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ON A DISTRIBUTION OF SUM OF IDENTICALLY
DISTRIBUTED CORRELATED GAMMA VARIABLES

by

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0. Abstract:

A distribution of a sum of identically distributed Gamma-variables correlated according to an 'exponential' autocorrelation law $\rho_{kj} = \rho^{|k-j|}$ ($k, j = 1 \dots n$) where ρ_{ij} is the correlation coefficient between the i -th and j -th random variables and $0 < \rho < 1$ is a given number is derived. This is performed by determining the characteristic roots of the appropriate variance - covariance matrix, using a special method, and by applying Robbins' and Pitman's result on mixtures of distributions to the case of Gamma-variables. An "approximate" distribution of the sum of these variables under the assumption that the sum itself is a Gamma variable is given. A comparison between an "exact" and an "approximate" distributions for certain values of the correlation coefficient, the number of variables in the sum and the values of parameters of the initial distributions are presented.

1. Introduction and general remarks.

Distribution of sum of correlated Gamma variables has a significant interest and many applications in engineering, insurance and other areas. Besides the applications mentioned in [1] and [2] we should like to note the usefulness of this distribution in problems connected with representation of precipitation amounts. Weekly or monthly, etc., precipitation amounts are regarded as sums of

¹This research was supported by the Mathematics Division of the Air Force Office of Scientific Research.

daily amounts, the latter being well fitted by Gamma distribution [3]. A particular solution for the case of a constant correlation between each pair of variables in the sum is presented in [1]. In this note we shall extend this result for the case of "an exponential autocorrelation scheme" between the variables, where each one of the variables has the marginal density function given by

$$f(x) = \frac{1}{\Gamma(r)\theta^r} e^{-\frac{x}{\theta}} x^{r-1}, \quad x \geq 0 \quad (1.1)$$

$$= 0 \quad x \leq 0$$

In meteorological applications it is sometimes assumed that the sum random variable is itself a Gamma variable and an "approximate" Gamma distribution whose first two moments are identical with the "exact" distribution of the sum is used [3], (see also, for example, [4] for an earlier application in another field). In the case of identically distributed r.v. and of stationary exponential correlation model (namely when the correlation coefficient between the i -th and the j -th r.v. in the sum is given by $\rho^{|i-j|}$ for all i and j , ρ being a given positive constant) it is easy to verify [3] that the appropriate parameters r_n and θ_n of the "approximate" Gamma variable, representing the sum r.v., are given by:

$$r_n = \frac{n^2}{n + \frac{2\rho}{1-\rho}(n - \frac{1-\rho^n}{1-\rho})} r \quad (1.2)$$

and

$$\theta_n = \frac{n + \frac{2\rho}{1-\rho}(n - \frac{1-\rho^n}{1-\rho})}{n} \theta \quad (1.3)$$

when n is the number variables in the sum and r and θ are the parameters of initial r.v. distributed according to (1.1). Since the first two moments

completely determine Gamma-distribution the "approximate" distribution is unique. The purpose of this note is to compare this "approximate" distribution (which is quite easy computable and applied) with an "exact distribution of the sum (which is much more difficult to be actually determined). We also note that, similar to the case treated in [1], the obtained "natural" exact distribution may not be unique, since the correlation scheme, in general, does not determine uniquely a multivariate Gamma distribution as it will be noted in some more detail in the following section.

2. A Joint Characteristic Function of Exponentially Correlated Gamma Variables

Consider the characteristic function $\phi(t_1, t_2, \dots, t_n)$,

$$\phi(t_1, t_2, \dots, t_n) = |I - r \Theta T V|^{-r} \quad (2.1)$$

where Θ and r are positive constants, I is the $n \times n$ identity matrix, T is the $n \times n$ diagonal matrix with the elements $t_{jj} = t_j$, and V is an arbitrary $n \times n$ positive definite matrix. This characteristic function leads to a joint probability density function whose marginals are given by (1.1) and whose matrix of second moments is some positive definite matrix V^* , say. In general, a joint density function with marginal given by (1.1) and a given covariance matrix V^* is not uniquely determined; (for example, as it is known, a multivariate Gamma distribution with constant correlation between the components is not unique [5]). An explicit expression for a joint distribution with a general matrix is rather complicated, but can be obtained in terms of a series of Laguerre polynomials by applying the derivations given in [6]. We will not present these expressions here since the main concern of this paper is the validity and applicability of approximations to the distribution of the sum of the component random variables. Following Gurland [1] we shall regard (2.1) as the characteristic function of a natural multivariate analogue of the Gamma distribution.

In particular, if the elements of V are given by $v_{ij} = \rho^{|i-j|}$, $i, j = 1, \dots, n$ ($0 < \rho < 1$ a given constant) it is easy to verify by differentiation of (2.1) that the corresponding elements of V^* will be given by $v_{ij}^* = r \theta^2 \rho^{2|i-j|}$, $i, j = 1, \dots, n$.

The characteristic function of the distribution of the sum of the random variables whose joint distribution has the characteristic function given in (2.1) is:

$$\phi(t) = \|I - i \theta t V\|^{-r} \quad (2.2)$$

and (2.3) may be expressed in the form

$$\phi(t) = \prod_{j=1}^n (1 - i \theta \lambda_j t)^{-r} \quad (2.3)$$

where the λ_j are the characteristic roots of the matrix V . In section 4 we shall describe a procedure for an explicit determination of these characteristic roots.

3. The Distribution of the Random Variables whose Characteristic Function is $\phi(t)$.

Robbins proved in [7] that if $\{c_k\}$ is a sequence of constants such that $c_k \geq 0$, $k = 1, \dots$ and $\sum_{k=0}^{\infty} c_k = 1$ and if $\{F_k(\cdot)\}$ is a sequence of distribution functions with corresponding sequence of characteristic functions $\{\phi_k(\cdot)\}$. then (i) $F(\cdot) = \sum c_k F_k(\cdot)$ is a distribution function, (ii) the characteristic function $\phi(\cdot)$ of $F(\cdot)$ is given by $\phi(\cdot) = \sum c_k \phi_k(\cdot)$, and (iii) for any Borel measurable function $g(\cdot)$ the following equality holds:

$$\int_{\mathcal{X}} g(x) dF(x) = \sum_k c_k \int_{\mathcal{X}} g(x) dF_k(x),$$
 where \mathcal{X} is the appropriate space. These results is valid for a general class of spaces and in particular for any finite dimensional Euclidean space. The distribution function of the Gamma variable with positive parameters r and θ denoted here by $F_r(\cdot)$ is given by:

$$F_r(x) \begin{cases} = \frac{1}{\Theta^r \Gamma(r)} \int_0^x e^{-\frac{u}{\Theta}} u^{r-1} du, & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (3.1)$$

The characteristic function of this distribution is

$$\phi^*(t) = (1 - i \Theta t)^{-r}. \quad (3.2)$$

Let Y be the random variable whose characteristic function is given by (2.4).

The Y has the same distribution as the random variable X ,

$$X = \sum_{j=1}^n \lambda_j X_j \quad (3.3)$$

where X_j , $j = 1, 2, \dots, n$, are independent identically distributed random variables, each following a Gamma distribution with parameters r and Θ as given in (3.1).

Let $\lambda^* = \min_j \lambda_j$. Then

$$\Pr(Y < y) = \Pr\left(\frac{X}{\lambda^*} < \frac{y}{\lambda^*}\right) \quad (3.4)$$

and the characteristic function of the random variable X/λ^* is

$$\psi(t) = \prod_{j=1}^n \left(1 - i \Theta \frac{\lambda_j}{\lambda^*} t\right)^{-r} \quad (3.5)$$

Using the method developed by Pitman and Robbins in [8], we express the right hand side of (3.5) in the form

$$\prod_{j=1}^n \left(1 - i \Theta \frac{\lambda_j}{\lambda^*} t\right)^{-r} = (1 - i \Theta t)^{-nr} \prod_{j=1}^n \left\{ \left(\frac{\lambda_j}{\lambda^*}\right) \left(1 - \left(1 - \frac{\lambda^*}{\lambda_j}\right) (1 - i \Theta t)^{-1}\right) \right\}^{-r} \quad (3.6)$$

Since,

$$\left| \left(1 - \frac{\lambda^*}{\lambda_j}\right) (1 - i \theta t)^{-1} \right| \leq 1 \quad (3.7)$$

for all real θ and t the binomial theorem may be applied to expand the r.h.s. of (3.6). We thus obtain

$$\prod_{j=1}^n \left(1 - i \theta \frac{\lambda_j}{\lambda^*} t\right)^{-r} = \sum_{k=0}^{\infty} c_k (1 - i \theta t)^{-nr+k} \quad (3.8)$$

where the coefficients c_k are determined by the identity

$$\prod_{j=1}^n \left\{ \left(\frac{\lambda_j}{\lambda^*}\right) \left[1 - \left(1 - \frac{\lambda^*}{\lambda_j}\right) z\right] \right\}^{-r} = \sum_{k=0}^{\infty} c_k z^k \quad (3.9)$$

It follows from (3.6) that all $c_k > 0$, and putting $(1 - i \theta t) = 1$ in (3.6) and (3.8) shows that $\sum c_k = 1$.

Thus the results of Robbins [7] apply for c_k determined by (3.9) and hence the r.h.s. of (3.8) is the characteristic function of the distribution $F(\cdot)$ given by

$$F(x) = \sum_{k=0}^{\infty} c_k F_{nr+k}(x) \quad (3.10)$$

where $F_{nr+k}(\cdot)$ ($k = 0, 1, \dots$) are defined by (3.1). Hence

$$\begin{aligned} \Pr(Y < y) &= \sum_{k=0}^{\infty} c_k F_{nr+k}(y/\lambda^*) = \\ &= \sum_{k=0}^{\infty} \frac{c_k}{\theta \Gamma(nr+k)} \int_0^{\frac{y}{\lambda^*}} \left(\frac{u}{\theta}\right)^{nr+k-1} e^{-\frac{u}{\theta}} du \end{aligned} \quad (3.11)$$

and the corresponding probability density function is given by:

$$f_Y(y) = \begin{cases} \frac{1}{\lambda^* \theta} \sum_{k=0}^{\infty} \frac{c_k}{\Gamma(nr+k)} \left(\frac{x}{\lambda^* \theta}\right)^{nr+k-1} e^{-\frac{x}{\theta \lambda^*}}, & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

(The term by term differentiation is justified by the uniform convergence of the series). The upper bound on the error of truncation is given by the following formula: for any integers $0 \leq p_1 \leq p_2$ and for any x :

$$0 \leq \Pr(Y < y) - \sum_{P_1}^{P_2} F_{nr+k}(y/\lambda^*) \leq \sup \left\{ F_{nr+k}(y/\lambda^*) \right\} \left(\sum_0^{P_1-1} c_k \right) + \\ + \sup \left\{ F_{nr+k}(y/\lambda^*) \right\} \left(1 - \sum_0^{P_1-1} c_k - \sum_{P_1}^{P_2} c_k \right) \leq 1 - \sum_{P_1}^{P_2} c_k.$$

We should like to point out that the expansions (3.8) and (3.9) are closely related to the general results on expansion of the hypergeometric functions of matrix variables into series of so-called "zonal polynomials" [10, 10a].

4. The Characteristic Roots of the Matrix V.

The inverse matrix of V , whose elements are $v_{ij} = \rho^{i-j}$, $i, j = 1, \dots, n$, is V^{-1} given by

$$V^{-1} = ((1+\rho^2)I - \rho^2 A - \rho B)(1-\rho^2)^{-1} \quad (4.1)$$

where I is the $n \times n$ identity matrix, A is the $n \times n$ matrix with elements $a_{11} = a_{nn} = 1$ and all other elements 0, and B is the $n \times n$ matrix with

$b_{ij} = 1$ for $|i-j| = 1$ and all other elements 0. Hence the characteristic roots of V^{-1} are $(1-\rho^2)^{-1} \gamma_j$, $j = 1, 2, \dots, n$, where the γ_j are those value of γ which satisfy the equation:

$$|(1 + \rho^2 - \gamma)I - \rho^2 A - \rho B| = \quad (4.2)$$

Let $\Delta_n(n, \rho)$ be the l.h.s. of (4.2). Expanding $A_n(A, \rho)$ by its minors we obtain

$$\begin{aligned} \Delta_n(\gamma, \rho) &= (1 - \gamma)^2 D_{n-2}(\gamma, \rho) - 2\rho^2(1-\gamma)D_{n-3}(\gamma, \rho) \\ &\quad + \rho^4 D_{n-4}(\gamma, \rho) \end{aligned} \quad (4.3)$$

where $D_n(\gamma, \rho)$ is the determinant of the $n \times n$ matrix U ,

$$U = (1 + \rho^2 - \gamma) - \rho B \quad (4.4)$$

Expanding $D_n(\gamma, \rho)$ by its minors we get the following difference equation

$$D_n(\gamma, \rho) - (1 + \rho^2 - \gamma) D_{n-1}(\gamma, \rho) + \rho^2 D_{n-2}(\gamma, \rho) = 0 \quad (4.5)$$

and, as it is well known, (see e.g. [27]), this difference equation has the solution

$$D_n(\gamma, \rho) = \rho^n \frac{\sin(n+1)\theta}{\sin \theta} \quad (4.6)$$

where $1 + \rho^2 - \gamma = 2\rho \cos \theta$

Substituting (4.6) in (4.3) and equating $\Delta_n(\gamma, \rho)$ to zero we obtain the equation

$$\begin{aligned} (1 - \gamma)^2 \rho^{n-2} \frac{\sin(n-1)\theta}{\sin \theta} - 2(1 - \gamma)\rho^{n-1} \frac{\sin(n-2)\theta}{\sin \theta} + \\ + \rho^n \frac{\sin(n-3)\theta}{\sin \theta} = 0 \end{aligned} \quad (4.7)$$

Using the complex representation of the sine function we get after multiplying (4.7) by $\sin \theta$ (excluding the value of θ for which $\sin \theta = 0$),

$$e^{\frac{i(n-3)\theta}{2}} ((1 - \gamma) e^{i\theta} - \rho) = \pm e^{-\frac{i(n-3)\theta}{2}} ((1 - \gamma) e^{-i\theta} - \rho), \quad (4.8)$$

and after a few algebraic manipulations the equation (4.8) reduces to

$$\sin \frac{(n+1)\theta}{2} = \rho \sin \frac{(n-1)\theta}{2} \quad (4.9)$$

and

$$\cos \frac{(n+1)\theta}{2} = \rho \cos \frac{(n-1)\theta}{2} \quad (4.10)$$

If θ_j , $j = 1, 2, \dots, n$ are the values of θ which satisfy one or the other of the equations (4.9) and (4.10), then the characteristic roots of the matrix V^{-1} are given by $\gamma_j = (1 - 2\rho \cos \theta_j + \rho^2)^{-1} (1 - \rho^2)$, and the characteristic roots of V are $\lambda_j = \gamma_j^{-1} = (1 - 2\rho \cos \theta_j + \rho^2) (1 - \rho^2)$, $j = 1, 2, \dots, n$.

Apparently it is not possible to express the solutions of (4.9) and (4.10) as explicit functions of ρ and θ . However, for $0 \leq \rho \leq 1$, it can be verified that the solutions of (4.9) are located one in each of the intervals $(\frac{2k-1}{n}\pi, \frac{2k}{n+1}\pi)$, $k = 1, 2, \dots, \frac{n}{2}$ for $\frac{n-1}{2}$ depending on whether n is even or odd; and that the solutions of (4.10) are one in each of the intervals $(\frac{2k-2}{n}\pi, \frac{2k-1}{n+1}\pi)$, $k = 1, 2, \dots, \frac{n}{2}$ or $\frac{n+1}{2}$ depending on whether n is even or odd.

Using this observation we can obtain the solutions of these equations.

Consider the iterative equations:

$$\theta_{jk} = \begin{cases} \frac{2}{n+1} (k\pi - \sin^{-1}(\rho \sin \frac{n-1}{2} \theta_{j-1,k})), & \text{for } k \text{ odd} \\ \frac{2}{n+1} (k\pi + \sin^{-1}(\rho \sin \frac{n-1}{2} \theta_{j-1,k})), & \text{for } k \text{ even} \end{cases} \quad (4.11)$$

where $\theta_{0k} \in (\frac{2k-1}{n}\pi, \frac{2k}{n+1}\pi)$, $-\frac{\pi}{2} \leq \sin^{-1}(\rho \sin \frac{n-1}{2} \theta_{j-1,k}) \leq \frac{\pi}{2}$, $j = 0, 1, \dots$,

and $k = 1, 2, \dots, \frac{n}{2}$ ($\frac{n-1}{2}$) for n even (odd); and

$$\theta_{jk} = \begin{cases} \frac{2}{n+1} ((k-1)\pi + \cos^{-1}(\rho \cos \frac{n-1}{2} \theta_{j-1,k})), & \text{for } k \text{ odd} \\ \frac{2}{n+1} ((k-1)\pi - \cos^{-1}(\rho \cos \frac{n-1}{2} \theta_{j-1,k})), & \text{for } k \text{ even,} \end{cases}$$

where $\theta_{ok} \in \left(\frac{2k-2}{n} \pi, \frac{2k-1}{n+1} \pi\right)$, $-\frac{\pi}{2} \leq \cos^{-1}(\rho \cos \frac{n-1}{2} \theta_{j-1,k}) \leq \frac{\pi}{2}$,
 $j = 0, 1, \dots$ and $k = 1, 2, \dots, \frac{n}{2} \left(\frac{n+1}{2}\right)$ for n even (odd).

The series of solutions obtained using this iterative procedure actually converges to the solutions of equations (4.9) and (4.10). This can easily be seen by means of the following theorem of differential calculus. (See e.g. [117]).

If $\theta_0 \in R$, where R is an interval containing a root of the equation $\theta = g(\theta)$, and if $g(\theta)$ is a differential function of θ throughout R and, in addition, $|g'(\theta)| < M < 1$ for all $\theta \in R$, then the sequence of iterative solutions of the equation $\theta_j = g(\theta_{j-1})$, starting with the point θ_0 , converges to a root of the equation.

In the case of the equations (4.11) and (4.12) the conditions of this theorem are clearly satisfied. In fact, θ_{ok} belongs, in each case, to an interval containing exactly one solution of (4.11) or (4.12) respectively. Moreover, the absolute values of the derivatives of (4.11) and (4.12) are:

$$\frac{n-1}{n+1} \left| \frac{\rho \cos \left(\frac{n-1}{2}\right) \theta}{\sqrt{1-\rho^2 \sin^2 \left(\frac{n-1}{2}\right) \theta}} \right| \quad (4.13)$$

and

$$\frac{n-1}{n+1} \left| \frac{\rho \sin \left(\frac{n-1}{2}\right) \theta}{\sqrt{1-\rho^2 \cos^2 \left(\frac{n-1}{2}\right) \theta}} \right| \quad (4.14)$$

which for all ρ , $|\rho| < 1$ and for all real θ , are, in each case, less than $\frac{n-1}{n+1} < 1$.

5. Comparison between the "approximate" and "exact" distributions of sums of identically distributed exponentially correlated Gamma-variables.

In order to obtain the distribution of sum of identically distributed Gamma variables correlated according to the stationary exponential law

($r(X_i, X_j) = \rho^{|i-j|}$, $i, j = 1, \dots, n$) we replace in (2.1) the matrix

$V = v_{ij} = \rho^{|i-j|}$ by the matrix $(\Sigma) = \{w_{ij}\} = \left\{ \rho^{\frac{|i-j|}{2}} \right\}$ so that the matrix $(\Sigma)^*$ which corresponds to the matrix V^* will have its elements $\{w_{ij}^*\} = \left\{ r \theta^2 \rho^{|i-j|} \right\}$, where $\lambda \theta^2$ is the common variance each of the gamma-variables X_i $i = 1, \dots, n$.

Using the UNIVAC 1105 of the Computation Center of the University of North Carolina we computed the parameters $\{c_k\}$ and λ^* of the distribution given by (3.11) for several different values of n , ρ , θ , x and compared several percentiles of this distribution with the corresponding percentiles of the "approximate" Gamma distribution with the parameters given by (1.2) and (1.3). The results of the computations are presented in Table 1 on page 12. We recall that the first two moments of these two distributions are identical; (this fact has been used for checking the correctness of the numerical calculations). Owing to computational difficulties and complications especially on computing the sequences $\{c_k\}$ we cannot claim higher accuracy than the second significant figure.

Comparing these two types of distributions we observe that the functional dependence of parameters on n follows the same pattern in both cases. The parameter r is asymptotically linear with n while θ is independent of n . The discrepancies between the corresponding distributions are relatively small and may be considered insignificant for most of the practical purposes in the cases of small values of ρ ($\rho \leq 0.3$) even for values of n as small as 5. The computations also indicate that for higher values of ρ ($\rho = 0.5$ and greater) the relatively large deviations at the lower tail decline sharply with the increase of the values of n . The number of c_k 's necessary to obtain the cumulative sum equal to 1 (up to 8-th significant figure) increases with both n and, even more rapidly, with ρ . For $n = 5$, $\rho = 0.2$ 100 c_k 's are sufficient, for $n=5$, $\rho = 0.5$ more than 200 are needed and for $n = 15$, $\rho = 0.5$ the number of the necessary c_k 's is over 300.

The number of iterative steps necessary to obtain the values of the characteristic roots λ_j with accuracy up to the 6-th significant digit also depends strongly on n and ρ and varies for the cases presented below between 5 and 18.

TABLE I

Comparison between the approximate and "exact" distributions of sum of identically distributed exponentially correlated Gamma variables

				Percentiles of exact distribution corresponding to the following percentiles of the approximate distribution					
n	ρ	r	θ	1	5	25	75	95	99
5	.2	1.92	.5	.62	4.14	24.81	75.73	94.80	98.75
10	.2	2.04	.5	.70	4.39	25.03	75.61	94.87	98.87
5	.3	1.94	.5	.47	3.75	24.67	75.93	94.68	98.63
10	.3	2.00	.5	.56	4.06	25.00	75.81	94.75	98.75
5		2.00		.28	3.18	24.60	76.24	94.77	98.72
10	.5	2.04	.5	.35	3.50	24.81	76.20		98.76
15		2.00		.42	3.74	25.10	76.0		98.8
5	.75	2.00		.18	3.01	23.8	76.6		98.6
5	.75	3.00	.5	.34	3.66	24.6	76.2		98.7
15	.332	.151	28.1	.10	1.40	20.1	77.5	95.2	98.70

*This case corresponds to a distribution of precipitation amounts at a certain station in .

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