QUOTA FULFILMENT IN FINITE POPULATIONS

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In choosing a sample it can happen that extra expense in involved if selection has to be made from a restricted section ('stratum') of a population. If a sample containing specified numbers of individuals from each stratum is required it may then be more economical to sample in two stages, the first sample being chosen at random from the whole population, and the second arranged to complete the sample so that it has the desired constitution. An earlier paper discussed the choice of size of first sample assuming that the population size is effectively infinite. This report extends the results to populations of finite size.

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QUOTA FULFILMENT IN FINITE POPULATIONS[1]

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1. Introduction.

Johnson [1] has discussed the choice of size (N) of an initial random sample from a population divided into k strata $\Pi_1, \Pi_2, \ldots, \Pi_k$, when the final aim is to obtain a sample containing specified numbers $m_1, m_2, \ldots, m_k$ respectively of individuals from these k strata $(\Sigma_{i=1}^k m_i = m_o)$. In [1] it was assumed that the strata sizes were effectively infinite (though in known ratios). In the present paper the same problem will be discussed for the case when the numbers of individuals in the strata $\Pi_1, \Pi_2, \ldots, \Pi_k$ are known to be $M_1, M_2, \ldots, M_k$ respectively. $(\Sigma_{i=1}^k M_i = M_o.)$

2. Optimum Size of Initial Sample.

As in [1], c will denote the cost per individual of an unrestricted random (without replacement) sample; c_i will denote the cost per individual of a random (without replacement) sample from stratum Π_i; and c_i' will denote the value (if any) of each excess individual beyond the quota requirement (m_i) from stratum Π_i.

Then, using the same approach as in [1], consider the expected net cost of increasing the size of the first (unrestricted) sample from N to (N + 1). If the first sample (of size N) contains $n_1, n_2, \ldots, n_k$ individuals from $\Pi_1, \Pi_2, \ldots, \Pi_k$ respectively (with, of course, $\Sigma_{i=1}^k n_i = N$), then the conditional increase in the expected net cost arising from the extra individual increasing the sample size from N to N + 1 is

[1]This research was supported by the Mathematics Division of the Air Force Office of Scientific Research.

$$(1) \qquad \Delta C(n_1, n_2, \ldots, n_k) = c - \sum_{n_i < m_i} \left(\frac{M_i - n_i}{M_o - N} \right) \cdot c_i - \sum_{n_i \geq m_i} \left(\frac{M_i - n_i}{M_o - N} \right) \cdot c_i' \; .$$

The (unconditional) increase in expected net cost is obtained by taking expected values with respect to the n_i's, giving

$$(2) \qquad \Delta C = c - \sum_{i=1}^{k} c_i \, \mathcal{E} \left[\frac{M_i - n_i}{M_o - N} \;\middle|\; n_i < m_i \right] \cdot \Pr \left[n_i < m_i \right]$$

$$- \sum_{i=1}^{k} c_i' \, \mathcal{E} \left[\frac{M_i - n_i}{M_o - N} \;\middle|\; n_i \geq m_i \right] \cdot \Pr \left[n_i \geq m_i \right]$$

where
$$\Pr\left[n_i < m_i\right] = \binom{M_o}{N}^{-1} \sum_{n_i = o}^{m_i - 1} \binom{M_i}{n_i} \cdot \binom{M_o - M_i}{N - n_i}$$

$$= \sum_{n_i = o}^{m_i - 1} p(M_o, N, M_i, n_i)$$

$$= P(M_o, N, M_i, m_i - 1)$$

using the notation of Lieberman and Owen [2].

Also

$$(3) \qquad \mathcal{E}\left[n_i \mid n_i < m_i\right] \Pr\left[n_i < m_i\right] = \sum_{n_i = o}^{m_i - 1} n_i p(M_o, N, M_i, n_i)$$

$$= \frac{NM_i}{M_o} \cdot P(M_i - 1, N - 1, M_i - 1, m_i - 2)$$

and

$$\mathcal{E}\left[n_i \mid n_i < m_i\right] \Pr\left[n_i < m_i\right] + \mathcal{E}\left[n_i \mid n_i \geq m_i\right] \Pr\left[n_i \geq m_i\right] = \mathcal{E}(n_i)$$

$$= NM_i/M_o \; .$$

Hence equation (2) can be written

$$\Delta C = c - (M_o - N)^{-1} \left[\sum_{i=1}^{k} c_i \left\{ M_i P(M_o,N,M_i,m_i-1) - (NM_i/M_o)P(M_o-1,N-1,M_i-1,m_i-2) \right\} \right.$$

$$\left. + \sum_{i=1}^{k} c_i' \left\{ M_i \left[1-p(M_o,N,M_i,m_i-1)\right] - (NM_i/M_o) \left[1-p(M_o-1,N-1,M_i-1,m_i-2)\right] \right\} \right]$$

$$= c - M_o^{-1} \sum_{i=1}^{k} M_i c_i' + \sum_{i=1}^{k} M_i (c_i - c_i') \left\{ P(M_o,N,M_i,m_i-1) - (N/M_o)P(M_o-1,N-1,M_i-1,m_i-2) \right\}$$

This can be further simplified by noting that

$$P(M_o,N,M_i,m_i-1) - (N/M_o)P(M_o-1,N-1,M_i-1,m_i-2)$$

$$= \binom{M_o}{N}^{-1} \left[\sum_{j=o}^{m_i-1} \binom{M_i}{j} \cdot \binom{M_o - M_i}{N - j} - \sum_{j=o}^{m_i-2} \binom{M_i-1}{j} \cdot \binom{M_o-M_i}{N-1-j} \right]$$

$$= \binom{M_o}{N}^{-1} \left[\binom{M_o-M_i}{N} + \sum_{j=1}^{m_i-1} \binom{M_o-M_i}{N-j} \left\{ \binom{M_i}{j} - \binom{M_i-1}{j-1} \right\} \right]$$

$$= \frac{M_o-N}{M_o} \cdot \binom{M_o-1}{N}^{-1} \left[\binom{M_i-1}{0} \binom{M_o-M_i}{N} + \sum_{j=1}^{m_i-1} \binom{M_i-1}{j} \binom{M_o-M_i}{N-j} \right]$$

(4) $$= (1 - N/M_o)P(M_o-1,N,M_i-1,m_i-1) \quad .$$

(This formula can also be established by considerations of probability.)

Using this relationship, the following expression for increase in expected net cost is obtained

(5) $$\Delta C = c - M_o^{-1} \sum_{i+1}^{k} M_i c_i' - M_o^{-1} \sum_{i=1}^{k} M_i (c_i - c_i') P(M_o-1,N,M_i-1,m_i-1) \quad .$$

For small values of N, ΔC is negative. As N increases, so does ΔC. The optimal value of N is the least value of N for which $\Delta C \geq 0$. This will, approximately, be the solution of the equation

(6) $$c - M_o^{-1} \sum_{i=1}^{k} M_i c_i' = M_o^{-1} \sum_{i=1}^{k} M_i (c_i - c_i') P(M_o - 1, N, M_i - 1, m_i - 1) \quad .$$

Two special cases of (6) will now be enumerated. If $c_i/c = d$ and $c_i'/c = d'$ for all i, then (6) becomes

(7) $$(1-d')/(d-d') = M_o^{-1} \sum_{i=1}^{k} M_i P(M_o - 1, N, M_i - 1, m_i - 1) \quad .$$

If, further, $M_i = M_o/k$, $m_i = m_o/k$ for all i, then

(8) $$(1-d')/(d-d') = P(M_o - 1, N, (M_o/k) - 1, (m_o/k) - 1) \quad .$$

A few optimum values of N obtained from (8) are given in Table 1. These

Table 1. Optimum Values of Initial Sample Size

k	m_o	M_o	d' \ d	1.25	1.5	2.0	2.5	3.0
2	50	100	0.9	53	55	57	58	59
			0.7	50	52	54	55	56
			0.25	47	49	51	53	54
			0.0	46	48	50	52	53
5	50	100	0.9	56	60	64	66	67
			0.7	49	54	58	60	62
			0.25	44	48	52	55	56
			0.0	42	46	50	53	55
10	50	100	0.9	59	65	70	73	74
			0.7	49	55	61	65	67
			0.25	40	46	53	57	59
			0.0	37	44	50	54	57

were obtained using [2]. They do not cover the same range of values of m_o, d and d' as Table 1 of [1]. However, the following approximate analysis shows how usefully accurate values of N can be obtained using the tables given in [1].

3. Some Approximations

To obtain an approximate explicit expression for N, use may be made of Wise's [3] approximation to the hypergeometric function

$$(9) \quad P(M, n, M', m) \doteq I_h\,(n-m,\, m+1)$$

where $h = 1 - \dfrac{M' - \frac{1}{2} m}{M + \frac{1}{2} - \frac{1}{2} n}$ and $I_p(r,s) = [B(r,s)]^{-1} \displaystyle\int_0^p t^{r-1}(1-t)^{s-1}\,dt$

is the incomplete Beta function ratio.

Using the approximation (9) in (8) leads to the equation

$$(10) \qquad 1 - I_{1-h}\left(\frac{m_o}{k},\, N - \frac{m_o}{k} + 1\right) \doteq \frac{1 - d'}{d - d'}$$

where

$$1 - h = \left[\frac{M_o}{k} - \frac{1}{2} - \frac{m_o}{2k}\right]\left[M_o - \frac{1}{2} - \frac{N}{2}\right]^{-1}.$$

Equation (10) can be expressed in the form

$$(11) \qquad \Pr\left[x < \frac{m_o}{k}\right] \doteq \frac{1 - d'}{d - d'}$$

where x is a binomial variable with parameters

$$(12) \quad \omega(M_o, N, m_o, k) = \left[\frac{M_o - \frac{1}{2} m_o}{k} - \frac{1}{2}\right]\left[M_o - \frac{1}{2} N - \frac{1}{2}\right]^{-1}, \text{ and } N.$$

Equation (11) differs from equation (8) in [1] (which does not use an approximation) only in that ω replaces k^{-1} as one binomial parameter. Note that if $N = m_o + k - 1$ then the two equations lead to identical results, so that when in [1] it is found that

(13) Optimal size of initial sample = (Size of final sample) +(Number of Strata) -1

the same size of initial sample will approximately be optimal whatever be the size, M_o, of the population.

Approximating the binomial distribution, in turn, by a normal distribution, the following approximate formula for N is obtained:

$$\frac{m_o}{k} - \frac{1}{2} - N\omega \doteq \lambda\sqrt{N\omega(1-\omega)}$$

where

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-\frac{1}{2}u^2} du = \frac{1-d'}{d-d'}$$

and ω is defined by (12).

Writing this equation explicitly in terms of N it can be arranged in the form

$$(14) \quad (m_o - \tfrac{1}{2}k)(M_o - \tfrac{1}{2}) - N(M_o - \tfrac{3k}{4}) \doteq \lambda\sqrt{N(M_o - \tfrac{m_o}{2} - \tfrac{k}{2})(M_o(k-1) - \tfrac{1}{2}(kN - m_o))} \, .$$

Neglecting terms of order m_o/M_o and assuming that N will be roughly of order m_o, (14) can be written

$$(15) \quad m_o - \tfrac{1}{2}k - N \doteq \lambda(1 - \frac{m_o}{2M_o})\sqrt{N(k-1)} \, .$$

This is the same as equation (9) in [1], with λ replaced by $\lambda(1 - \frac{m_o}{2M_o})$, suggesting that the approximation

$$N \sim m_o - \tfrac{1}{2}k - \lambda(1 - \frac{m_o}{2M_o})\sqrt{(k-1)(m_o - \tfrac{1}{2}k)} + \tfrac{1}{2}\lambda^2(1 - \frac{m_o}{2M_o})^2 (k-1)^2$$

might be appropriate.

The two conditions

(a) if $N \sim m_o + k - 1$, N does not vary much with changes in M_o (see (12) above), and

(b) if $\lambda = 0$, $N \sim (m_o - \frac{1}{2}k)(M_o - \frac{1}{2})(M_o - \frac{3k}{4})^{-1}$ (see (13) above)

suggest the slightly modified form

$$(16)\quad N \sim m_o - \frac{1}{2}k + \frac{m_o}{2M_o}\left(\frac{3k}{2} - 1\right) - \lambda\left(1 - \frac{m_o}{2M_o}\right)\sqrt{(k-1)(m_o - \frac{1}{2}k)} + \frac{1}{2}\lambda^2\left(1 - \frac{m_o}{2M_o}\right)(k-1)^2 .$$

So a first approximation to N can be obtained from the infinite population value with

$$\frac{1-d'}{d-d'} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda(1-m_o/2M_o)} e^{-\frac{1}{2}u^2}\,du .$$

This value can then be adjusted by adding

$$\frac{m_o}{2M_o}\left[\frac{3k}{2} - 1 + \frac{1}{2}\lambda^2\left(1 - \frac{m_o}{2M_o}\right)(k-1)^2\right] .$$

Some explicit values of this correction, for different values of m_o/M_o are shown in Table 2.

TABLE 2: Corrective additions to approximate optimum value of N

m_o/M_o	Correction
1/2	$0.375k - 0.25 + 0.094\,\lambda^2(k-1)^2$
1/3	$0.25k - 0.17 + 0.070\,\lambda^2(k-1)^2$
1/4	$0.19k - 0.125 + 0.055\,\lambda^2(k-1)^2$
1/5	$0.15k - 0.1 + 0.045\,\lambda^2(k-1)^2$
1/6	$0.125k - 0.08 + 0.038\,\lambda^2(k-1)^2$
1/8	$0.094k - 0.06 + 0.029\,\lambda^2(k-1)^2$
1/10	$0.075k - 0.05 + 0.024\,\lambda^2(k-1)^2$

Nominal values of $(1-d')/(d-d')$ corresponding to actual values of $(1-d')/(d-d')$ and various values of the sampling fraction m_o/M_o are shown in Table 3.

TABLE 3. Nominal values of $(1-d')/(d-d')$

Actual Value of $(1-d')/(d-d')$	m_o/M_o = 0.6	0.5	0.4	0.3	0.2	0.1
0.05	0.13	0.11	0.09	0.08	0.07	0.06
0.1	0.19	0.17	0.15	0.14	0.13	0.11
0.15	0.23	0.22	0.20	0.19	0.18	0.16
0.2	0.27	0.26	0.25	0.24	0.23	0.21
0.25	0.32	0.30	0.29	0.28	0.27	0.26
0.3	0.36	0.35	0.34	0.33	0.32	0.31
0.35	0.39	0.39	0.38	0.37	0.37	0.36
0.4	0.43	0.42	0.42	0.41	0.41	0.40
0.45	0.47	0.46	0.46	0.46	0.46	0.45
0.5	0.50	0.50	0.50	0.50	0.50	0.50

[For $(1-d')/(d-d')$ greater than 0.5, enter the table with $1 - (1-d')/(d-d')$ and use $1 -$ (tabulated nominal value of $(1-d')/(d-d')$).]

As an example of the use of this table, suppose $d = 3$, $d' = 0.5$ and $m_o/M_o = 0.4$. Then $(1-d')/(d-d') = 0.2$ but we use the nominal value 0.25. This could be done, for example, by entering interpolating in Table 1 of [1] with $d = 3$, $d' = 1/3$.

4. Comparison of Expected Net Costs

The expected net cost is

$$(17)\qquad C = Nc + \sum_{i=1}^{k} \Big[c_i \mathcal{E}\{m_i - n_i \,|\, n_i < m_i\} \Pr[n_i < m_i]$$

$$- c_i' \mathcal{E}\{n_i - m_i \,|\, n_i \geq m_i\} \Pr[n_i \geq m_i] \Big] .$$

Since

$$\mathcal{E}\{m_i - n_i | n_i < m_i\} \Pr[n_i < m_i] - \mathcal{E}\{n_i - m_i | n_i \geq m_i\} \Pr[n_i \geq m_i]$$

$$= \mathcal{E}\{m_i - n_i\} = m_i - N\,M_i/M_o$$

equation (17) can be written

$$C = Nc + \sum_{i=1}^{k} \Big[(c_i - c_i')\mathcal{E}\{m_i - n_i | n_i < m_i\} \Pr[n_i < m_i] + c_i'(m_i - N\,M_i/M_o)\Big]$$

$$= N(c - M_o^{-1}\sum_{i=1}^{k} M_i c_i') + \sum_{i=1}^{k} m_i c_i' + \sum_{i=1}^{k} (c_i - c_i')\Big[m_i P(M_o, N, M_i, m_i - 1)$$

$$- (N\,M_i/M_o) P(M_o - 1, N - 1, M_i - 1, m_i - 2)\Big]$$

Using (4) this can be expressed as

$$C = N(c - M_o^{-1}\sum_{i=1}^{k} M_i c_i') + \sum_{i=1}^{k} m_i c_i' + \sum_{i=1}^{k} (c_i - c_i')\Big[M_i(1 - N/M_o) P(M_o - 1, N, M_i - 1, m_i - 1)$$

$$- (M_i - m_i) P(M_o, N, M_i, m_i - 1)\Big] .$$

If N satisfies (6) then

$$C = M_o(c - M_o^{-1}\sum_{i=1}^{k} M_i c_i') + \sum_{i=1}^{k} m_i c_i' - \sum_{i=1}^{k} (M_i - m_i)(c_i - c_i') P(M_o, N, M_i, m_i - 1)$$

$$(18)\qquad = M_o c - \sum_{i=1}^{k} (M_i - m_i)\Big[c_i P(M_o, N, M_i, m_i - 1) + c_i'\{1 - P(M_o, N, M_i, m_i - 1)\}\Big]$$

If $c_i/c = d$ and $c_i'/c = d'$, (18) becomes

(19) $$C = c\left[M_o - \sum_{i=1}^{k} (M_i - m_i)\left[d\, P(M_o,N,M_i,m_i-1) + d'\left\{1 - P(M_o,N,M_i,m_i-1)\right\}\right]\right]$$

If, further, $M_i = M_o/k$ and $m_i = m_o/k$ then we have

$$C = c\left[M_o - (M_o - m_o)\left[d\, P(M_o,N,(\frac{M_o}{k}, \frac{m_o}{k} - 1) + d'\left\{1 - P(M_o,N,\frac{M_o}{k}, \frac{m_o}{k} - 1)\right\}\right]\right]$$

$$= c\left[m_o + (M_o - m_o)\left\{1 - d' - (d-d')P(M_o,N,\frac{M_o}{k}, \frac{m_o}{k} - 1)\right\}\right]$$

(20) $$= c\left[m_o + (M_o - m_o)(d-d')\left\{P(M_o-1,N,\frac{M_o}{k} - 1, \frac{m_o}{k} - 1) - P(M_o,N,\frac{M_o}{k}, \frac{m_o}{k} - 1)\right\}\right]$$

The approximation (11) to $P(M_o,N,\frac{M_o}{k}, \frac{m_o}{k} - 1)$ is the same as the approximation to $P(M_o - 1, N, \frac{M_o}{k} - 1, \frac{m_o}{k} - 1)$, but with ω increased by

$$\frac{M_o - \frac{M_o}{k}}{(M_o - \frac{1}{2}N - \frac{1}{2})(M_o - \frac{1}{2}N + \frac{1}{2})} \doteq \frac{M_o(k-1)}{k(M_o - \frac{1}{2}N)^2} .$$

So

$$P(M_o-1,N,\frac{M_o}{k} - 1, \frac{m_o}{k} - 1) - P(M_o, N, \frac{M_o}{k}, \frac{m_o}{k} - 1)$$

$$\doteq \frac{M_o(k-1)}{k(M_o - \frac{1}{2}N)^2} \; \frac{N!}{(\frac{m_o}{k} - 1)!(N - \frac{m_o}{k})!} \; \omega^{m_o/k-1}(1-\omega)^{N-m_o/k}$$

with ω given by (12).

Inserting this approximation in (20) we find

$$(21)\quad C = c\left[m_o + (M_o - m_o)(d-d')(k-1)k^{-2} m_o M_o \left(\frac{M_o - \frac{1}{2} m_o}{k} - \frac{1}{2}\right)^{-1} \left(M_o - \frac{1}{2} N + \frac{1}{2}\right)^{-1} \cdot \binom{N}{m_o/k} \omega^{m_o/k} (1-\omega)^{N-m_o/k} \right].$$

The ratio of C to the cost of sampling separately from each stratum ($m_o\, c\, d$) is

$$(22)\quad \frac{1}{d} + \left(1 - \frac{d'}{d}\right)\left(1 - \frac{m_o}{M_o}\right)\left(1 - \frac{m_o + k}{2M_o}\right)^{-1} \left(1 - \frac{N-1}{2M_o}\right)^{-1} \left(\frac{k-1}{k}\right) \cdot \binom{N}{m_o/k} \omega^{m_o/k} (1-\omega)^{N-m_o/k} .$$

When M_o is large this becomes approximately

$$\frac{1}{d} + \left(1 - \frac{d'}{d}\right)\left(\frac{k-1}{k}\right) \binom{N}{m_o/k} \left(\frac{1}{k}\right)^{m_o/k} \left(\frac{k-1}{k}\right)^{N-m_o/k}$$

which agrees with (12) in [1].

Very roughly, it appears that the excess of the ratio over $1/d$ is inversely proportional to (initial sample size)$^{1/2}$ for given sets of values of k, d'/d and m_o.

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