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PRESENT VALUE OF A RENEWAL PROCESS

by

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May 1963

This research was supported by a grant from the NATO Research Grants Programme under a joint arrangement between the Institute of Probability of the University of Rome, the Institute of Statistics of the University of Paris and the Department of Statistics of the University of North Carolina.

Institute of Statistics
No. 366

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0. Summary.

This paper studies the present cost C of a renewal process, defined as the sum of the values of the costs of the replacements considered at the starting time of the renewal process, with a compound interest.

The characteristic function of C is found when the inter-arrival times are negatively exponentially distributed; the asymptotic properties of C as the rate of interest tends to zero are studied in the general case.

1. Introduction.

Let us consider a renewal process with inter-arrival times X_1, X_2, \dots identically and independently distributed with distribution function $F(x)$ ($F(0) = 0$). Starting at time $T_0 = 0$ we will have a replacement at each of the instants $T_1 = X_1, T_2 = X_1 + X_2, \dots$.

We suppose that each renewal has a constant cost, which we assume equal to 1. So one will have to pay one (dollar, say) at time T_1 , one at time T_2 , and so on. It is of interest to study the present value, at time 0, of these payments, assuming compound interest.

The present value A_0 , at time 0, of a sum A at time T , is given by:

$$(1) \quad A_0 = e^{-\rho T} A$$

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where ρ is the force of interest (see, for instance, [37]). So the value C_i , at time 0, of a replacement which will take place at time T_i , is given by

$$(2) \quad C_i = e^{-\rho T_i}.$$

The present value, at time 0, of the total cost of the renewal process, which will be denoted by C , is

$$(3) \quad \begin{aligned} C &= \sum_{i=1}^{\infty} C_i \\ &= \sum_{i=1}^{\infty} e^{-\rho \sum_{j=1}^i X_j} \end{aligned}$$

Let us consider the random variables Y_i defined by

$$(4) \quad Y_i = e^{-\rho X_i}.$$

They are independent and identically distributed; we will denote by α_r their moments. If X is a random variable with distribution function $F(x)$, and $\phi_X(t)$ is the characteristic function of X , we have

$$(5) \quad \begin{aligned} \alpha_r &= E Y_j^r \\ &= E e^{-\rho r X} \\ &= \phi_X(i r \rho). \end{aligned}$$

With the exception of the trivial case $P(X = 0) = 1$, we shall have:

$$(6) \quad 0 < \alpha_r < 1.$$

With the introduction of the Y_j , we can write:

$$(7) \quad C = \sum_{i=1}^{\infty} \frac{1}{\prod_{j=1}^i Y_j},$$

and the moments of C are easily expressed in terms of the α_r ; putting

$$(8) \quad \gamma_r = E(C^r),$$

we have

$$\begin{aligned} \gamma_1 &= E \sum_{i=1}^{\infty} \frac{1}{\prod_{j=1}^i Y_j} \\ &= \sum_{i=1}^{\infty} \alpha_1^i \end{aligned}$$

$$(9) \quad = \frac{\alpha_1}{1 - \alpha_1}$$

$$\begin{aligned} \gamma_2 &= E \left(\sum_{i=1}^{\infty} \frac{1}{\prod_{j=1}^i Y_j} \right)^2 \\ &= 2 E \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} \frac{1}{\prod_{j=1}^i Y_j} \frac{1}{\prod_{s=1}^k Y_s} + E \sum_{i=1}^{\infty} \left(\frac{1}{\prod_{j=1}^i Y_j} \right)^2 \\ &= 2 E \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} \frac{1}{\prod_{j=1}^i Y_j^2} \frac{1}{\prod_{j=i+1}^k Y_j} + E \sum_{i=1}^{\infty} \frac{1}{\prod_{j=1}^i Y_j^2} \\ &= 2 \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} \alpha_2^i \alpha_1^{k-i} + \sum_{i=1}^{\infty} \alpha_2^i \\ &= 2 \frac{\alpha_2}{1 - \alpha_2} \frac{\alpha_1}{1 - \alpha_1} + \frac{\alpha_2}{1 - \alpha_2} \\ &= \frac{\alpha_2(1 + \alpha_1)}{(1 - \alpha_2)(1 - \alpha_1)}. \end{aligned}$$

$$(10) \quad \sigma^2(C) = \frac{\alpha_2 - \alpha_1^2}{(1-\alpha_1)^2 (1-\alpha_2)} .$$

The first two moments are finite, because of (6). It can be easily seen that all the moments of C are finite; in fact, as for γ_2 , each moment can be expressed as the sum of a finite number of terms which are all finite. The same result could be obtained by an argument similar to the one used to prove that N_T (i.e. the number of renewals between 0 and T) has finite moments of all orders (see [2], p. 245).

From (3) we obtain moreover

$$(11) \quad C = e^{-\rho X_1} (1 + C')$$

where C' has the same distribution as C , and is independent of X_1 .

2. The case of the negative exponential distribution.

A particular case of great importance in applications is given by the negative exponential distribution, which will be studied in this section. In this case, we have

$$dF(x) = \lambda e^{-\lambda x} dx \quad (x > 0)$$

with $\lambda > 0$. We have also

$$\begin{aligned} \phi_X(t) &= \frac{\lambda}{\lambda - i t} , \\ \alpha_r &= \phi_X(i \rho r) \\ &= \frac{\lambda}{\lambda + r \rho} , \\ EC &= \frac{\lambda}{\rho} , \\ \sigma^2(C) &= \frac{\lambda}{2\rho} . \end{aligned}$$

In order to study the distribution of C , we can use (11). For the characteristic function of C we have

$$\begin{aligned}\phi_C(t) &= E e^{i t C} \\ &= E e^{i t e^{-\rho X_1} + i t e^{-\rho X_1} C'} = E_{X_1} E' [e^{i t e^{-\rho X} + i t e^{-\rho X} C'} | X=X_1] \\ &= E e^{i t e^{-\rho X}} \phi_C(t e^{-\rho X}),\end{aligned}$$

therefore

$$(12) \quad \phi_C(t) = \int_0^{\infty} e^{i t e^{-\rho x}} \phi_C(t e^{-\rho x}) d F(x).$$

Introducing the density function of the negative exponential distribution, we obtain, by a change of variables:

$$\begin{aligned}\phi_C(t) &= \int_0^{\infty} e^{i t e^{-\rho x}} \phi_C(t e^{-\rho x}) \lambda e^{-\lambda x} dx \\ &= \frac{\lambda}{\rho} \int_0^t e^{i y} \phi_C(y) y^{\frac{\lambda}{\rho} - 1} t^{-\frac{\lambda}{\rho}} dy.\end{aligned}$$

In order to solve this integral equation, we transform it into a differential equation, by differentiating on both sides after multiplying by $t^{\frac{\lambda}{\rho}}$. From

$$t^{\frac{\lambda}{\rho}} \phi_C(t) = \frac{\lambda}{\rho} \int_0^t e^{i y} \phi_C(y) y^{\frac{\lambda}{\rho} - 1} dy$$

we have

$$\frac{\lambda}{\rho} t^{\frac{\lambda}{\rho} - 1} \phi_C(t) + t^{\frac{\lambda}{\rho}} \phi_C'(t) = \frac{\lambda}{\rho} e^{i t} \phi_C(t) t^{\frac{\lambda}{\rho} - 1}$$

and hence

$$\phi'_C(t) = \frac{\lambda}{\rho} \frac{e^{it} - 1}{t} \phi_C(t) ,$$

with the condition

$$\phi_C(0) = 1 .$$

We obtain finally

$$(13) \quad \phi_C(t) = \exp \int_0^t \frac{\lambda}{\rho} \int_0^x \frac{e^{ix} - 1}{x} dx .$$

From this expression we can derive the moments of C; expanding the exponentials in series, we have

$$\begin{aligned} \phi_C(t) &= \exp \int_0^t \frac{\lambda}{\rho} \int_0^x \left(1 - \frac{x}{2!} + i \frac{x^2}{3!} - \dots \right) dx \\ &= \exp \int_0^t \frac{\lambda}{\rho} \left(i t - \frac{t^2}{4} - \frac{1}{3} \frac{t^3}{3!} + \dots \right) dt \\ &= 1 + \frac{\lambda}{\rho} \left(i t - \frac{t^2}{4} - \frac{1}{3} \frac{t^3}{3!} + \dots \right) \\ &\quad + \frac{1}{2} \frac{\lambda^2}{\rho^2} \left(i t - \frac{t^2}{4} - \frac{1}{3} \frac{t^3}{3!} + \dots \right)^2 + \dots . \end{aligned}$$

Thus the first moments are

$$\gamma_1 = \frac{\lambda}{\rho} , \quad \gamma_2 = \frac{\lambda}{2\rho} + \frac{\lambda^2}{\rho^2} , \quad \text{i.e. } \sigma^2(C) = \frac{\lambda}{2\rho} .$$

The same results could have been obtained from (9), (10) .

An interesting question concerning the random variable C is its asymptotic behaviour as ρ tends to zero. For the negative exponential distribution the study of this problem is made very easy by formula (13). Let us consider the new

variable Z defined as

$$Z = \frac{C - EC}{\sigma(C)} = \frac{C - \gamma_1}{\sigma}.$$

Using (13) we have

$$\phi_Z(t) = \exp\left[-\frac{\gamma_1}{\sigma} it + \frac{\lambda}{\rho} \int_0^{\frac{t}{\sigma}} \frac{e^{ix} - 1}{x} dx\right].$$

Thus

$$\begin{aligned} \log \phi_Z(t) &= \frac{-\gamma_1}{\sigma} it + \frac{\lambda}{\rho} \left(i \frac{t}{\sigma} - \frac{t^2}{4\sigma^2} - \frac{1}{3} i \frac{t^3}{\sigma^3} + \dots \right) \\ &= -\frac{\lambda}{\rho} \left(\frac{2\rho}{\lambda}\right)^{\frac{1}{2}} t i + \frac{\lambda}{\rho} \left(\frac{2\rho}{\lambda}\right)^{\frac{1}{2}} t i - \frac{\lambda}{\rho} \frac{2\rho}{\lambda} \frac{t^2}{4} - \frac{1}{3} \frac{\lambda}{\rho} \left(\frac{2\rho}{\lambda}\right)^{\frac{3}{2}} \frac{t^3}{3!} + \dots \\ &= -\frac{t^2}{2} - i \frac{2^{\frac{3}{2}}}{3} \lambda^{-\frac{1}{2}} \rho^{\frac{1}{2}} \frac{t^3}{3!} + \dots \end{aligned}$$

All the terms in the last expression, except the first one, contain ρ with a positive exponent; the series is uniformly convergent for ρ in any interval of the form $0 \leq \rho \leq \Delta$; hence we have:

$$\lim_{\rho \rightarrow 0} \log \phi_Z(t) = -\frac{t^2}{2}.$$

Thus we have proved:

Theorem 1. If X has a negative exponential distribution, C is asymptotically normally distributed as ρ tends to zero.

3. The general case.

It can be easily seen that the negative exponential distribution is the only one which permits a simple solution of the integral equation (12) by a transformation into a differential equation. In the general case, since the integral equation

(12) is a homogeneous Volterra equation of second kind, the solution must be searched for among the singular solutions. Thus it appears rather difficult to find by this method the distribution of C for distributions of X other than the negative exponential one.

On the other hand, if we want to study the asymptotic properties of the distribution of C , other means are available; for instance, the investigation of the behaviour of the moments of C .

Let us first establish some lemmas.

Lemma 1. If, for an integer k , $\beta_k = E X^k < \infty$, then

$$(14) \quad \alpha_r = 1 - r \beta_1 \rho + \frac{r^2}{2} \beta_2 \rho^2 + \dots + (-1)^k \frac{r^k}{k!} \beta_k \rho^k \theta(\rho)$$

where $0 \leq \theta(\rho) \leq 1$, and $\lim_{\rho \rightarrow 0} \theta(\rho) = 1$.

Proof: We have

$$e^{-r\rho x} = \sum_{i=0}^{k-1} (-1)^i \frac{r^i}{i!} x^i \rho^i + (-1)^k \frac{r^k}{k!} x^k \rho^k \theta_1(x, \rho),$$

where

$$e^{-\rho r x} \leq \theta_1(x, \rho) \leq 1.$$

If we integrate this equation, and use the mean value theorem, then we find

$$\begin{aligned} \int_0^{+\infty} e^{-r\rho x} dF(x) &= \sum_{i=1}^{k-1} (-1)^i \frac{r^i}{i!} \beta_i \rho^i + (-1)^k \frac{r^k}{k!} \rho^k \int_0^{+\infty} x^k \theta_1(x, \rho) dF(x) \\ &= \sum_{i=1}^{k-1} (-1)^i \frac{r^i}{i!} \beta_i \rho^i + (-1)^k \frac{r^k}{k!} \rho^k \beta_k \theta(\rho), \end{aligned}$$

where $0 \leq \theta(\rho) \leq 1$. Moreover

$$\theta(\rho) = \frac{1}{\beta_k} \int_0^{+\infty} x^k \theta_1(x, \rho) dF(x) .$$

Hence it follows from the hypotheses and the properties of $\theta_1(x, \rho)$:

$$\lim_{\rho \rightarrow 0} \theta(\rho) = 1 .$$

The lemma is thus proved.

Lemma 2. If $E X^k < \infty$ ($k \geq 1$) then, for every $t \geq 0$,

$$\lim_{\rho \rightarrow 0} \rho^{-k} E e^{-\rho t X} (e^{-\rho X} - \alpha_1)^h = \begin{cases} E(X - \beta_1)^k & \text{if } h = k \\ 0 & \text{if } h > k \end{cases}$$

Proof: We can write, as in lemma 1,

$$e^{-\rho X} = 1 - \rho x \theta_1(x, \rho) ,$$

where $e^{-\rho X} \leq \theta_1(x, \rho) \leq 1$; and

$$\alpha_1 = 1 - \rho \beta_1 \theta(\rho) ,$$

where $0 \leq \theta(\rho) \leq 1$ and $\lim_{\rho \rightarrow 0} \theta(\rho) = 1$.

Then

$$\begin{aligned} \rho^{-k} E e^{-\rho t X} (e^{-\rho X} - \alpha_1)^h &= E e^{-\rho t X} (e^{-\rho X} - \alpha_1)^{h-k} \left(\frac{e^{-\rho X} - \alpha_1}{\rho} \right)^k \\ &= E e^{-\rho t X} (e^{-\rho X} - \alpha_1)^{h-k} \left[\frac{1 - \rho X \theta_1(x, \rho) - 1 + \rho \beta_1 \theta(\rho)}{\rho} \right]^k \\ &= E e^{-\rho t X} (e^{-\rho X} - \alpha_1)^{h-k} \left[\beta_1 \theta(\rho) - X \theta_1(x, \rho) \right]^k . \end{aligned}$$

Moreover, for $h-k \geq 0$,

$$\begin{aligned} |e^{-\rho t X} (e^{-\rho X} - \alpha_1)^{h-k} [\beta_1 \theta(\rho) - X \theta_1(X, \rho)]^k| &\leq |\beta_1 \theta(\rho) - X \theta_1(X, \rho)|^k \\ &\leq 2^k |\beta_1^k \theta^k(\rho) + X^k \theta_1^k(X, \rho)| \\ &\leq 2^k (\beta_1^k + X^k). \end{aligned}$$

Since the expectation of $2^k (\beta_1^k + X^k)$ is finite, we can interchange the expectation and lim signs, and the lemma is proved.

Let us consider now the central moments $\bar{\gamma}_r$ of C . We have

$$\bar{\gamma}_r = E(C - EC)^r = E(C - \gamma_1)^r$$

From (11) we obtain a recurrence formula for the moments γ_r :

$$\begin{aligned} \gamma_r &= E C^r \\ &= E e^{-r \rho X_1} (1 + C')^r \\ &= E e^{-r \rho X_1} E(1 + C')^r \\ &= \alpha_r E \sum_{j=0}^r \binom{r}{j} C'^j \\ &= \alpha_r \sum_{j=0}^r \binom{r}{j} \gamma_j \end{aligned}$$

By the well-known relations between moments and central moments

$$\begin{aligned} \bar{\gamma}_r &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \gamma_1^{r-j} \gamma_j, \\ \gamma_r &= \sum_{j=0}^r \binom{r}{j} \bar{\gamma}_j \gamma_1^{r-j}, \end{aligned}$$

we may derive a recurrence formula for the central moments of C :

$$\begin{aligned}
\bar{V}_r &= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \gamma_1^{r-j} \gamma_j \\
&= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \gamma_1^{r-j} \alpha_j \sum_{t=0}^j \binom{j}{t} \gamma_t \\
&= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \gamma_1^{r-j} \alpha_j \sum_{t=0}^j \binom{j}{t} \sum_{i=0}^t \binom{t}{i} \bar{V}_i \gamma_1^{t-i} \\
&= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \gamma_1^{r-j} \alpha_j \sum_{i=0}^j \bar{V}_i \sum_{t=i}^j \frac{j!}{(j-t)! i! (t-i)!} \gamma_1^{t-i} \\
&= \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \gamma_1^{r-j} \alpha_j \sum_{i=0}^j \bar{V}_i \binom{j}{i} (1 + \gamma_1)^{j-i} .
\end{aligned}$$

Substituting γ_1 by (9), we have:

$$\begin{aligned}
\bar{V}_r &= \sum_{i=0}^r \sum_{j=i}^r (-1)^{r-j} \binom{r}{j} \frac{\alpha_1^{r-j}}{(1-\alpha_1)^{r-j}} \alpha_j \bar{V}_i \binom{j}{i} \frac{1}{(1-\alpha_1)^{j-i}} \\
&= \sum_{i=0}^r \bar{V}_i \frac{1}{(1-\alpha_1)^{r-i}} \sum_{j=i}^r (-1)^{r-j} \binom{r}{j} \binom{j}{i} \alpha_1^{r-j} \alpha_j \\
&= \sum_{i=0}^r \binom{r}{i} \frac{1}{(1-\alpha_1)^{r-i}} \bar{V}_i \sum_{j=i}^r \binom{r-i}{j-i} \alpha_j (-\alpha_1)^{r-j} \\
&= \alpha_r \bar{V}_r + \sum_{i=0}^{r-1} \binom{r}{i} \frac{1}{(1-\alpha_1)^{r-i}} \bar{V}_i \sum_{j=i}^r \binom{r-i}{j-i} \alpha_j (-\alpha_1)^{r-j} .
\end{aligned}$$

We obtain finally

$$(15) \quad \bar{V}_r = \frac{1}{1 - \alpha_r} \sum_{i=0}^{r-1} \binom{r}{i} \frac{1}{(1-\alpha_1)^{r-i}} \bar{V}_i \sum_{j=i}^r \binom{r-i}{j-i} \alpha_j (-\alpha_1)^{r-j} .$$

Since

$$\begin{aligned} \sum_{j=i}^r \binom{r-i}{j-i} \alpha_j (-\alpha_1)^{r-j} &= \sum_{j=i}^r \binom{r-i}{j-i} E e^{-\rho j X} (-\alpha_1)^{r-j} \\ &= E \sum_{j=i}^r \binom{r-i}{j-i} e^{-\rho j X} (-\alpha_1)^{r-j} \\ &= E e^{-\rho i X} (e^{-\rho X} - \alpha_1)^{r-i}, \end{aligned}$$

(15) can also be written as

$$(15') \quad \bar{\gamma}_r = \frac{1}{1 - \alpha_r} \sum_{i=0}^{r-1} \binom{r}{i} \frac{1}{(1 - \alpha_1)^{r-i}} \bar{\gamma}_i E e^{-\rho i X} (e^{-\rho X} - \alpha_1)^{r-i}.$$

We can now proceed to study the convergence of the central moments $\bar{\gamma}_r$.

Theorem 2. If, for $r > 0$, $E X^{\frac{r}{2} + 1} < \infty$, then

$$(16) \quad \lim_{\rho \rightarrow 0} \rho^{\frac{r}{2}} \bar{\gamma}_r = K_r$$

where

$$K_r = \begin{cases} \frac{(\beta_2 - \beta_1^2)^{\frac{r}{2}} r!}{2^r (\frac{r}{2})! \beta_1^{\frac{r}{2}}} & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd} \end{cases}$$

Proof: We will prove this theorem by induction. Clearly (16) holds for $r = 0, 1$, since $\gamma_0 = 1$, $\gamma_1 = 0$. Now let us assume that it holds for $0, 1, \dots, r-1$, with $r \geq 2$, and prove it must consequently hold for r .

From (15') we have

$$(17) \quad \rho^{\frac{r}{2}} \bar{Y}_r = \sum_{i=1}^{r-1} \rho^{\frac{r}{2}} T_i$$

where

$$\begin{aligned} \rho^{\frac{r}{2}} T_i &= \rho^{\frac{r}{2}} \frac{1}{1-\alpha_r} \frac{1}{(1-\alpha_1)^{r-1}} \binom{r}{i} \bar{Y}_i E e^{-\rho i X} (e^{-\rho X} - \alpha_1)^{r-i} \\ &= \binom{r}{i} \frac{\rho}{1-\alpha_r} \left(\frac{\rho}{1-\alpha_1} \right)^{r-i} \rho^{\frac{1}{2}} \bar{Y}_i \rho^{-\left(\frac{r}{2} + 1 - \frac{i}{2}\right)} E e^{-\rho i X} (e^{-\rho X} - \alpha_1)^{r-i}. \end{aligned}$$

By lemma 1 we obtain:

$$\lim_{\rho \rightarrow 0} \frac{\rho}{1-\alpha_r} = \frac{1}{r \beta_1},$$

so that

$$\lim_{\rho \rightarrow 0} \rho^{\frac{r}{2}} T_i = \binom{r}{i} \frac{1}{r \beta_1} \frac{1}{\beta_1^{r-i}} K_i \lim_{\rho \rightarrow 0} \rho^{-\left(\frac{r}{2} + 1 - \frac{i}{2}\right)} E e^{-\rho i X} (e^{-\rho X} - \alpha_1)^{r-i}.$$

Since $\frac{r}{2} + 1 - \frac{i}{2} \leq \frac{r}{2} + 1$, we have

$$E X^{\frac{r}{2} + 1 - \frac{i}{2}} < \infty$$

and we can use lemma 2. So, if $i < r-2$, we have $\frac{r}{2} + 1 - \frac{i}{2} < r-i$, and then

$$\lim_{\rho \rightarrow 0} \rho^{\frac{r}{2}} T_i = 0.$$

If on the contrary $i = r-2$, i.e. $\frac{r}{2} + 1 - \frac{i}{2} = r-i$, we have:

$$\begin{aligned} \lim_{\rho \rightarrow 0} \rho^{\frac{r}{2}} T_{r-2} &= \binom{r}{2} \frac{1}{r} \frac{1}{\beta_1^3} K_{r-2} E(X - \beta_1)^2 \\ &= \binom{r}{2} \frac{1}{r} \frac{\beta_2 - \beta_1^2}{\beta_1^3} K_{r-2}. \end{aligned}$$

Finally, for $i = r - 1$, we can write

$$E e^{-(r-1)\rho X} (e^{-\rho X} - \alpha_1) = \alpha_r - \alpha_1 \alpha_{r-1}$$

and, by lemma 1, we have

$$\begin{aligned} \lim_{\rho \rightarrow 0} \rho^{-\frac{3}{2}} (\alpha_r - \alpha_1 \alpha_{r-1}) &= \\ &= \lim_{\rho \rightarrow 0} \rho^{-\frac{3}{2}} \left\{ 1 - r\beta_1\rho + o_1(\rho^2) - [1 - (r-1)\beta_1\rho + o_2(\rho^2)] [1 - \beta_1\rho + o_3(\rho^2)] \right\} \\ &= \lim_{\rho \rightarrow 0} \rho^{-\frac{3}{2}} [1 - r\beta_1\rho - 1 + \beta_1\rho + (r-1)\beta_1\rho + o(\rho^2)] \\ &= 0 \end{aligned}$$

Hence

$$\lim_{\rho \rightarrow 0} \rho^{\frac{2r}{2}} T_{r-1} = 0 .$$

We thus have:

$$\begin{aligned} \lim_{\rho \rightarrow 0} \rho^{\frac{2r}{2}} \bar{Y}_r &= \sum_{i=0}^{r-1} \lim_{\rho \rightarrow 0} \rho^{\frac{2i}{2}} T_i \\ &= \binom{r}{2} \frac{1}{r} \frac{\beta_2 - \beta_1^2}{\beta_1^3} K_{r-2} . \end{aligned}$$

If r is odd, so is $r-2$, and then $K_r = K_{r-2} = 0$. If r is even, we have

$$\begin{aligned}
\binom{r}{2} \frac{1}{r} \frac{\beta_2 - \beta_1^2}{\beta_1^3} K_{r-2} &= \binom{r}{2} \frac{1}{r} \frac{\beta_2 - \beta_1^2}{\beta_1^3} \frac{(\beta_2 - \beta_1^2)^{\frac{r}{2} - 1} (r-2)!}{2^{r-2} \left(\frac{r}{2} - 1\right)! \beta_1^{\frac{r}{2} (r-2)}} \\
&= \frac{1}{2^r} \frac{r!}{\left(\frac{r}{2}\right)!} \frac{(\beta_2 - \beta_1^2)^{\frac{r}{2}}}{\beta_1^{\frac{r}{2} r}} = K_r .
\end{aligned}$$

The theorem is thus proved.

Theorem 2 enables us to establish a sufficient condition for the asymptotic normality of C .

Theorem 3. If all the moments of X are finite, then the distribution of $(C - EC)/\sigma(C)$ tends to the standard normal distribution as ρ decreases to zero.

Proof: Theorem 2 gives, for $r = 2$:

$$\lim_{\rho \rightarrow 0} \rho \sigma^2(C) = \frac{\beta_2 - \beta_1^2}{2 \beta_1^3}$$

Thus we have:

$$\begin{aligned}
\lim_{\rho \rightarrow 0} E \left[\frac{C - EC}{\sigma(C)} \right]^r &= \lim_{\rho \rightarrow 0} \frac{\bar{Y}_r}{\sigma^r(C)} \\
&= \lim_{\rho \rightarrow 0} \frac{\rho^{\frac{r}{2}} \bar{Y}_r}{\rho^{\frac{r}{2}} \sigma^r(C)} = \left(\frac{2 \beta_1^3}{\beta_2 - \beta_1^2} \right)^{\frac{r}{2}} K_r \\
&= \frac{1}{2^{\frac{r}{2}}} \frac{r!}{\left(\frac{r}{2}\right)!}
\end{aligned}$$

which is equal to the r -th moment of the normal variate $N(0,1)$. Thus the moments of $(C - EC)/\sigma(C)$ converge to the moments of the normal distribution, and this is sufficient to ensure that the distribution of the variate converge to the normal.

distribution (see, for instance, [1] p. 110). The theorem is thus proved.

The asymptotic normality is of course an important feature of C ; it permits us to approximate C with a normal variate when ρ is small. It would be interesting to have some necessary condition in order that the asymptotic distribution be normal; unfortunately a condition of this kind appears rather difficult to establish.

However, the proof given above requires as an essential condition the existence of all the moments of X ; so that one should expect that, if this condition is dropped, the moments of $(C - EC)/\sigma(C)$ do not converge to the moments of the normal variate. That this is actually true will be seen later.

Lemma 3. If $EX < \infty$ and

$$\lim_{\rho \rightarrow 0} \rho^{-t} E (e^{-\rho X} - \alpha_1)^k = 0 \quad (0 < t \leq k)$$

then, for every $q < t$:

$$EX^q < \infty$$

Proof: We will show first that, under the hypotheses above, we have:

$$(18) \quad \lim_{\rho \rightarrow 0} \rho^{-t} E |e^{-\rho X} - \alpha_1|^k = A$$

where $0 \leq A < \infty$. In fact, we can write:

$$(19) \quad \rho^{-t} E (e^{-\rho X} - \alpha_1)^k = \rho^{-t} \int_0^{-\frac{1}{\rho} \log \alpha_1} (e^{-\rho x} - \alpha_1)^k dF(x) + \rho^{-t} \int_{-\frac{1}{\rho} \log \alpha_1}^{+\infty} (e^{-\rho x} - \alpha_1)^k dF(x)$$

Now, since $\beta_1 = E X < \infty$,

$$\begin{aligned} \lim_{\rho \rightarrow 0} \left(-\frac{1}{\rho} \log \alpha_1 \right) &= - \lim_{\rho \rightarrow 0} \frac{1}{\alpha_1} \frac{d \alpha_1}{d \rho} \\ &= \lim_{\rho \rightarrow 0} \frac{\int_0^{+\infty} x e^{-\rho x} dF(x)}{\int_0^{+\infty} e^{-\rho x} dF(x)} \\ &= \beta_1 \end{aligned}$$

Moreover

$$\begin{aligned} \rho^{-t} |e^{-\rho x} - \alpha_1|^k &= |e^{-\rho x} - \alpha_1|^{k-t} |\beta_1 \theta(\rho) - x \theta_1(x, \rho)|^t \\ &\leq 2^t (\beta_1^t + x^t). \end{aligned}$$

Hence

$$\lim_{\rho \rightarrow 0} \rho^{-t} \int_0^{\beta_1} \frac{1}{\rho} \log \alpha_1 (e^{-\rho x} - \alpha_1)^k dF(x) = \frac{A}{2},$$

where

$$\frac{A}{2} = \begin{cases} \int_0^{\beta_1} (\beta_1 - x)^t dF(x) & \text{if } k = t \\ 0 & \text{if } k > t. \end{cases}$$

Since, by hypothesis, the left side of (19) tends to zero, we have also

$$\lim_{\rho \rightarrow 0} \rho^{-t} \int_{-\frac{1}{\rho} \log \alpha_1}^{+\infty} (e^{-\rho x} - \alpha_1)^k dF(x) = -\frac{A}{2}.$$

We have finally

$$\rho^{-t} E|e^{-\rho X} - \alpha_1|^k = \rho^{-t} \left| \int_0^{\frac{1}{\rho} \log \alpha_1} (e^{-\rho x} - \alpha_1)^k dF(x) \right| + \rho^{-t} \int_{\frac{1}{\rho} \log \alpha_1}^{+\infty} (e^{-\rho x} - \alpha_1)^k dF(x) \left| - \frac{1}{\rho} \log \alpha_1 \right|, \quad 18$$

and (18) follows.

Now, since $\lim_{\rho \rightarrow 0} \alpha_1 = 1$, given a number K with $0 < K < 1$, by making ρ small enough, we can make $\alpha_1 > K$. Then we have:

$$\begin{aligned} E|e^{-\rho X} - \alpha_1|^k &\geq \int_{\frac{1}{\rho} \log \frac{1}{K}}^{+\infty} (\alpha_1 - e^{-\rho x})^k dF(x) \\ &\geq (\alpha_1 - K)^k \int_{\frac{1}{\rho} \log \frac{1}{K}}^{+\infty} dF(x) \\ &= (\alpha_1 - K)^k [1 - F(\frac{1}{\rho} \log \frac{1}{K})]. \end{aligned}$$

If we put $\frac{1}{\rho} \log \frac{1}{K} = x$, we obtain

$$\begin{aligned} x^t [1 - F(x)] &= (\log \frac{1}{K})^t \rho^{-t} [1 - F(\frac{1}{\rho} \log \frac{1}{K})] \\ &\leq \frac{(\log \frac{1}{K})^t}{(\alpha_1 - K)^k} \rho^{-t} E|e^{-\rho X} - \alpha_1|^k. \end{aligned}$$

Then from (18) it follows that

$$\overline{\lim}_{x \rightarrow +\infty} x^t [1 - F(x)] \leq (\log \frac{1}{K})^t (1 - K)^{-k} A.$$

This establishes the lemma.

Theorem 4. If $E X^2 < \infty$, then in order that all the moments of $(C - EC)/\sigma(C)$ converge to the moments of the normal distribution $N(0,1)$, it is necessary that all the moments of X be finite.

Proof: The proof will be by induction. Since $E X^2 < \infty$, the convergence of $E[(C - EC)/\sigma(C)]^k$ to the k -th central moment of the normal variate $N(0,1)$ is equivalent to

$$(20) \quad \lim_{\rho \rightarrow 0} \rho^{\frac{k}{2}} \bar{\gamma}_k = K_k$$

We will assume that $E X^{\frac{r}{2}} < \infty$ for an integer $r \geq 4$, and that (20) holds for every integer k , and we will prove that $E X^{\frac{r+1}{2}} < \infty$. The theorem will thus be proved.

With the notation of theorem 2, (17) holds; since (20) holds for $k < r$, we have again:

$$\lim_{\rho \rightarrow 0} \rho^{\frac{r}{2}} T_i = \binom{r}{i} \frac{1}{r \beta_1} \frac{1}{\beta_1^{r-i}} K_i \lim_{\rho \rightarrow 0} \rho^{-(\frac{r}{2} + 1 - \frac{i}{2})} E e^{-\rho i X} (e^{-\rho X} - \alpha_1)^{r-i}.$$

For $i \geq 2$, that is $\frac{r}{2} + 1 - \frac{i}{2} \leq \frac{r}{2}$, we have $E X^{\frac{r}{2} + 1 - \frac{i}{2}} < \infty$.

Hence, for $i \geq 2$, the conclusions of theorem 2 hold, that is

$$\lim_{\rho \rightarrow 0} \rho^{\frac{r}{2}} T_i = \begin{cases} K_r & \text{if } i = r-2 \\ 0 & \text{if } i \neq r-2 \end{cases}$$

Moreover, for $i = 1$, $\gamma_1 = 0$; thus

$$\lim_{\rho \rightarrow 0} \rho^{\frac{r}{2}} \bar{\gamma}_r = \sum_{i=0}^{r-1} \lim_{\rho \rightarrow 0} \rho^{\frac{r}{2}} T_i = K_r + \lim_{\rho \rightarrow 0} \rho^{\frac{r}{2}} T_0.$$

Since (20) holds for $k = r$, the last limit above must be zero, that is:

$$\lim_{\rho \rightarrow 0} \rho^{2r} \frac{1}{1 - \alpha_r} \frac{1}{(1 - \alpha_1)^r} E(e^{-\rho X} - \alpha_1)^r = 0.$$

$$\lim_{\rho \rightarrow 0} \rho^{-(\frac{r}{2} + 1)} E(e^{-\rho X} - \alpha_1)^r = 0.$$

By lemma 3, this implies that $E X^q < \infty$ for every $q < \frac{r}{2} + 1$, in particular for $q = \frac{r+1}{2}$. The theorem is thus proved.

4. Acknowledgements. I would like to thank Dr. Walter L. Smith for proposing this problem and for his kind suggestions.

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