	.9369	2	0
8	5	.9	m.s.	-	2	-
8	10	.1	2/100	.8844	1	1
8	10	.9	m.s.	-	0	-
8	50	.7	0/100	1.088	0	0
8	50	.9	m.s.	-	0	-
10	5	.7	5/100	1.0368	1	0
10	10	.5	0/100	1.0014	0	0
10	10	.9	m.s.	-	0	-
10	50	.3	0/100	.9534	0	0

a/ The proportion of least squares samples that did not converge.
b/ The residual variance, s^2 ($\hat{\sigma}^2$).
c/ The number of samples where r(1,0) is greater than 10.
d/ The number of least squares samples where the increment is greater than 10.
e/ Sample not used.
f/ Missing sample or least squares estimate was not calculated.
g/ Also some of r(1,1.5) and r(1,1) were > 10.
h/ Contains an error.

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APPENDIX A

A PROOF THAT THE ESTIMATOR PROPOSED BY MONROE IS r(1,0)

Portman (1961) showed that the estimator of ρ proposed by Monroe (1949) is a ratio of quadratic terms.

Following the development of Monroe, Portman wrote the model as

$$E(y) = w = \alpha - \beta\rho^{x} .$$

This is the general solution of the first order differential equation in w,

$$\frac{dw}{dx} = (\alpha - w)\, c \qquad \text{where} \quad \rho = e^{c} .$$

The model can be generated exactly by the first order linear difference equation of the form:

$$w_i - w_{i-1} = \beta_1 + \beta_2 w_i . \qquad (1)$$

If this equation is summed over all values of w_i and there are equal increments of the independent variable, one obtains

$$w_j = \beta_0 + \beta_1 x_j + \beta_2 X_j , \qquad (2)$$

where

$$X_j = \sum_{i=1}^{j} w_i \qquad x_j = 0, 1, \ldots, n-1$$

$$i,j = 1, \ldots, n \qquad n = \text{the number of observations}$$

$$\beta_o = w_0 \qquad \alpha = \frac{-\beta_1}{\beta_2}$$

$$\beta = \frac{\beta_0}{\beta_2 - 1} - \frac{\beta_1}{\beta_2} \qquad \rho = \frac{1}{1-\beta_2} .$$

Since the parameters, β's of (2), enter linearly, they can be estimated from actual data, $y_i = w_i + \epsilon_i$, by the least squares method if n is equal to, or greater than, 3.

Portman (1961) took this development and showed how the estimator of r could be written as a ratio of quadratic terms. She developed the normal equations in matrix form as follows:

$$KK'\,\hat{\underline{\beta}} = K\underline{y}$$

where

$$\hat{\underline{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} \qquad \underline{y} = \begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-1} \end{bmatrix} \qquad K' = \left[\underline{1}, \underline{X}, \underline{y}\right]$$

and

$$\underline{1} = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \qquad \underline{X} = \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ n-1 \end{bmatrix} \qquad \underline{y} = T\underline{y} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & 0 & \\ \cdot & \cdot & \cdot & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \underline{y} .$$

The matrices K' and $\underline{\beta}$ can be partitioned as follows:

$$K' = [S \,|\, T\underline{y}] \qquad \text{where } s = [\underline{1}, \underline{X}]$$

$$\hat{\underline{\beta}} = \left[\frac{\hat{\Theta}}{\hat{\beta}_2}\right] \qquad \hat{\Theta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} .$$

Then the normal equations are:

$$S'S\hat{\theta} + S'T\underline{y}\hat{\beta}_0 = S'\underline{y} \tag{3}$$

$$\underline{y}'T'S\hat{\theta} + \underline{y}'T'T\underline{y}\hat{\beta}_2 = \underline{y}'T'\underline{y} \; . \tag{4}$$

Premultiplying (3) by $\underline{y}'T'S(S'S)^{-1}$ and subtracting the results from (4) gives

$$\hat{\beta}_2 = \frac{\underline{y}'T'[I_n - S(S'S)^{-1}S']\underline{y}}{\underline{y}'T'[I_n - S(S'S)^{-1}S']T\underline{y}} = \frac{\underline{y}'T'P\underline{y}}{\underline{y}'T'PT\underline{y}} \; .$$

Now $$\hat{\rho} = r_M = \frac{1}{1-\beta_2} = \frac{\underline{y}'T'PT\underline{y}}{\underline{y}'T'P(T-I_n)\underline{y}} \; .$$

This is the compact form which Portman (1961) used to develop her approximate bias and variance formulae. It is very similar to the form used by Patterson (1958). With further simplification the equivalence of the two can be shown.

The element of the k^{th} row and ℓ^{th} column of $S(S'S)^{-1}S'$ is needed first. It is:

$$\Sigma x^2 - x_k \Sigma x - x_\ell \Sigma x + n x_k x_\ell \; .$$

Using this, we can see that the i^{th}, j^{th} element of $T'P$ is given by:

$$(t'p)_{ij} = \begin{cases} 1 - \frac{u}{h} & i \leq j \\ -\frac{u}{h} & i > j \end{cases}$$

where $u = (n-i+1)\sum_1^n x^2 - \sum_1^n x \sum_i^n x - (n-i+1)\, x_j \sum_1^n x + nx_j \sum_i^n x$

$h = n \Sigma x^2 - (\Sigma x)^2$.

When $i = 1$ $\quad (t'p)_{1j} = 0 \quad$ for all j.

Now T'PT = (T'P)T. Thus, using (T'P) and multiplying on the right by T gives us:

(a) $(t'pt)_{1k} = 0$ for all k .

(b) $(t'pt)_{\ell 1} = 0$ for all ℓ .

(c) otherwise

$$(t'p)_{\ell k} = \begin{cases} (n-k+1) - \frac{f}{h} & \text{if } k \geq \ell \\ (n-\ell+1) - \frac{f}{h} & \text{if } k < \ell \end{cases}$$

where

$$f = [n-\ell+1)(n-k+1) \Sigma x^2 - (n-k+1) \sum_1^n x \sum_\ell^n x - (n-\ell+1) \sum_1^n x \sum_k^n x + n \sum_k^n x \sum_\ell^n x].$$

Therefore,

$$T'PT = \left[\begin{array}{c|ccccc} 0 & 0 & . & . & . & 0 \\ \hline 0 & & & & & \\ \cdot & & & & & \\ \cdot & & & W & & \\ \cdot & & & & & \\ 0 & & & & & \end{array}\right] \quad \text{where } w_{ij} = (t'pt)_{(i-1),(j-1)} \cdot$$

Then

$$w_{ij} = \begin{cases} (n-j) - \frac{m}{h} & \text{if } j \geq i \\ (n-i) - \frac{m}{h} & \text{if } j < i \end{cases}$$

where $m = [(n-i)(n-j) \sum_0^{n-1} x^2 - (n-j) \sum_0^{n-1} x \sum_i^{n-1} x - (n-i) \sum_0^{n-1} x \sum_j^{n-1} x + n \sum_i^{n-1} x \sum_j^{n-1} x]$

now $\sum_0^{n-1} x^2 = \frac{n(n-1)(2n-1)}{6}$ $\sum_0^{n-1} x = \frac{n(n-1)}{2}$ $\sum_\rho^{n-1} x = \frac{n(n-1)}{2} - \frac{\rho(\rho-1)}{2}$.

After some simplification, we find that

$$w_{ij} = \begin{cases} \dfrac{i(n-j)}{n} - \dfrac{3ij(n-i)(n-j)}{n(n^2-1)} & i \leq j \\[2ex] \dfrac{j(n-i)}{n} - \dfrac{3ij(n-i)(n-j)}{n(n^2-1)} & i > j \end{cases} .$$

By comparing this with (3), we see that

$$W = D$$

or that

$$T'PT = \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & D & \\ 0 & & & \end{array}\right] .$$

Similarly

$$T'P(T-I_n) = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 0 \\ \hline & & & 0 \\ & D & & \vdots \\ & & & 0 \end{array}\right] .$$

Thus

$$r_M = \frac{\underline{y}'T'PT\underline{y}}{\underline{y}'T'P(T-I_n)\underline{y}} = \frac{\underline{y}_1'\ D\ \underline{y}_1}{\underline{y}_0'\ D\ \underline{y}_1} = r(1,0)$$

or the estimator proposed by Monroe (1949) is r(1,0) in Patterson's (1958) notation.

APPENDIX B

FORMULAE FOR PATTERSON'S APPROXIMATIONS

Formulae used in the calculation of the approximate bias and variance of Patterson's quadratic estimators of r are given below.

n=4

$$F_0 = \frac{1}{10}[3 - 2\rho - 2\rho^2 - 2\rho^3 + 3\rho^4]$$

$$F_1 = \frac{1}{100}[14 - 6\rho - 16\rho^2 - 16\rho^3 + 14\rho^4]$$

$$F_2 = \frac{1}{100}[-1 + 4\rho - 6\rho^2 + 4\rho^3 - \rho^4]$$

n=5

$$F_0 = \frac{1}{10}[4 - \rho^2 - 6\rho^3 - \rho^4 + 4\rho^6]$$

$$F_1 = \frac{1}{100}[24 + 12\rho - 14\rho^2 - 44\rho^3 - 14\rho^4 + 12\rho^5 + 24\rho^6]$$

$$F_2 = \frac{1}{100}[4 + 12\rho - 9\rho^2 - 14\rho^3 - 9\rho^4 + 12\rho^5 + 4\rho^6]$$

n=6

$$F_0 = \frac{1}{210}[100 + 40\rho + 28\rho^2 - 112\rho^3 - 112\rho^4 - 112\rho^5 + 28\rho^6 + 40\rho^7 + 100\rho^8]$$

$$F_1 = \frac{4}{(210)^2}[3850 + 4270\rho + 1456\rho^2 - 5404\rho^3 - 8344\rho^4 - 5404\rho^5 + 1456\rho^6 + 4270\rho^7 + 3850\rho^8]$$

$$F_2 = \frac{4}{(210)^2}[1225 + 3220\rho + 721\rho^2 - 2464\rho^3 - 5404\rho^4 - 2464\rho^5 + 721\rho^6 + 3220\rho^7 + 1225\rho^8]$$

n=7

$$F_0 = \frac{1}{28}[15 + 10\rho + 11\rho^2 - 8\rho^3 - 14\rho^4 - 28\rho^5 - 14\rho^6 - 8\rho^7 + 11\rho^8 + 10\rho^9 + 15\rho^{10}]$$

$$F_1 = \frac{1}{(28)^2}[364 + 560\rho + 476\rho^2 - 112\rho^3 - 728\rho^4 - 1120\rho^5 - 728\rho^6 - 112\rho^7 + 476\rho^8 + 560\rho^9 + 364\rho^{10}]$$

$$F_2 = \frac{1}{(28)^2}[154 + 420\rho + 322\rho^2 - 532\rho^4 - 728\rho^5 - 532\rho^6 + 322\rho^8 + 420\rho^9 + 154\rho^{10}]$$

$$n=8 \quad F_0 = \frac{1}{84}[49 + 42\rho + 54\rho^2 + 2\rho^3 - 21\rho^4 - 84\rho^5 - 84\rho^6 - 84\rho^7 - 21\rho^8 + 2\rho^9 + 54\rho^{10} + 42\rho^{11} + 49\rho^{12}]$$

$$F_1 = \frac{1}{(84)^2}[4116 + 7644\rho + 8736\rho^2 + 3948\rho^3 - 3780\rho^4 - 12600\rho^5 - 16128\rho^6 - 12600\rho^7 - 3780\rho^8 + 3948\rho^9 + 8736\rho^{10} + 7644\rho^{11} + 4116\rho^{12}]$$

$$F_2 = \frac{1}{(84)^2}[2058 + 5880\rho + 6468\rho^2 + 3864\rho^3 - 2898\rho^4 - 9072\rho^5 - 12600\rho^6 - 9072\rho^7 - 2898\rho^8 + 3864\rho^9 + 6468\rho^{10} + 5880\rho^{11} + 2058\rho^{12}]$$

$$n=9 \quad F_0 = \frac{1}{180}[112 + 112\rho + 157\rho^2 + 62\rho^3 + 17\rho^4 - 136\rho^5 - 186\rho^6 - 276\rho^7 - 186\rho^8 - 136\rho^9 + 17\rho^{10} + 62\rho^{11} + 157\rho^{12} + 112\rho^{13} + 112\rho^{14}]$$

$$F_1 = \frac{1}{(180)^2}[22848 + 48048\rho + 64428\rho^2 + 50448\rho^3 + 12828\rho^4 - 45024\rho^5 - 94104\rho^6 - 118944\rho^7 - 94104\rho^8 - 45024\rho^9 + 12828\rho^{10} + 50448\rho^{11} + 64428\rho^{12} + 48048\rho^{13} + 22848\rho^{14}]$$

$$F_2 = \frac{1}{(180)^2}[12768 + 37968\rho + 50298\rho^2 + 44868\rho^3 + 11298\rho^4 - 32784\rho^5 - 77364\rho^6 - 94104\rho^7 - 77364\rho^8 - 32784\rho^9 + 11298\rho^{10} + 44868\rho^{11} + 50298\rho^{12} + 37968\rho^{13} + 12768\rho^{14}]$$

$$n=10 \quad F_0 = \frac{1}{165}[108 + 120\rho + 178\rho^2 + 108\rho^3 + 78\rho^4 - 64\rho^5 - 132\rho^6 - 264\rho^7 - 264\rho^8 - 264\rho^9 - 132\rho^{10} - 64\rho^{11} + 78\rho^{12} + 108\rho^{13} + 178\rho^{14} + 120\rho^{15} + 108\rho^{16}]$$

$$F_1 = \frac{1}{(165)^2}[22572 + 51876\rho + 77088\rho^2 + 76098\rho^3 + 49038\rho^4 - 6600\rho^5 - 69696\rho^6 - 126324\rho^7 - 148104\rho^8 - 126324\rho^9 - 69696\rho^{10} - 6600\rho^{11} + 49038\rho^{12} + 76098\rho^{13} + 77088\rho^{14} + 51876\rho^{15} + 22572\rho^{16}]$$

$$F_2 = \frac{1}{(165)^2}[13662 + 41976\rho + 62403\rho^2 + 67188\rho^3 + 42603\rho^4 - 1320\rho^5 - 58806\rho^6 - 104544\rho^7 - 126324\rho^8 - 104544\rho^9 - 58806\rho^{10} - 1320\rho^{11} + 42603\rho^{12} + 67188\rho^{13} + 62403\rho^{14} + 41976\rho^{15} + 13662\rho^{16}]$$