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INCOMPLETE MULTIRESPONSE DESIGNS

by

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This paper discusses in detail a simple example of an incomplete multiresponse design, defined therein. General formulae have also been presented to enable an experimenter to analyze a certain wide class of such designs, of which the example is a member.

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1. Introduction

In an earlier paper [3], Roy and Srivastava introduced two classes of designs which are specially suited to multiresponse experiments. Here we shall consider general classes of designs which are characterized by the fact that not all the variates are measured on each experimental unit. Such a situation arises when either it is either physically impossible or otherwise inadvisable or inconvenient to measure each response on each unit. As an example, suppose an anthropologist has unearthed a number of skulls, each of which is in a partly mutilated condition. Then the kinds of measurements that he can take on a skull may differ from one skull to the other.

A sketch of the general theory of incomplete-multiresponse (I-M) designs is outside the scope of this paper. For that the reader is referred to [4]. Another general discussion using likelihood ratio tests will be found in [1]. In what follows we shall discuss a very simple example of an I-M design to illustrate the general theory. At the end a few remarks will be made on the general theory itself and on the new combinatorial research problems which it poses.

2. A Simple Example of an I-M Design

Suppose we have $p=3$ variates, and $u=8$ Sets $S_i (i = 1, 2, \dots, 8)$ of experimental units. For each i we choose $p_i (\leq p)$ variates, each of which is measured on each experimental unit in the set S_i . This choice of variates is called the

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variate-wise design and is denoted by D_1 . In the present example, let D_1 be as shown below:

Set	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
Variates	1,2	1,2	1,3	1,3	2,3	2,3	1,2,3	1,2,3

Each set S_i is supposed to have a (univariate block-treatment design D_{2i} defined over it. For simplicity again, we suppose D_{2i} the same for all i , being a BIBD with parameters (v, r, λ) where v is the total number of treatments. The above then fixes the I-M design of our example, viz. the $(u+1)$ -tuple $(D_1, D_{21}, \dots, D_{2u})$.

Let the p 'true' response to the different treatments be denoted by the $v \times p$ matrix

$$(1) \quad \xi_{v \times p} = \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1p} \\ \dots & \dots & \dots & \dots \\ \xi_{v1} & \xi_{v2} & \cdots & \xi_{vp} \end{bmatrix}$$

$$= [\xi_1, \xi_2, \dots, \xi_p], \text{ say,}$$

where ξ_{jl} denotes the true value of the l -th response to the j -th treatment. Let $\xi^{(i)}$ be a $v \times p_i$ matrix whose p_i columns are those p_i columns (in order) of ξ which correspond to the p_i variates studied on the set S_i . In the example, we thus have

$$(2) \quad \xi^{(5)} = \begin{bmatrix} \xi_{22} & \xi_{32} \\ \dots & \dots \end{bmatrix}$$

It is easy to see that there exists a $p \times p_i$ matrix B_i such that

$$(3) \quad \xi^{(i)} = \xi B_i, \quad i = 1, 2, \dots, u.$$

Let \underline{y}'_{is} denote the $1 \times p_i$ vector of observations on the s -th unit in S_i ($s = 1, 2, \dots, N_i; i = 1, 2, \dots, u$). Here we have $N_i = vr$, for all i . Furthermore,

assume that

$$(4) \quad \text{Var} (Y'_{-is}) = \Sigma_{(i)} (p_i \times p_i)$$

$$\text{Cov} (Y_{-is}, Y_{-i's'}) = 0_{p_i p_i}, \text{ if } s \neq s', i = 1', \text{ or}$$

$$\text{if } i = i',$$

where $\Sigma_{(i)}$ is the $p_i \times p_i$ matrix obtained by taking the appropriate p_i rows and columns of a covariance matrix $\Sigma(p \times p)$, to be called the population dispersion matrix of all the p variates. In our example, we have, say,

$$(5) \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ \text{Sym.} & & \sigma_{33} \end{bmatrix}$$

$$\Sigma_{(3)} = \begin{bmatrix} \sigma_{11} & \sigma_{13} \\ \text{Sym} & \sigma_{33} \end{bmatrix} \quad \text{etc.}$$

It is easy to see that

$$(6) \quad \Sigma_{(i)} = B_i' \Sigma B_i, \quad i = 1, 2, \dots, u.$$

Also we shall make the usual assumption about treatment effects viz.

$$J_{1V} \underline{\xi}_l = 0, \quad l = 1, 2, \dots, p.$$

Now consider a fixed set of units S_i and the design D_{2i} . Define

$$(7) \quad b_i = \text{number of blocks in } S_i,$$

T_{ijl} = total (for the l -th variate) response to the j -th treatment over all units in S_i to which j -th treatment was allotted by D_{2i} .

B_{ijl} = Total (for the l -th variate) response over all units in the g -th block in the set S_i .

n_{igj} = number of units in the g -th block in S_i , to which j -th treatment is allotted under D_{2i} .

K_{ig} = number of units in the g -th block in S_i .

r_{ij} = number of units in S_i under treatment j .

(In our example, let $K_{ij} = K$, $r_{ij} = r$ for all i, g, j .)

$$Q_{ijl} = T_{ijl} - \frac{1}{K} \sum_{g=1}^{b_i} n_{ijg} B_{igl}, \text{ and finally}$$

$$Q_{il} = \begin{bmatrix} Q_{il1} \\ \vdots \\ Q_{ilv} \end{bmatrix}$$

From the theory of block-treatment designs it follows that for all $(j \neq j')$ and all i ,

$$(8) \quad (Q_{il}) = C_i \xi_l$$

$$\text{Var} (Q_{il}) = C_i, \quad ,$$

where C_i is a $v \times v$ matrix with elements given by

$$C_i(j, j') = -\frac{1}{K} \sum_{g=1}^{b_i} n_{igj} n_{igj'} = -\mu_{ijj'}, \text{ say, } = -\frac{1}{K}, \text{ here,}$$

$$C_i(j, j) = r_{ij} \left(1 - \frac{1}{K}\right) = r \left(1 - \frac{1}{K}\right), \text{ here.}$$

Also define for convenience,

$$(9) \quad Q_i = (Q_{i1}, Q_{i2}, \dots, Q_{ip_i}) = \begin{bmatrix} Q'_{i1} \\ Q'_{i2} \\ \vdots \\ Q'_{iv} \end{bmatrix}, \text{ say,}$$

so that

$$(10) \quad \begin{pmatrix} Q_i \\ v \times p_i \end{pmatrix} = C_i \begin{pmatrix} \xi^{(i)} \\ v \times v \end{pmatrix}, \quad i = 1, 2, \dots, u.$$

The general theory is concerned with those I-M designs for which there exist known matrices F_1, F_2, \dots, F_m such that

$$(11) \quad C_i = \alpha_{i1} F_1 + \alpha_{i2} F_2 + \dots + \alpha_{im} F_m, \quad i = 1, 2, \dots, u,$$

where α_{iq} are known scalars. This condition is satisfied in the present illustration and we have

$$(11a) \quad m = 2, \quad F_1 = I_v, \quad F_2 = J_{vv} - I_v, \\ \alpha_{i1} = r(1 - \frac{1}{K}), \quad \alpha_{i2} = -\frac{\lambda}{K}, \quad \text{for all } i.$$

So far as α 's are concerned, we first observe that using (7) one can write

(10) in the form

$$(12) \quad (Q_i) = [C_i + \sigma_i J_{vv}] \xi^{(i)}, \quad i = 1, 2, \dots, v,$$

where σ_i 's are arbitrary constants. Suppose

$$(13) \quad F_1 + F_2 + \dots + F_m = J_{vv},$$

as is the case in the example. Then one can write

$$(14) \quad C_i = \alpha'_{i1} F_1 + \alpha'_{i2} F_2 + \dots + \alpha'_{im} F_m,$$

where

$$(15) \quad \alpha'_{iq} = \alpha_{iq} + \sigma_i \quad \text{for all } i \text{ and } q.$$

Define for $q = 1, 2, \dots, m$,

$$Q = (Q_1, Q_2, \dots, Q_u)$$

$$L_q = (\alpha'_{1q} B_1, \dots, \alpha'_{uq} B_u).$$

Now

$$\begin{aligned}
 (Q_i) &= C_i \xi B_i \\
 &= \left(\sum_{q=1}^m \alpha'_{iq} F_q \right) \xi B_i \\
 &= \sum_{q=1}^m F_q \xi (\alpha'_{iq} B_i) \quad ,
 \end{aligned}$$

so that

$$\begin{aligned}
 (16) \quad (Q) &= (F_1 \xi, F_2 \xi, \dots, F_m \xi) \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{bmatrix} \\
 &= (F_1 \xi, F_2 \xi, \dots, F_m \xi) \begin{matrix} L \\ \text{mp} \times \Sigma p_i \end{matrix} \quad , \text{ say } .
 \end{aligned}$$

Consider the matrix LL' ($\text{mp} \times \text{mp}$). In our example,

$$\text{mp} = 6, \text{ and } \Sigma p_i = 18 \quad ,$$

so that $\text{mp} < \Sigma p_i$. Also by direct calculation

$$(17) \quad LL' = \begin{bmatrix} \sum_{i=1}^u \alpha'_{i1}{}^2 D_i & \sum_{i=1}^u \alpha'_{i1} \alpha'_{i2} D_i \\ \sum_{i=1}^u \alpha'_{i1} \alpha'_{i2} D_i & \sum_{i=1}^u \alpha'_{i2}{}^2 D_i \end{bmatrix}$$

where D_i is a $p_i \times p_i$ diagonal matrix containing unity at those p_i places which correspond to the p_i variates studied on S_i , and zero elsewhere. Now

$$\begin{aligned}
 (18) \quad \alpha'_{i1} &= r(1 - \frac{1}{k}) + \sigma_i \quad \text{say} \\
 \alpha'_{i2} &= -\frac{\lambda}{k} + \sigma_i \quad \text{say} \quad .
 \end{aligned}$$

It can be shown that the matrix LL' will be singular if we take all σ 's to be zero. A convenient choice here would be

$$(19) \quad \begin{aligned} \sigma_i &= \frac{\lambda}{k}, \quad i = 1, 3, 5, 7 \\ &= 0, \quad i = 2, 4, 6, 8. \end{aligned}$$

Then we get

$$\begin{aligned} \alpha'_{i1} &= \frac{\lambda v}{k} = v\alpha, \quad \text{say, } i = 1, 3, 5, 7 \\ &= \frac{\lambda(v-1)}{k} = (v-1)\alpha, \quad i = 2, 4, 6, 8. \\ \alpha'_{i2} &= 0, \quad i = 1, 3, 5, 7 \\ &= -\alpha, \quad i = 2, 4, 6, 8. \end{aligned}$$

Hence

$$\sum_{i=1}^u \alpha'^2_{i1} D_i = 3 \{ v^2 \alpha^2 + (v-1)^2 \alpha^2 \} I_3 = 3\alpha^2(2v^2 - 2v+1) I_3$$

$$\sum_{i=1}^u \alpha'_{i1} \alpha'_{i2} D_i = -3\alpha^2(v-1) I_3$$

$$\sum_{i=1}^u \alpha'^2_{i2} D_i = 3\alpha^2 I_3,$$

so that

$$LL' = 3\alpha^2 \begin{bmatrix} (2v^2 - 2v + 1) - (v-1) & \\ - (v-1) & 1 \end{bmatrix} \otimes I_3.$$

Hence LL' is nonsingular and

$$(20) \quad (LL')^{-1} = \frac{1}{3\alpha^2 v^2} \begin{bmatrix} 1 & +(v-1) \\ +(v-1) & (2v^2-2v+1) \end{bmatrix} \otimes I_3 = [H_1, H_2], \text{ say,}$$

Thus we can write

$$[Q L' (LL')^{-1}] = (F_1 \xi, F_2 \xi),$$

or

$$(21) \quad \begin{bmatrix} Q & L' & H_1 \\ Q & L' & H_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \xi.$$

Now

$$\begin{aligned} L' H_1 &= (L'_1 L'_2) H_1 \\ &= \begin{bmatrix} \alpha'_{11} & B'_1 & \alpha'_{12} & B'_1 \\ \alpha'_{u1} & B'_u & \alpha'_{u2} & B'_u \end{bmatrix} \cdot \frac{1}{3\alpha^2 v^2} \begin{bmatrix} 1 \\ +(v-1) \end{bmatrix} \otimes I_3 \end{aligned}$$

$$= \frac{1}{3\alpha^2 v^2} \begin{bmatrix} (\alpha'_{11} + \overline{v-1} \alpha'_{12}) & B'_1 \\ (\alpha'_{u1} + \overline{v-1} \alpha'_{u2}) & B'_u \end{bmatrix} \otimes I_3$$

$$= \frac{1}{3\alpha^2 v^2} \begin{bmatrix} \alpha v B'_1 \\ 0 \\ \alpha v B'_3 \\ 0 \\ \alpha v B'_5 \\ 0 \\ \alpha v B'_7 \\ 0 \end{bmatrix}.$$

Hence

$$(22) \quad Q L' H_1 = \frac{1}{3\alpha v} \left[\begin{array}{c} \Sigma \\ 1,3,5,7 \end{array} Q_i B'_i \right]$$

$$= \begin{bmatrix} Z_{11} \\ Z_{21} \\ \vdots \\ Z_{v1} \end{bmatrix},$$

where

$$(23) \quad Z_{j1} = (Z_{j11}, Z_{j12}, Z_{j13}) ,$$

and

$$(24) \quad Z_{j11} = \int (1)(Q_{1j1} + Q_{3j1} + Q_{7j1}) + (0)(Q_{2j1} + Q_{4j1} + Q_{8j1})_7 \cdot \frac{1}{3\alpha v}$$

$$Z_{j12} = \int (1)(Q_{1j2} + Q_{5j2} + Q_{7j2}) + (0)(Q_{2j2} + Q_{6j2} + Q_{8j2})_7 \cdot \frac{1}{3\alpha v}$$

$$Z_{j13} = \int (1)(Q_{3j3} + Q_{5j3} + Q_{7j3}) + (0)(Q_{4j3} + Q_{6j3} + Q_{8j3})_7 \cdot \frac{1}{3\alpha v}$$

Similarly

$$Q L' H_2 = \frac{1}{3\alpha^2 v^2} \left[\begin{array}{c} \{ +\alpha'_{11}(v-1) + (2v^2 - 2v + 1)\alpha'_{12} \} B'_1 \\ \{ +\alpha'_{u1}(v-1) + (2v^2 - 2v + 1)\alpha'_{u2} \} B'_u \end{array} \right]$$

$$= \frac{1}{3\alpha v} \left[+ (v-1) \sum_{1,3,5,7} (Q_i B'_i) - \sum_{2,4,6,8} (Q_i B'_i) \right]$$

$$= \begin{bmatrix} Z_{12} \\ Z_{22} \\ \dots \\ Z_{v2} \end{bmatrix}, \text{ say } ,$$

where Z_{j2} are defined analogous to Z_{j1} , except that the coefficients 1 and (0) are replaced by $(v-1)$ and (-1) respectively. It is easy to see that

$$\sum_{j=1}^v Z_{j1} = \sum_{j=1}^v Z_{j2} = 0_{13} \quad .$$

We therefore consider the $2(v-1) \times 3$ matrix

$$(25) \quad Z = \begin{bmatrix} Z_{21} \\ \vdots \\ Z_{v1} \\ \vdots \\ Z_{22} \\ \vdots \\ Z_{v2} \end{bmatrix} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_2 \end{bmatrix}, \text{ say}$$

and first examine the $2(v-1)$ vectors for linear independence.

We observe that the Q_{ijl} ($i=1,2,\dots,u$; $j=2,3,\dots,v$; $l=1,2,\dots,p$), (considered as linear functions of original observations y 's) are all linearly independent, and so is Z_{22}, \dots, Z_{v2} . Also for any j , Z_{j1} and Z_{j2} are linearly independent, since the matrix

$$(26) \quad \begin{bmatrix} 1 & 0 \\ (v-1) & -v \end{bmatrix}$$

is nonsingular. Thus, no Z_{j1} could depend on any linear function of Z_{j2} , which shows that the rows of Z are linearly independent.

Let us now calculate the variances and covariances of the Z_{je} . We recall that

$$\begin{aligned} \text{covar}(Q_{ijl}, Q_{i'j'l'}) &= -\alpha \sigma_{ll'}, \text{ if } i=i', j \neq j' \\ &= \alpha v \sigma_{ll'}, \text{ if } i=i', j=j' \\ &= 0, \text{ otherwise,} \end{aligned}$$

for $i = 1, 2, \dots, 8; j = 1, 2, \dots, v; l = 1, 2, 3$.

Thus

$$\begin{aligned} \text{var}(Z_{j11}) &= \frac{\sigma_{11}}{9\alpha^2 v} \cdot 3\alpha v = \frac{1}{3\alpha v} \sigma_{11} \\ \text{cov}(Z_{j11}, Z_{j12}) &= \frac{\sigma_{12}}{9\alpha^2 v} \cdot \alpha v \cdot 2 = \frac{1}{3\alpha v} \left(\frac{2}{3} \sigma_{12}\right), \text{ etc. ,} \end{aligned}$$

Hence

$$(27) \quad \text{var}(Z_{j1}) = \frac{1}{3\alpha v} \Sigma^*, \quad j = 2, 3, \dots, v,$$

where

$$\Sigma^* = \begin{bmatrix} \sigma_{11} & \frac{2}{3} \sigma_{12} & \frac{2}{3} \sigma_{13} \\ & \sigma_{22} & \frac{2}{3} \sigma_{23} \\ \text{Sym.} & & \sigma_{33} \end{bmatrix}.$$

Similarly an easy calculation shows that

$$\begin{aligned} \text{var}(Z_{j21}) &= \frac{1}{9\alpha^2 v} \left[3(v-1)^2 + 3v^2 \right] \alpha v \sigma_{11} \\ &= \frac{1}{3\alpha v} (2v^2 - 2v + 1) \sigma_{11}, \end{aligned}$$

and, in general, that

$$(29) \quad \text{var}(Z_{j2}) = \frac{(2v^2 - 2v + 1)}{3\alpha v} \cdot \Sigma^*.$$

Exactly in the same way, we further have

$$\text{cov}(Z_{j1}, Z_{j2}) = \frac{(v-1)}{3\alpha v} \Sigma^*,$$

and for $j \neq j'$

$$\begin{aligned} \text{cov}(Z_{j1}, Z_{j'1}) &= - \frac{1}{3\alpha v^2} \Sigma^* , \\ \text{cov}(Z_{j2}, Z_{j'2}) &= - \frac{(2v^2 - 2v + 1)}{3\alpha v^2} \Sigma^* , \\ \text{cov}(Z_{j1}, Z_{j'2}) &= - \frac{(v-1)}{3\alpha v^2} \Sigma^* . \end{aligned}$$

The above can be written in the compact form

$$\begin{aligned} (30) \quad \text{Var} \begin{bmatrix} Z_{21} \\ \dots \\ Z_{v1} \\ \hline Z_{22} \\ \dots \\ Z_{v2} \end{bmatrix} &= \frac{1}{3\alpha v^2} \begin{bmatrix} 1 & & & & (v-1) \\ & & & & \\ & & & & \\ & & & & \\ (v-1) & & & & (2v^2 - 2v + 1) \end{bmatrix} \otimes \Sigma^* \\ &\otimes \begin{bmatrix} v & -1 & -1 & \dots & -1 \\ & v & -1 & \dots & -1 \\ & & v & \dots & -1 \\ \text{Sym.} & & & \cdot & v \end{bmatrix} \otimes \Sigma^* \\ &= W \otimes \Sigma^* , \text{ say } . \end{aligned}$$

(v-1)x(v-1)

Clearly the two matrices on the r-h-s are positive definite, and so is Σ^* . In order to reduce (30) to the usual model for multivariate analysis of variance (MANOVA), we first factorize as below:

$$W = RR' ,$$

which is possible since W is p.d. Then

$$(31) \quad \text{Var}[\underline{R}^{-1}\underline{Z}] = I_{2(v-1)} \otimes \Sigma^* ,$$

$$(32) \quad \begin{bmatrix} R^{-1}Z \end{bmatrix} = \begin{matrix} R^{-1} \\ 2(v-1) \times 2(v-1) \end{matrix} \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \begin{matrix} \\ 2(v-1) \times (v-1) \end{matrix} \xi_{(v-1) \times p} ,$$

where F_{10} and F_{20} are obtained by cutting out the first row and column of F_1 and F_2 respectively, and ξ_0 is obtained from ξ by cutting out the first row.

We then find the estimate of ξ_0 as

$$(33) \quad \hat{\xi}_0 = \left\{ (F'_{10}, F'_{20}) R^{-1} R^{-1} \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \right\}^{-1} \begin{bmatrix} F'_{10}, F'_{20} \end{bmatrix} R^{-1} R^{-1} Z \\ = \left\{ (F'_{10}, F'_{20}) W^{-1} \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \right\}^{-1} \begin{bmatrix} F'_{10}, F'_{20} \end{bmatrix} W^{-1} Z ,$$

where the matrix inside the curly brackets can be shown to be positive definite.

Now

$$W^{-1} = 3\alpha v^2 \begin{bmatrix} 1 & (v-1) \\ (v-1) & (2v^2 - 2v + 1) \end{bmatrix}^{-1} \otimes \begin{bmatrix} v & -1 & -1 & \dots & -1 \\ & v & -1 & \dots & -1 \\ & & & \ddots & \\ \text{Sym.} & & & & v \end{bmatrix}^{-1} \\ = \frac{3\alpha}{2(v+1)} \begin{bmatrix} (2v^2 - 2v + 1) & -(v-1) \\ -(v-1) & 1 \end{bmatrix} \otimes \begin{bmatrix} 3 & 1 & 1 & \dots & 1 \\ & 3 & 1 & \dots & 1 \\ & & & \ddots & \\ \text{Sym.} & & & & 3 \end{bmatrix} , \text{ and}$$

$$F_{10} = I_{v-1} , \quad F_{20} = J_{v-1, v-1} - I_{v-1} ,$$

and hence a substitution in (33) gives

$$\hat{\xi}_0 = \begin{bmatrix} (1 - \frac{1}{2v})I_{v-1} + v'J_{v-1, v-1} & (1 - \frac{1}{2v})I_{v-1} + v''J_{v-1, v-1} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} ,$$

where v' , v'' are certain constants depending upon the chosen values of σ 's, and on the value of v etc. Thus one can take as an estimate of ξ :

$$(34) \quad \hat{\xi} = \left(1 - \frac{1}{2v}\right) \begin{bmatrix} z_{11} \\ \vdots \\ z_{v1} \end{bmatrix} + \frac{1}{2v} \begin{bmatrix} z_{12} \\ \vdots \\ z_{v2} \end{bmatrix}$$

If one wants to test a linear hypothesis regarding ξ , one can use the standard multiresponse analysis of variance tests. Thus the above method of analysis leads one to a valid test procedure.

In the model (31), consider the nature of Σ^* . This matrix is the same as Σ , except that the covariances in Σ^* are exactly two thirds of those in Σ . This means that if ρ_{ll}^* is a typical correlation obtained from Σ^* , then

$$(35) \quad \rho_{ll}^* \leq \frac{2}{3} .$$

In a situation like the one here, where the absolute values of the correlation have (known) upper limits other than unity, it is not known at present which of the three procedures (viz. largest root test, trace test, and likelihood ratio tests) would have greater power.

3. We shall now present the general formulae for the analysis of a multiresponse design, particularly the general conditions on the I-M design so that this analysis can go through, and various other crucial points in brief. The formulae given here have been reproduced from [4].

Procedure for Analysis

(36) (i) First calculate C_1, C_2, \dots, C_u . Verify whether there exist matrices F_1, F_2, \dots, F_m such that (11) is satisfied, and such that

$$\sum_{q=1}^m F_q = J_{vv}$$

(ii) Next obtain the matrices $L_q L_q'$, ($q, q' = 1, 2, \dots, m$) and hence LL' in terms of α_{iq}' ($i = 1, 2, \dots, u; q = 1, 2, \dots, m$). Choose σ_i ($i = 1, 2, \dots, u$) in such a way that LL' is nonsingular.

(iii) Notice that LL' is nonsingular if and only if Δ_ℓ ($\ell = 1, 2, \dots, p$) are so, where

$$\Delta_\ell = \begin{bmatrix} \sum_{i \in U_\ell} \alpha_{i1}^{\prime 2} & \dots & \sum_{i \in U_\ell} \alpha_{i1}' \alpha_{im}' \\ \dots & \dots & \dots \\ \sum_{i \in U} \alpha_{im}' \alpha_{i1}' & \dots & \sum_{i \in U} \alpha_{im}^{\prime 2} \end{bmatrix},$$

where U_ℓ is the collection of sets (among S_1, S_2, \dots, S_u) on which the ℓ -th response is measured.

$$\pi_\ell = \Delta_\ell^{-1} = \begin{bmatrix} \pi_{11}^\ell & \dots & \pi_{1m}^\ell \\ \dots & \dots & \dots \\ \pi_{m1}^\ell & \dots & \pi_{mm}^\ell \end{bmatrix}, \quad \ell = 1, 2, \dots, p,$$

$$H_{qq'} = \text{diag} (\pi_{qq'}^1, \pi_{qq'}^2, \dots, \pi_{qq'}^p), \quad (q, q' = 1, 2, \dots, m),$$

and observe that

$$(LL')^{-1} = \begin{bmatrix} H_{11} & H_{12} & \dots & H_{1m} \\ \dots & \dots & \dots & \dots \\ H_{m1} & H_{m2} & \dots & H_{mm} \end{bmatrix} = \sqrt{H_1, H_2, \dots, H_m} \\ = H, \text{ say}$$

(iv) Calculate

$$Z_q^* = Q L' H_q = \begin{bmatrix} Z_{1q} \\ Z_{2q} \\ \vdots \\ Z_{vq} \end{bmatrix} \text{ say, } q = 1, 2, \dots, m.$$

and

$$Z_q = \begin{bmatrix} Z_{2q} \\ \vdots \\ Z_{vq} \end{bmatrix}.$$

(v) Calculate the quantities on the l.h.s below and verify whether they can be factorized in the form shown on the r.h.s.

$$(a) \sum_{i \in (U_{\ell'} \cap U_{\ell})} (r_{ij} - \mu_{ijj}) \left(\sum_{q=1}^m \alpha_{iq}^{\ell} \pi_{q'}^{\ell} \right) \left(\sum_{q=1}^m \alpha_{iq}^{\ell} \pi_{qq'}^{\ell} \right) = w_{qq'}^{jj} \cdot \gamma_{\ell\ell'}$$

$$(b) \sum_{i \in (U_{\ell'} \cap U_{\ell})} (-\mu_{ijj'}) \left(\sum_{q'=1}^m \alpha_{iq'}^{\ell} \pi_{q'}^{\ell} \right) \left(\sum_{q=1}^m \alpha_{iq}^{\ell} \pi_{qq'}^{\ell} \right) = w_{qq'}^{jj'} \cdot \gamma_{\ell\ell'}$$

for all $\ell, \ell' = 1, 2, \dots, p$; $q, q' = 1, 2, \dots, m$; $j, j' = 1, 2, \dots, v$.

(vi) Verify whether the $m(v-1) \times m(v-1)$ matrix

$$W = \left((w_{qq'}^{jj'}) \right)$$

is positive definite.

(vii) Also verify whether Z_{jq} ($j = 2, \dots, v$; $q = 1, 2, \dots, m$) are linearly independent. (It should be noted that this also will depend on the choice of σ_1).

(viii) If all the above goes through, then the following holds:

$$E \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_m \end{bmatrix} = \begin{bmatrix} F_{10} \\ \cdot \\ \cdot \\ \cdot \\ F_{m0} \end{bmatrix} \xi_0,$$

where the F_{q0} and ξ_0 are defined as earlier. Also

$$\text{Var} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_m \end{bmatrix} = W \otimes \Sigma^*,$$

where

$$\Sigma^* = ((\sigma_{\ell\ell'}^*))$$

$$\sigma_{\ell\ell'}^* = \gamma_{\ell\ell'}; \sigma_{\ell\ell}; \ell, \ell' = 1, 2, \dots, p.$$

From here on the analysis proceeds as in the example above.

Any multiresponse design for which the above analysis can go through is called regular. It is beyond the scope of this paper to go into the detailed characterization and properties of regular I-M designs, and the new and interesting areas of research they lead to. However, some brief comments on some of these aspects will be offered.

It turns out that for a regular design m should be large. In case the above properties hold, it can be easily checked that

$$(37) \quad mp \leq p_1 + p_2 + \dots + p_u.$$

Hence we should take m as near $\left[\frac{p_1 + p_2 + \dots + p_u}{p} \right]$ as possible, where $\lfloor a \rfloor$ denotes the largest integer less than a .

Notice that except for (36) (ii), we have not put any other condition on the matrices F_1, F_2, \dots, F_m . However in special situations, the proper choice of these matrices may be very important. This choice determines how the different treatments stand relative to each other in the over all design. It will also affect the facility with which the matrices involved in the final estimate of ξ can be inverted. If one chooses F_1, F_2, \dots, F_m as the m association matrices arising out of a (generalized or ordinary) partially balanced association scheme with $(m-1)$ associate classes (though it is not suggested that this should necessarily be done), then one interesting problem for research would be to study PB association schemes with large m , and their use in I-M designs.

However the central problem of I-M designs is to have matrices F_1, F_2, \dots, F_m , the design D_1 , and the choice of σ_i such that (i) not only is (36) satisfied but also that (ii) Σ^* is (in some sense) close to Σ , (iii) the variances of

the treatment contrasts of interest are small and the power of the MANOVA test procedure that we use is relatively large.

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REFERENCES

- [1] Monahan, I. P. (1961). Incomplete-Variable Designs, Unpublished Thesis, V.P.I., Blacksburg, Virginia.
- [2] Roy, S. N. (1957). Some Aspects of Multivariate Analysis, Wiley, New York.
- [3] Roy, S. N. and Srivastava, J. N. (1962). Hierarchical and p-Block Multi-response Designs and Their Analysis, Institute of Statistics, Mimeo Series No.
- [4] Srivastava, J. N. (1962). General Theory of Incomplete Multiresponse Designs, Institute of Statistics, Mimeo Series No.