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HIERARCHICAL AND p-BLOCK MULTIRESPONSE DESIGNS AND THEIR ANALYSIS

by

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This paper introduces two classes of multiresponse designs: (a) hierarchical designs, and (b) designs with p block systems. The former will be useful for those situations where the responses could be arranged in a descending order of importance, so that with respect to any two responses it is known which one needs to be measured on a larger number of experimental units. The latter class is called for when the nature and the pattern of heterogeneity in the experimental material differs from one response to the other. The analysis of such designs together with other properties have also been discussed.

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1. Introductory Remarks on Multiresponse Designs in General

In any planned experiment what we capture first under our sampling scheme is a set of experimental units and according as we study each unit on one or several responses the experiment is called a uniresponse or multiresponse experiment. As already stated elsewhere, the overall objective of the experiment being what it is, namely the study of the relationship between the set of responses and the set of factors and also (in a sense) among the responses themselves, an appropriate design for the allocation of the units has to be made with an eye to this overall objective. Till now, while choosing a design not much consideration has been given to the multiresponse aspect of the experimentation, the choice being usually made as if we had only a single response under study and we wanted to carry out that study in a reasonably efficient manner (defined and described in standard books). Also till now almost all those that have considered the analysis and interpretation of multiresponse experiments have merely taken over such designs, and then assumed that each experimental unit is studied on all responses or characteristics and then carried out the analysis accordingly. (The only exception, known to the authors, is Monahan [1].) However, a little reflection and even a slight acquaintance with experimental situations will indicate that in many such cases it is neither necessary nor even feasible to study each unit on all characteristics. This immediately suggests that every design has two facets, one relative to the treatments or factor-level combinations and the other relative to the variates or responses. For uniresponse

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problems the latter facet, of course, is absent. The first one is concerned with the allocation of the experimental units over the treatments or factor-level combinations but even here the other aspect (relative to variates) may well play an indirect role. In other words, even this allocation may have to be made in different ways according as the overall objective is one of uniresponse or multiresponse study. This paper discusses two classes of multiresponse designs, to be designated as the hierarchical and the p-block designs. For the analysis of these two classes of designs, a step-down procedure has been suggested.

2. General Description of Hierarchical Designs and the Problems Under that Class

Suppose that on consideration of relative cost or difficulty or on other considerations based on a priori information, the different variates or responses have been arranged in a descending order of importance (for operating purposes) such that for any pair of variates it is known which one needs to be measured on a larger number of experimental units. Suppose there are p variates that are named $V_1, V_2, \ldots, V_p$ in the above order, and suppose these are measured respectively on the sets of units $U_1, U_2, \ldots, U_p$ that are not necessarily disjoint. Let U_j consist of n_j experimental units.

The multiresponse design determined by $(U_1, U_2, \ldots, U_p)$ will be called hierarchical if and only if $U_1 \supseteq U_2 \supseteq \cdots \supseteq U_p$. Consider the set U_j. This, as usual, can be divided into blocks through homogeniety or similar considerations for the variate V_j. The allocation of treatments together with the set of blocks over U_j defines a (univariate) design D_j over U_j. The hierarchical multiresponse design will then be denoted by

$$\mathscr{D} = (D_1, D_2, \ldots, D_p) , \qquad (1)$$

under the condition that

(2) $\qquad U_j \supseteq U_{j+1}\,, \quad (j = 1,2,\dots,p-1).$

In set notation define

(3) $\qquad U'_j = U_j - U_{j+1}, \qquad j = 1, 2, \dots, p-1\,,$

$$U'_p = U_p\ .$$

Then for $j = 1, 2, \dots, p$, U'_j represents the set of

(4) $\qquad n'_j = n_j - n_{j+1}\ , \quad (n_{p+1} = 0)\ ,$

experimental units on which only the variates $V_1, V_2, \dots, V_j$ are studied. To delineate the problem and describe the procedure we have to use some further notation as follows. Let $\underline{Z}^j_r$ denote the $n'_j x1$ vector of observations on the r-th variate $(r = 1,2,\dots,j;\ \ j = 1,2,\dots,p)$, that are made on the n'_j units in U'_j. If $\underline{Y}_j(j = 1,2,\dots,p)$ denotes the $n_j x1$ vector of observations on the n_j units studied under the j-th variate, then we can write

(5) $$\underline{Y}_j = \begin{bmatrix} \underline{Z}^j_j \\ \underline{Z}^{j+1}_j \\ \vdots \\ \underline{Z}^p_j \end{bmatrix} .$$

Also let Z^j_{rs} $(j = 1,2,\dots,p;\ r = 1,2,\dots,j;\ \ s = 1,2,\dots,n'_j)$ denote the s-th element of $\underline{Z}^j_r$. On each unit in $U'_j(j = 1,2,\dots,p)$ the j observed values (on the different variates) will be supposed to follow a multivariate normal distribution with a jxj dispersion matrix Σ_j. It is clear, that for all

$j \leq p$, Σ_j is the $(j x j)$ top left hand submatrix of the total $(p x p)$ matrix Σ_p ($\equiv \Sigma$, say), corresponding to all the p variates. Given any single variate V_j, the expected values of the observations on V_j will be supposed to involve the $m_j x 1$ vector of m_j parameters $\underline{\xi}_j$, with components $(\xi_{j1},\ldots,\xi_{jm_j})$ such that

$$(6) \qquad E\left(Z^k_{js}\right) = \underline{a}^{k'}_{js} \quad \underline{\xi}_j ,$$

for all permissible k, j and s(i.e., $j = 1,2,\ldots,k$; $k = 1,2,\ldots,p$; $s = 1,2,\ldots,n'_j$), where $\underline{a}^k_{js}$ is an $m_j x 1$ column vector of known constants given to us by the design $\mathcal{D}_j$.

The null hypothesis H_o on the ξ's is then given by

$$(7) \qquad H_o: \quad \bigcap_{j=1}^{b} \left[\underline{\Phi}_j \equiv C_j \quad \underline{\xi}_j = \underline{0} \right] \equiv \bigcap_{j=1}^{b} H_{oj} \quad \text{(say)} .$$

where C_j is a $s_j x m_j$ matrix, and $\underline{0}$ is an $s_j x 1$ vector. Notice two important points. The first is that the expectation model here is not the same as the usual multivariate model. The second is that, as will be shown later, the attempt to obtain a step-down procedure for testing the H_o of (7), transforms H_o in a natural manner into an H^*_o given by

$$(8) \qquad H^*_o: \quad \bigcap_{j=1}^{p} \left[\underline{\psi}_j \equiv \left(D_j \quad \underline{\theta}_j\right) = \underline{0} \right] = \bigcap_{j=1}^{p} H^*_{oj} \quad \text{(say)},$$

where D_j is a known $s_j x m_j$ matrix, $\underline{0}$ an $s_j x 1$ vector, and $\underline{\theta}_j$ an $m_j x 1$ vector of unknown parameters given by

(9) $$\underline{\theta}_{j+1} = \begin{bmatrix} \underline{\xi}_1^j \\ \underline{\xi}_2^j \\ \vdots \\ \underline{\xi}_j^j \\ \underline{\xi}_{j+1} \end{bmatrix}, \quad j = 0, 1, 2, \ldots, p-1 ;$$

with $\underline{\theta}_1 = \underline{\xi}_1$,

(10) $$\underline{\xi}_r^j = -\underline{\xi}_r \, \beta_{jr} , \quad r = 1, 2, \ldots, j;$$

where β_{jr} is the r-th element of the vector $\underline{\beta}_j$, which is given by

(11) $$\underline{\beta}_j = \Sigma_j^{-1} \begin{bmatrix} \sigma_{1,j+1} \\ \vdots \\ \sigma_{j,j+1} \end{bmatrix}$$

Furthermore, it also turns out that on any unit on which at least up to (j+1) variates have been studied, the (j+1)-th variate, given the others is conditionally distributed as a normal variate with variance given by $|\Sigma_{j+1}|/|\Sigma_j| = \sigma_{j+1}^2$ say, $(j = 0, 1, 2, \ldots, p-1)$, and the conditional expectation of $\underline{Y}_{j+1}$ given by

(12) $$E(\underline{Y}_{j+1} \mid \underline{Y}_1, \ldots, \underline{Y}_j) = B_{j+1} \, \underline{\theta}_{j+1} + Y_j \, \beta_j \text{ (say)} , \quad j = 1, \ldots, p-1 ,$$

where Y_j is a matrix to be defined in the next section in terms of $(\underline{Y}_1, \ldots, \underline{Y}_j)$, and where B_{j+1}, of dimensions $(n_{j+1}) x (\sum_{\ell=1}^{j+1} m_\ell)$, is given by

(13) $$B_{j+1} = \left[\begin{array}{cccc|c} A_1^{j+1} & A_2^{j+1} & \cdots & A_j^{j+1} & A_{j+1}^{j+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \;\cdot\;\cdot \\ A_1^p & A_2^p & & A_j^p & A_{j+1}^p \end{array}\right] ,$$

and the $(n'_k \times m_\ell)$ matrix A^k_ℓ by

$$(14) \qquad A^k_\ell = \begin{bmatrix} (\underline{a}^k_{\ell 1})' \\ (\underline{a}^k_{\ell 2})' \\ (\underline{a}^k_{\ell n'_k})' \end{bmatrix} , \quad (\ell \leq k; \ k = 1, 2, \ldots, p) .$$

For j=o, i.e. at the first stage, we get in place of (12),

$$(15) \qquad E(\underline{Y}_1) = B_1\underline{\theta}_1 = B_1\underline{\xi}_1 = \begin{bmatrix} A^1_1 \\ A^2_1 \\ \vdots \\ A^p_1 \end{bmatrix} \underline{\xi}_1$$

We now have the problem of testing the hypothesis $H^*_o \equiv \bigcap_{j=1}^{p} H^*_{oj}$ of (8) under the model given by (15) and (12) and note that the testability of H_{oj} of (7) is thrown back on that of its equivalent of H^*_{oj}. Assume that H^*_{oj} is testable (the conditions for which will be discussed later), denote by $\hat{\underline{\psi}}_j$ the usual least squares estimate of $\underline{\psi}_j$ of (8) under the model (12), by W_j the the dispersion matrix of $\hat{\underline{\psi}}_j$, by ρ'_j the rank of B_j, by ρ' the rank of $[Y_j : B_j]$ and by τ_j the rank of D_j. Also let s^2_j be the usual error mean square in the least squares estimation at this stage. Then it turns out that under H^*_{oj}

$$(16) \qquad F_j = \frac{(n_j - \rho'_j)}{\tau_j} \times \frac{(\hat{\underline{\psi}}'_j \ W^{-1}_j \ \hat{\underline{\psi}}_j)}{s^2_j} ,$$

has an F-distribution with d.f. τ_j and $(n_j - \rho'_j)$, and that the F_j's (under H^*_o) are independently distributed. The stepdown procedure suggested is as follows:

(17) Accept H_o, i.e., H_o^* if $F_j \leq \mu$, $j = 1, 2, \ldots, p$, and reject otherwise, where μ is given by

(18) $$\prod_{i=1}^{p} \text{Prob} [F_j \leq \mu \mid H_{oj}^*] = 1 - \alpha .$$

In the next section we sketch the mathematical justification for the procedure proposed.

3. A Sketch of the Mathematical Justification of the Procedure Proposed (Assuming Testability of H_{oj}^*'s)

Toward this end we try to find the conditional distribution of $\underline{Y}_{j+1}$, given $\underline{Y}_1, \ldots, \underline{Y}_j$. Put

(19) $$\underline{Y}_{jj'} = \begin{bmatrix} \underline{Z}_j^{j'} \\ \underline{Z}_j^{j'+1} \\ \vdots \\ \underline{Z}_j^{p} \end{bmatrix} , \quad j \leq j' = 1, 2, \ldots, p ,$$

whence $\underline{Y}_{jj} = \underline{Y}_j$, $j = 1, 2, \ldots, p$.

Also we have (in terms of p.d.f.)

(20) $$\text{Prob.} [\underline{Y}_{j+1} \mid \underline{Y}_1, \ldots, \underline{Y}_j] = P[\underline{Y}_{j+1,j+1} \mid \underline{Y}_{1,j+1}, \underline{Y}_{2,j+1}, \ldots, \underline{Y}_{j,j+1}]$$

$$= \frac{P[\underline{Y}_{1,j+1}, \underline{Y}_{2,j+1}, \ldots, \underline{Y}_{j,j+1}, \underline{Y}_{j+1,j+1}]}{P[\underline{Y}_{1,j+1}, \underline{Y}_{2,j+1}, \ldots, \underline{Y}_{j,j+1}]} .$$

To comp te the right side of (20) we observe that the p.d.f. of the vector $(Z_{1s}^{j+1}, Z_{2s}^{j+1}, \ldots, Z_{j+1,s}^{j+1})$ is

(21) $$\text{Const. } \exp\left(\frac{-1}{2}\left[\underline{u}_1' \Sigma_{j+1}^{-1} \underline{u}_1\right]\right),$$

where

(22) $$\underline{u}_1' = \left\{\left[Z_{1s}^{j+1} - (\underline{a}_{1s}^{j+1})' \underline{\xi}_1\right], \left[Z_{2s}^{j+1} - (\underline{a}_{2s}^{j+1})' \underline{\xi}_2\right], \ldots, \left[Z_{j+1,s}^{j+1} - (\underline{a}_{j+1,s}^{j+1})' \underline{\xi}_{j+1}\right]\right\}$$

whence the conditional distribution of $Z_{j+1,s}^{j+1}$, given $Z_{1s}^{j+1}, \ldots, Z_{j,s}^{j+1}$, is

(23) $$\text{const. } \exp\left[\frac{(-1)}{2}\left(\underline{u}_1' \Sigma_{j+1}^{-1} \underline{u}_1 - \underline{u}_2' \Sigma_j^{-1} \underline{u}_2\right)\right],$$

where $\underline{u}_2'$ is obtained by cutting out the last component of $\underline{u}_1'$. It is checked that this conditional distribution is normal with a variance σ_{j+1}^2 equal to $|\Sigma_{j+1}|/|\Sigma_j|$, and (conditional) mean given by

(24) $$E(Z_{j+1,s}^{j+1} | Z_{1s}^{j+1}, Z_{2s}^{j+1}, \ldots, Z_{js}^{j+1}) = E^c(Z_{j+1,s}^{j+1}) \quad \text{(say)}$$

$$= (\underline{a}_{j+1,s}^{j+1})' \underline{\xi}_{j+1} - \left[(\underline{a}_{1s}^{j+1})' \underline{\xi}_1, (\underline{a}_{2s}^{j+1})' \underline{\xi}_2, \ldots, (\underline{a}_{js}^{j+1})' \underline{\xi}_j\right] \underline{\beta}_j$$

$$+ (Z_{1s}^{j+1}, Z_{2s}^{j+1}, \ldots, Z_{js}^{j+1}) \underline{\beta}_j,$$

$\underline{\beta}_j$ being the vector defined earlier (11). Thus we can write

(25) $$E^c(Z_{j+1}^{j+1}) = A_1^{j+1} \underline{\xi}_1^j + A_2^{j+1} \underline{\xi}_2^j + \ldots + A_j^{j+1} \underline{\xi}_j^j + A_{j+1}^{j+1} \underline{\xi}_{j+1}$$

$$+ Z_j^{j+1} \underline{\beta}_j,$$

where A_k^{j+1} and $\underline{\xi}_k^j (k=1,2,\ldots,j)$ have been already defined by (9)-(11) and (14), $\underline{Z}_{j+1}^{j+1}$ is the vector defined by

$$(26) \qquad \underline{Z}_{j+1}^{j+1} = \begin{bmatrix} Z_{11}^{j+1} \\ \cdots \\ Z_{1,n'_{j+1}}^{j+1} \end{bmatrix} \qquad (j = 0, 1, 2, \ldots, p-1) .$$

and where Z_j^k stands for the $n'_k \times j$ matrix

$$(27) \qquad Z_j^k = \begin{bmatrix} Z_{11}^k & Z_{21}^k & \cdots & Z_{j1}^k \\ \cdot & \cdot & \cdots & \cdot \\ Z_{1,n'_k}^k & Z_{2,n'_k}^k & \cdots & Z_{j,n'_k}^k \end{bmatrix} \qquad \begin{matrix} (k=j+1,\ldots,p), \\ (j=1,2,\ldots,p-1). \end{matrix}$$

Hence we have the conditional expectation equation for $\underline{Y}_{j+1}$, given by (12) and it turns out that the Y_j occuring in the right side of (12) is given by

$$(28) \qquad Y_j = \begin{bmatrix} Z_j^{j+1} \\ Z_j^{j+2} \\ \vdots \\ Z_j^{j} \end{bmatrix} .$$

From the above it is easy to check the statements made in Section 2 on the conditional distribution of $\underline{Y}_{j+1}$, given $\underline{Y}_1, \underline{Y}_2, \ldots, \underline{Y}_j$ and also about the independence of the F_j's under H_0^*. For the latter note that although the F_{j+1} itself involves $(\underline{Y}_1,\ldots,\underline{Y}_j)$, its distribution under the null hypothesis is a central F and is independent of $(\underline{Y}_1,\ldots,\underline{Y}_j)$, and hence of $F_1,\ldots, F_j$. The question of the testability of H_{0j}^*'s and eventually of H_0^* and hence of H_0 is one of great interest and will indicate the extent to which the D_j's of (8) and hence the C_j's of (7) might differ for different variates. This will be discussed at some length in section 5 following section 4 which introduces the notion of p-block designs in general (for p variates) and considers, in particular, a hierarchical p-block design. The testability condition for this is the one discussed in section 5, from which follows as a slightly special case what is wanted for the present section. One preliminary remark at this point might be helpful. The testability condition for H^*_{oj} under the model (12) is given by

$$(29) \qquad \text{Rank} \begin{bmatrix} B_j & Y_j \\ D_j & 0 \end{bmatrix} = \text{Rank} \; [B_j \; Y_j] \; .$$

4. Multiresponse Designs with p-block Systems

To fix our ideas, let us start with an example. Suppose we want to experiment on v varieties of wheat, the characteristics under study being the yield and the susceptibility to pests. These two variates are well known to be correlated. Before we proceed further one might raise the question: why study the susceptibility, since the ultimate yield of a variety is all that matters. This has several answers, one being that pest incidence may or may not be uniformly and independently controllable for all varieties. A variety which may be a poor yielder because of being susceptible to a large number of pests, may become the highest yielder when under pest control.

Now suppose that we have $b\,k(=n$, say) experimental units, and suppose that a fertility gradient suggests dividing these into b blocks of k units each. Suppose further that with respect to soil fertility, etc. the blocks are very homogeneous and between-block differences are very large, so that a BIB design (say) with these blocks will be quite efficient. The degrees of freedom for error in the ordinary analysis of variance are n-v-b+1. The power of the F-test for the equality of yields is a decreasing function of b, provided that the other parameters are kept constant. However, the introduction of blocks causes the error variance to drop so much that ordinarily this would offset the decrease in power due to increase of b, and, in fact, would push the power up. The soil fertility has, however, no effect on the pest incidence, and, therefore, the above blocks cannot be expected to decrease the error variance for the pest susceptibility, and therefore their introduction may actually decrease the power of the test relative to that variate. In other words, for the second variate we may use the completely randomized design (with b=1). Thus here we should have two block-systems, the first system consisting of b blocks and the second having just one block.

Multiresponse designs with p block-systems essentially deal with differential heterogeneity in the experimental material with respect to the various characteristics under study. If the appropriate stratification of this material relative to one response is very different from the appropriate stratification relative to another response, then two separate systems of blocks are called for corresponding to these two variates.

The analysis of the above multiresponse design with two block-systems (and in general of designs with p blcok-systems, if there are p responses) though not a special case of the usual model of multivariate linear hypotheses and analysis of variance, can be handled by suitably generalizing the step-down procedure. Instead of discussing the analysis for the particular situation when, on each unit, all the p variables are actually observed, we shall present the same for the more general case where a hierarchical design (in the sense of the last section) is defined over the units.

5. Testability Conditions of Hierarchical Designs with p-Block Systems

Consider the conditions

$$(30) \qquad \text{Rank}\ (B_j) = \text{Rank}\ \binom{B_j}{D_j}\ , \quad j = 1, 2, \ldots, p\ .$$

These have been obtained from (29) by cutting out the parts containing Y_{j-1}, which are stochastic variables. The conditions (30) will hence forth be called the "controllable part" of the testability conditions.

We first examine how far (30) will imply (29). Notice that $(p_j' - p_j)$ independent columns exist in Y_j which are linearly independent of cols. of B_j. Also it is clear that

$$(31) \qquad \text{Rank}\ \begin{pmatrix} B_j & Y_j \\ D_j & 0 \end{pmatrix} \geq \text{Rank}\ (B_j, Y_j)\ .$$

If (30) holds for some j, and (29) does not hold, then in (31) we must have an inequality. Now Y_j consists of n_j vectors which are distributed independently according to a nonsingular j-variate normal distribution. Thus the probability that the j vectors in Y_j will be mutually independent and also linearly independent of the column vectors of B_j, is unity. Hence with probability one, i.e. almost everywhere, we must have

$$(32) \qquad \text{Rank}\ (B_j, Y_j) = j + \text{Rank}\ (B_j), \text{ for all } j.$$

$$\text{Rank}\ \begin{pmatrix} B_j & Y_j \\ D_j & 0 \end{pmatrix} \leq \text{Rank}\ \binom{B_j}{D_j} + \text{Rank}\ \binom{Y_j}{0}$$

$$= \text{Rank}\ (B_j) + j, \text{ a.e.}$$

$$= \text{Rank}\ (B_j, Y_j), \text{ a.e.}$$

Comparing (32) with (31) we thus find that, almost everywhere, the conditions (30) imply (29), a.e.

Hence in our further discussion, we need to consider, only the conditions (30), which represent the controllable part of the testability conditions.

Let us now go back to (13) and write

$$(33) \qquad B_k^j = \begin{bmatrix} A_k^j \\ A_k^{j+1} \\ A_k^p \end{bmatrix} ,$$

for $1 \leq k \leq j$, $j = 1, 2, \ldots, p$. Then for all j,

$$(34) \qquad B_j = (B_1^j, B_2^j, \ldots, B_{j-1}^j, B_j^j) .$$

From (30), the above partitioning of B_j induces a partitioning in D_j, say

$$(35) \qquad D_j = (D_1^j, D_2^j, \ldots, D_j^j)$$

Since our discussion is completely general, we may now assume that corresponding to each variate there exists one system of blocks. The situation where we have just one system of blocks for all variates is evidently a particular case of the above. Similarly for each variate we may have a different set of treatment effects. The vectors of parameters for the j-th variate $\underline{\xi}_j$ can therefore be written

$$(36) \qquad \underline{\xi}_j = \begin{bmatrix} \underline{\xi}_{jb} \\ \underline{\xi}_{jt} \end{bmatrix} , \; j = 1, 2, \ldots p$$

where $\underline{\xi}_{jb}$ (say $m_{jb} \times 1$) is the set of parameters corresponding to the blocks, and $\underline{\xi}_{jt}$ ($m_{jt} \times 1$ say) is the set of parameters representing the treatment effects. We have then

(37) $$m_{jb} + m_{jt} = m_j, \; j = 1, 2, \dots p$$

Corresponding to (36), the vectors $\underline{\xi}_k^j$ ($1 \leq k \leq j$, $j = 1, 2, \dots p$) can then be partitioned in the form

(38) $$\underline{\xi}_k^j = \begin{bmatrix} \underline{\xi}_{kb}^j \\ \underline{\xi}_{kt}^j \end{bmatrix}$$

Now relative to the partitioning (36) and (38) of the vectors $\underline{\xi}_j$ and $\underline{\xi}_k^j$, we have the corresponding partitioning of the matrices B_k^j, this partitioning being obtained from (15). Thus we write, for $1 \leq k \leq j$, $j = 1, 2, \dots p$

(39) $$B_k^j = (B_{kb}^j \; , \; B_{kt}^j)$$

This, in turn, induces a partitioning on the D_k^j ,

(40) $$D_k^j = (D_{kb}^j \; , \; D_{kt}^j)$$

If the hypothesis H_o concerns the treatments only, then so does the hypothesis H^*_o. Hence for $1 \leq k \leq j$, $j = 1, 2, \dots p$, we have

(41) $$D_{kb}^j = 0 \; ,$$

and

$$D_k^j = (0, D_{kt}^j) \; ,$$

where 0 denotes a zero matrix of proper order.

The controllable part of the testability condition can then be written, for $1 \leq j \leq p$,

(42) $$\text{Rank } (B_{1b}^j, B_{2b}^j, \dots, B_{jb}^j, B_{1t}^j, B_{2t}^j, \dots, B_{jt}^j)$$

$$= \text{Rank} \left(\begin{array}{cccc|ccc} B_{1b}^j & B_{2b}^j & \dots & B_{jb}^j & B_{1t}^j & \dots & B_{jt}^j \\ 0 & 0 & \dots & 0 & D_{1t}^j & \dots & D_{jt}^j \end{array} \right)$$

Two cases now arise

Case I For $j = 1, 2, \ldots p$,

$$B^j_{1t} = \ldots = B^j_{jt} = B^j_t \quad \text{say} \tag{43}$$

This is likely to arise in a great majority of experiments, where we have a fixed set of treatments, the nature of the expected response for each individual treatment being exactly the same for each of the p variables under consideration. In fact it appears that up till now in statistical literature, this has been the model assumed for treatment effects in multivariate experiments.

An examination of (42) shows that for testability a.e. under the step-down procedure, we must have

$$D^j_{1t} = D^j_{2t} = \ldots = D^j_{jt} = D^j_t \quad \text{say} \ . \tag{44}$$

Thus we find that under the present model we cannot, at the j-th stage ($j = 2, \ldots p$) test different hypothesis for the variates $V_1, V_2, \ldots, V_j$. The fact that under (43), the j-th stage model (12) degenerates in such a way that for testability the conditions (44) become necessary, will be referred to as the "intrinsic confounding" of the j variables in the j-th stage model.

Let us now consider the effect of the phenomenon of "intrinsic confounding" on the testability of the hypothesis H_o. Because of (38) and (39) H^*_o reduces to

$$D^j_t \, (\underline{\xi}^j_{1t} + \ldots + \underline{\xi}^j_{j-1,t} + \underline{\xi}_{jt}) = 0, \; j = 1, 2, \ldots p \tag{45}$$

But H_o is

$$C_{jt} \, \underline{\xi}_{jt} = 0, \; j = 1, 2, \ldots p \ .$$

Writing in full, H_o^* is

(46)
$$\begin{bmatrix} D_t^1 & 0 & 0 & & 0 & 0 \\ \beta_{21} D_t^2 & D_t^2 & 0 & & 0 & 0 \\ \beta_{31} D_t^3 & \beta_{32} D_t^3 & D_t^3 & \cdot & 0 & 0 \\ & & & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & \cdot & \\ & & & & \cdot & \\ \beta_{p1} D_t^p & \beta_{p2} D_t^p & & & \beta_{p,p-1} D_t^p & D_t^p \end{bmatrix} \begin{bmatrix} \xi_{1t} \\ \xi_{2t} \\ \xi_{3t} \\ \\ \\ \\ \xi_{pt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

while H_o is

(47)
$$\begin{bmatrix} C_{1t} & 0 & \cdots & 0 \\ 0 & C_{2t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & C_{pt} \end{bmatrix} \begin{bmatrix} \xi_{1t} \\ \vdots \\ \xi_{pt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

For any matrix M let V(M) denote the vector space generated by the rows of M. Then a sufficient condition that H_o implies H_o^* is that the vector space generated by the rows of

(48)
$$\begin{bmatrix} D_t^j \\ D_t^{j+1} \\ \cdot \\ \cdot \\ \cdot \\ D_t^p \end{bmatrix}$$

be the same as $V(C_{jt})$. Similarly a sufficient condition that H_o^* implies H_o is that

$$(49)\quad \begin{array}{ll}\text{(i)} & V(D_t^1) \supseteq V(D_t^2) \ \ldots \ \supseteq V(D_t^p) \\ \text{(ii)} & V(D_t^j) = V(C_{jt}), \quad j = 1, 2, \ldots p .\end{array}$$

Combining the two, we find that (49) is a sufficient condition that H_o and H_o^* are equivalent. Since (46) involves β's which are unknown, we do not possess a set of necessary conditions for the equivalence of H_o and H_o^*.

Case II. $B_{1t}^j, B_{2t}^j, \ldots, B_{pt}^j$ are not all equal

An important example of this situation is the following one: Let the suffix t in the above matrices vary over the total number $N(=s_1 x s_2 x \ldots x s_m)$ of treatment combinations of a factorial experiment in which there are m factors, the k-th factor being at level s_k. However though the total number of treatments actually tried in the experiment is N, it may be plausible to assume that certain higher order interactions are zero for each of the p variables involved. A situation where $B_{1t}^j, \ldots, B_{jt}^j$ may not be equal will arise, if we assume for example that for variate V_1 all 2-factor and higher order interactions are zero, others being not negligible, for variate V_2, all 4-factor and higher interactions are zero, and others are not negligible, etc. In such cases even $m_{1t}, m_{2t}, \ldots m_{pt}$ will not be equal.

For testability, condition (42) will have to be verified. In many cases the matrices $B_{1t}^j, \ldots, B_{jt}^j$ may have a part (as for example, corresponding to the main effects) say B_t^{j*} which is common to all of them. This will lead to difficulties of a nature similar to those encountered under Case I.

One remark will be needed here. For certain special cases, the problems under Case I may be handled (and have been handled) by methods available in the present statistical literature using the known designs which are essentially meant for univariate problems. But the problems under Case II are amenable only under the present development. The use of appropriate multiresponse designs is necessary however for all multiresponse experiments that we wish to conduct in an efficient manner.

6. Nature of the Multivariate Designs for an Important Special Case

Consider now the particular case when (47) is satisfied. Then in H_o^*, we can take

$$(50) \qquad D_t^j = C_{jt}, \quad j = 1, 2, \ldots p .$$

For this special case, the conditions (42) reduce to the following, in view of (43):

$$(51) \quad \text{Rank } (B_{1b}^j, B_{2b}^j, \ldots, B_{jb}^j, B_t^j) = \text{Rank } \begin{bmatrix} B_{1b}^j & B_{2b}^j & \cdots & B_{jb}^j & B_t^j \\ 0 & 0 & & 0 & C_{jt} \end{bmatrix}$$

This shows that at the j-th stage, the design behaves as if there are

$$(52) \qquad m_{1b} + m_{2b} + \cdots + m_{jb}$$

block effects. If there is only one viz. the j-th variable, in the picture, then the blocks entering into the analysis would have been those which correspond to B_{jb}^j. This shows that our method of analysis introduces in effect a larger number of blocks at the j-th stage (for all j) than are present there from the consideration of the j-th variable alone. Thus we have effectively a larger amount of latitude now. For example, let us consider a matrix C_{jt} having $m_{jt}-1$ linearly independent contrasts. If we considered the j-th variable only, ignoring the rest, then the condition for testability would have been

$$\text{Rank}\quad B^{j}_{jb},\ B^{j}_{t} \quad = \quad \text{Rank}\begin{pmatrix} B^{j}_{jb} & B^{j}_{t} \\ 0 & C_{jt} \end{pmatrix} \qquad , \tag{53}$$

or equivalently that the j-th stage design should be connected. Comparing (53) with (51), we find that the connectedness is needed now in terms of a much larger number of block effects. However, we should still expect the over-all design to be quite complicated, since now we have j blocks passing through each unit. Further the concept of connectedness will have to be redefined in terms of j systems of blocks in contrast to the present definition of connectedness in terms of just one system of blocks.

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