UNIVERSITY OF NORTH CAROLINA
Department of Statistics
Chapel Hill, N. C.

SOME LIMIT DISTRIBUTIONS CONNECTED WITH FIXED INTERVAL ANALYSIS

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J. Sethuraman

Indian Statistical Institute and University of North Carolina

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SOME LIMIT DISTRIBUTIONS CONNECTED WITH FIXED INTERVAL ANALYSIS

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J. Sethuraman[1]

UNIVERSITY OF NORTH CAROLINA and

INDIAN STATISTICAL INSTITUTE

Summary: The proofs of some theorems (stated in Sethuraman (1963)) on the limiting distributions of some statistics that enter in the method of Fixed Interval Analysis are presented.

1. Introduction

Let $(\underset{\sim}{Y}, X)$ be a random variable taking values in $(\mathcal{Y} \times \mathcal{X})$ where $\mathcal{Y}$ is E_k the Euclidean space of k dimensions and $\mathcal{X}$ is a measurable space. Let $E_1, E_2, \ldots, E_g$ be g disjoint measurable sets in $\mathcal{X}$ whose union is the whole space $\mathcal{X}$.

$(\underset{\sim}{y}_1, x_1), (\underset{\sim}{y}_2, x_2), \ldots, (\underset{\sim}{y}_n, x_n)$ are n independent observations on $(\underset{\sim}{Y}, X)$. The number of x_i's that fall in E_j is n_j, $j = 1, \ldots, g$. $\underset{\sim}{u}_j$ is defined by the relation

$$\underset{\sim}{u}_j = \Sigma^j \underset{\sim}{y}_i / n_j \qquad j = 1, \ldots g$$

where Σ^j is the summation over all "i" such that x_i is in E_j.

Throughout this paper it is assumed that

$$V(\underset{\sim}{Y}) < \infty \tag{1}$$

and

$$\text{Prob. } (X \in E_j) = \pi_j > 0 \qquad j = 1, \ldots g \tag{2}$$

where for any random variable $\underset{\sim}{Z}$, $v(\underset{\sim}{Z})$ denotes the variance covariance matrix of $\underset{\sim}{Z}$.

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The following theorem is established in section 3.

Theorem 1. The asymptotic distribution of $(\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_g)$ is the distribution of g independent normal distributions.

This theorem plays a fundamental role in the method of Fixed Interval Analysis, for instance see Sethuraman (1963). Interpreted in Sample Survey language this theorem, among other things, states that the post-stratified stratum means are independently distributed in the limit.

2. Notations, definitions and preliminaries.

Let $\underset{\sim}{Y}(E_j)$, called the conditional random variable of $\underset{\sim}{Y}$ given that X is in E_j, denote a random variable on $\mathcal{Y}$ with the distribution defined by Prob. $(\underset{\sim}{Y}(E_j) \in A) =$ Prob. $(\underset{\sim}{Y} \in A, X \in E_j)/$Prob. $(X \in E_j)$. For any random variable $\underset{\sim}{Z}$, $E(\underset{\sim}{Z})$ denotes the vector of expectations of $\underset{\sim}{Z}$. Define

$$E(\underset{\sim}{Y}(E_j)) = \underset{\sim}{\mu}_j \tag{3}$$

$$V(\underset{\sim}{Y}(E_j)) = \Sigma_j \tag{4}$$

$$p_j = n_j/n \tag{5}$$

$$\sqrt{n}\,(\underset{\sim}{u}_j - \underset{\sim}{\mu}_j) = \underset{\sim}{\eta}_j(n) \tag{6}$$

$$\sqrt{n}\,(\underset{\sim}{p}_j - \underset{\sim}{\pi}_j) = \underset{\sim}{\zeta}_j(n) \tag{7}$$

$$j = 1, \ldots g .$$

Let $\{ \xi_n(\cdot\, , \theta)\}$, $n = 0, 1, \ldots$ be a sequence of families of probability distributions on the Borelsubsets of E_m (or more generally, of any topological space) and θ vary in a compact topological space K.

Definition. $\{ \xi_n(\cdot\, , \theta)\}$ is said to converge weakly, uniformly and continuously (in other words, in the UC* sense) to $\xi_o(\cdot\, , \theta)$ with respect to θ in K

if for every bounded continuous function $h(y)$ on E_m

$$\int g(y) \quad \xi_n(dy, \theta) \dashrightarrow \int g(y)\, \xi_o(dy, \theta) \text{ uniformly in } \theta$$

and

$$\int g(y) \quad \xi_o(dy, \theta) \text{ is a continuous function of } \theta .$$

The following theorem found in Sethuraman (1961) will be used in section 3.

Theorem 2. Let (Y_n, X_n) be a sequence of random variables on $(E_m \times S)$ where S is a complete separable metric space. Let the conditional probability measure of Y_n given that $X_n = x$ be denoted by $\xi_n(\cdot, x)$ and the marginal distribution of X_n be μ_n. Let $\xi_n(\cdot, x)$ converge in the UC^* sense to $\xi_o(\cdot, x)$ with respect to x in any compact subset of S and μ_n converge weakly to μ_o. Then the joint distribution of (Y_n, X_n) converges weakly to the distribution determined by $\xi_o(\cdot, x)$ and μ_o or, more precisely, to the distribution of (Y_o, X_o) where

$$\text{Prob.} \left\{ Y_o \in A, \quad X_o \in B \right\} = \int_B \xi_o(A,x)\, \mu(dx) .$$

The following lemma, which is immediate, is useful in establishing the UC^* convergence of a special sequence of families of distributions

Let

$$\begin{array}{l} \underset{\sim}{Z}_{11}, \ldots\ldots, \underset{\sim}{Z}_{1k_1}(\theta) \\ \underset{\sim}{Z}_{21}, \ldots\ldots\ldots\ldots, \underset{\sim}{Z}_{2k_2}(\theta) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \underset{\sim}{Z}_{n1}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \underset{\sim}{Z}_{nk_n}(\theta) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

be a triangular scheme of random variables in E_m where the variables in any row are identically and independently distributed. Assume that $E(\underset{\sim}{Z}_{n1}) = \underset{\sim}{\nu}_n$ and $V(\underset{\sim}{Z}_{n1}) = V_n$ are finite and that $V_n \longrightarrow V$ as $n \longrightarrow \infty$. Again let

$\inf_{\theta} k_n(\theta) \longrightarrow \infty$ as $n \longrightarrow \infty$.

Let $MN(\underset{\sim}{\alpha}, L)$ stand for the multivariate normal distribution with mean vector $\underset{\sim}{\alpha}$ and variance covariance matrix L.

Lemma 1. The sequence of families of distributions of $\left\{ (\underset{\sim}{Z}_{n_1} + \dots + \underset{\sim}{Z}_{nk_n(\theta)} - k_n(\theta)\, \underset{\sim}{\nu}_n)/\sqrt{k_n(\theta)} \right\}$ converges in the UC^* sense to the distribution $MN(0, V)$.

3. Main theorems.

We first prove the following lemma.

Lemma 2. The distributions of $(\underset{\sim}{\eta}_1(n), \dots \underset{\sim}{\eta}_g(n))$ given that $\underset{\sim}{\zeta}(n) = \underset{\sim}{z}$, $\sum_1^g z_i = 0$ converges in the UC^* sense to the distribution $MN(0, \Lambda)$ with respect to z in any closed bounded subset of E_g, where

$$\Lambda = \begin{pmatrix} \frac{1}{\pi_1}\Sigma_1 & 0 & & 0 \\ 0 & \frac{1}{\pi_2}\Sigma_2 & & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & \frac{1}{\pi_g}\Sigma_g \end{pmatrix} \tag{8}$$

Proof: The event $\underset{\sim}{\zeta}(n) = \underset{\sim}{z}$ is equivalent with probability one to the event $n_i = [n\,\pi_i + \sqrt{n}\, z_i]$ $i = 1, \dots, g$, since

$$\text{Prob.} \left\{ n\pi_i + \sqrt{n}\,\zeta_i(n) = [n\pi_i + \sqrt{n}\,\zeta_i(n)],\ i = 1, \dots, g \right\} = 1.$$

The conditional distribution of $\underset{\sim}{y}_1, \dots, \underset{\sim}{y}_n$ given that $n_i = [n\pi_i + \sqrt{n}\, z_i]$, $i = 1, \dots, g$ is the distribution of g independent samples of size $n_1, \dots, n_g$ on $\underset{\sim}{Y}(E_1), \dots, \underset{\sim}{Y}(E_g)$, respectively. $\sqrt{\frac{n_1}{n}}\, \underset{\sim}{\eta}_1(n), \dots, \sqrt{\frac{n_g}{n}}\, \underset{\sim}{\eta}_g(n)$ are the normalized means of these g independent samples. For $\underset{\sim}{z}$ in a closed bounded subset of E_g we note that $\inf_{i,\underset{\sim}{z}} [n\pi_i + \sqrt{n}\, z_i] \longrightarrow \infty$ as $n \longrightarrow \infty$. Thus all the conditions of lemma 1 are satisfied. Further $[n\pi_i + \sqrt{n}\, z_i]/n$ tends to π_i, uniformly in $\underset{\sim}{z}$ in any closed bounded subset of E_g. Hence the conditional distributions of $(\underset{\sim}{\eta}_1(n), \ldots, \underset{\sim}{\eta}_g(n))$ given that $\underset{\sim}{\zeta}(n) = \underset{\sim}{z}$ converges in the UC^* sense to the distribution $MN(0, \Lambda)$ with respect to $\underset{\sim}{z}$ in any closed bounded subset of E_g.

Theorem 3. The joint distribution of $(\underset{\sim}{\eta}_1(n), \ldots, \underset{\sim}{\eta}_g(n), \underset{\sim}{\zeta}(n))$ converges weakly to the distribution $MN(0, B)$ where

$$B = \left(\begin{array}{c|c} \Lambda & 0 \\ \hline 0 & C \end{array}\right) \tag{9}$$

where

$$C = \begin{pmatrix} \pi_1(1-\pi_2) & -\pi_1\pi_2 & \cdots & -\pi_1\pi_g \\ -\pi_1\pi_2 & \pi_2(1-\pi_2) & \cdots & -\pi_2\pi_g \\ \cdot & \cdot & \cdots & \cdot \\ -\pi_1\pi_g & -\pi_2\pi_g & \cdots & \pi_g(1-\pi_g) \end{pmatrix}. \tag{10}$$

Proof: This theorem is an immediate consequence of Theorem 2, lemma 2 and the observation that the distribution of $\underset{\sim}{\zeta}(n)$ converges weakly to the distribution $MN(0, C)$.

Proof of theorem 1: Theorem 1 is contained in theorem 3.

4. References

1. Sethuraman, J. (1961). Some limit theorems for joint distributions, Sankhyā, Series A, 23, 379-386.
2. Sethuraman, J. (1963). Fixed Interval Analysis and Fractile Analysis, submitted to the Mahalanobis Birthday Volume.