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A NOTE ON THE PARAMETERS OF PBIB ASSOCIATION SCHEMES

by

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October 1963

This paper presents several results on association scheme parameters of partially balanced designs, dealing mostly with divisibility properties and based on the standard identities relating these parameters. This shows that these identities have not yet been fully exploited, even in the simplest and most familiar case of two associate classes.

This research was supported in part by the National Science Foundation, Grant No. GP-1660.

Institute of Statistics
Mimeo Series No. 375

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1. Introduction. In an m-class partially balanced incomplete block (PBIB) design [2], any two distinct treatments are related as first, second, ..., or m-th associates in accordance with certain rules, and the resulting classification of pairs of treatments is called an association scheme [3]. Parameters, including v, n_i, p^i_{jk}, which depend only on the association relation between treatments and are common to all designs having a given association scheme, are called association scheme parameters. Other parameters, including b, r, k, λ_i, depend in addition on the arrangement of the treatments into blocks. Known results on two-class association scheme parameters, reviewed in this section with some changes in arrangement and notation, are used in Section 2 to prove some new relations. Dependent as they are on known necessary conditions, our theorems will not provide any new proofs of the nonexistence of particular designs. However, they are in a form which is convenient for application and are oriented toward the fundamental problem of the connection between number-theoretic properties of the parameters and combinatorial structure of the designs.

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[1]Prepared with the partial support of the National Science Foundation, Grant GP-1660.

The association scheme parameters are non-negative integers which satisfy the familiar relations

(1.1) $$n_1 + n_2 = v - 1 \quad ,$$

(1.2) $$p^1_{11} + p^1_{12} + 1 = p^2_{11} + p^2_{12} = n_1 \quad ,$$

(1.3) $$p^1_{12} + p^1_{22} = p^2_{12} + p^2_{22} + 1 = n_2 \quad ,$$

(1.4) $$n_1 p^1_{12} = n_2 p^2_{11}, \quad n_1 p^1_{22} = n_2 p^2_{12} \quad .$$

The following are immediate consequences.

(1.5) $$n_1 p^1_{12} + n_2 p^2_{12} = n_1 n_2 \quad ,$$

(1.6) $$0 \le p^1_{12} \le n_1 - 1 \quad , \quad 0 \le p^2_{12} \le n_2 - 1 \quad .$$

We take the four integers n_i, p^i_{12} as fundamental parameters, subject to (1.5) and (1.6). If (1.1), (1.2), (1.3) are then taken as definitions of the remaining parameters it is easy to verify that they are all non-negative integers and that (1.4) is satisfied.

If N is the $v \times b$ incidence matrix of the design, then [4] the $v \times v$ symmetric matrix NN' has only three distinct characteristic roots $\theta_0, \theta_1, \theta_2$, with multiplicities $\alpha_0, \alpha_1, \alpha_2$ respectively, where $\Sigma\, \alpha_i = v$. θ_0 may be expressed

(1.7) $$\theta_0 = r + \lambda_1 n_1 + \lambda_2 n_2 \quad ,$$

and $\alpha_0 = 1$ if NN' is irreducible (equivalently, if the design is connected). Also,

(1.8)
$$\theta_1 = r + \lambda_1 t + \lambda_2(-t-1) ,$$
$$\theta_2 = r + \lambda_1 (-s-1) + \lambda_2 s ,$$

(1.9)
$$\alpha_1 = [sn_1 + (s+1)n_2]/\Delta^{\frac{1}{2}} ,$$
$$\alpha_2 = [(t+1)n_1 + tn_2]/\Delta^{\frac{1}{2}}$$

where

(1.10)
$$s = \tfrac{1}{2}(\Delta^{\frac{1}{2}} - \gamma - 1) ,$$
$$t = \tfrac{1}{2}(\Delta^{\frac{1}{2}} + \gamma - 1) ,$$

(1.11)
$$\gamma = p^2_{12} - p^1_{12} ,$$

(1.12)
$$\Delta = \gamma^2 + 2p^1_{12} + 2p^2_{12} + 1 .$$

s, t, α_1, α_2, γ, and Δ are association scheme parameters. The fact that

(1.13) α_1 and α_2 are integral

turns out to be an additional constraint on n_i, p^i_{12}.

We need the following remarks about two special families of association schemes.

For a given v, an association scheme of group divisible (GD) type [1] exists for each integer n which is a proper divisor of v, and the smaller of n_1, n_2 is equal to $n-1$. Moreover, the association scheme for given v, n is unique, and there exist no other GD schemes. A necessary and sufficient condition for a two-class scheme to be of GD type is

(1.14)
$$p^i_{12} = 0 , \qquad i = 1 \text{ or } 2 .$$

Association schemes of cyclic type are defined in terms of their combinatorial structure [3] and have parameters which can be expressed as follows in terms of an integer q.

$$p_{12}^1 = p_{12}^2 = q \quad , \quad n_1 = n_2 = \alpha_1 = \alpha_2 = 2q \;, \quad v = \Delta = 4q + 1 \quad .$$

Knowledge of their existence is incomplete, though they are known to exist whenever v is a prime. Following a usage suggested by R. H. Bruck, we use the name pseudo-cyclic for all two-class designs having these parameters; the existence of pseudo-cyclic designs which do not have the structure of cyclic designs has not been investigated. It follows from Theorems 5.3 and 5.5 of [4] that

(1.15) In a two-class association scheme not of pseudo-cyclic type, Δ must be a perfect square.

2. Relations among parameters. Using (1.10), (1.11) and (1.12), we observe that s and t are non-negative. We calculate

(2.1) $\Delta^{\frac{1}{2}} = s + t + 1 \;,\qquad \gamma = t - s \;,$

(2.2) $p_{12}^{1} = s(t + 1) \;,\qquad p_{12}^2 = (s + 1)\, t \;.$

If $\Delta^{\frac{1}{2}}$ is an integer, (2.1) shows that $2s$ and $2t$ are integers which must be even in view of (2.2). Hence, using (1.15), s and t are non-negative integers for all two-class association schemes not of pseudo-cyclic type, and the parameters of such schemes may be expressed in terms of $s, t, n_1, n_2,$ subject to (1.5), (1.6) and (1.13). From (1.14), the scheme is of GD type if and only if $st = 0$.

Among various consequences of (2.2), we note that the product of p_{12}^1 and p_{12}^2 is divisible by 4 and that if neither is zero, their ratio is between $\frac{1}{2}$ and 2, since

$$\tfrac{1}{2} \leq s/(s+1) < s(t+1)/(s+1)t < (t+1)/t \leq 2 \quad .$$

While (2.2) clarifies the nature of p_{12}^i, it does not make the other restrictions unnecessary. In (a), (b), (c) below, (1.5), (1.6), (1.13) respectively are stated for the p_{12}^i values corresponding to $s = 1$, $t = 3$, with examples to show that each of the three conditions may be violated by n_1, n_2 values which satisfy the other two.

(a) $4n_1 + 6n_2 = n_1n_2$, violated by $n_1 = 9$, $n_2 = 8$;

(b) $n_1 \geq 5$, $n_2 \geq 7$, violated by $n_1 = 18$, $n_2 = 6$;

(c) $(n_1 + 2n_2)/5$ is an integer, violated by $n_1 = 12$, $n_2 = 8$.

All three conditions are satisfied by the values $n_1 = 8$, $n_2 = 16$ and $n_1 = n_2 = 10$, which correspond to known association schemes.

The following makes use of (1.9), (2.2) and (1.1) - (1.4).

$$\begin{aligned}\Delta\, \alpha_1\alpha_2 &= [n_1 s + n_2(s + 1)]\,[n_1(t + 1) + n_2 t] \\ &= (n_1)^2\, p^1_{12} + n_1n_2\,(2st + s + t + 1) + (n_2)^2\, p^2_{12} \\ &= n_1n_2\, p^2_{11} + n_1n_2(p^1_{12} + p^2_{12} + 1) + n_1n_2\, p^1_{22} \\ &= n_1n_2\,(n_1 + n_2 + 1) \qquad ,\end{aligned}$$

giving

$$(2.3) \qquad vn_1n_2 = \Delta\, \alpha_1\, \alpha_2 \qquad ,$$

an interesting relations which seems to have received little notice.

THEOREM 1. A two-class association scheme with v equal to a prime must be of pseudo-cyclic type.

PROOF. Let $v = p$, a prime. Then n_1, n_2, α_1, α_2 are positive integers less than p. p but not p^2 is a divisor of vn_1n_2; from (2.3) the same is true of $\Delta\, \alpha_1\, \alpha_2$ and hence of Δ. Therefore Δ is not a square and by (1.15) the scheme is of pseudo-cyclic type.

COROLLARY. There are no two-class association schemes with v equal to a prime of the form $4m + 3$.

THEOREM 2. In a two-class association scheme the products vn_i, n_1n_2, $n_ip^i_{jk}$ are even integers, $i, j, k = 1, 2$.

PROOF. n_1 and n_2 are both even for pseudo-cyclic type schemes. For a scheme not of pseudo-cyclic type, first suppose that n_1 and n_2 are both odd. Then $sn_1 + (s + 1)n_2$ is odd and from (1.9), $\Delta^{\frac{1}{2}}$ is odd. (2.1) shows that s and t are of the same parity, from (2.2) p^i_{12} are both even and from (1.5) n_1n_2 is even, a contradiction. Therefore n_1 and n_2 are not both odd, and two and hence all three terms of (1.5) are even. If either of n_i, say n_1, is odd, the remaining products involving it may be expressed $vn_1 = n_1(n_1 + 1) + n_1n_2$, $n_1p^1_{22} = n_2p^2_{12}$, $n_1p^1_{11} = n_1(n_1-1) - n_1p^1_{12}$ with the aid of (1.1), (1.4), and (1.2), where the right hand side in each equation is even.

We remark that parity conditions related to those of Theorem 2 can be proved for association schemes with any number of classes by suitable enumeration of elements in the symmetric matrix NN'.

THEOREM 3. In a two-class association scheme not of GD type, n_1 and n_2 are not relatively prime.

PROOF. Let $n_1 = m_1d$, $n_2 = m_2d$, where d is the greatest common divisor of n_1, n_2. Then (1.5) leads to

$$(2.4) \qquad m_1p^1_{12} + m_2p^2_{12} = m_1m_2d \quad .$$

Each term in this equation must be divisible by each of the relatively prime integers m_1, m_2. Hence there exist non-negative integers u, w such that

$$(2.5) \qquad p^1_{12} = um_2 \quad , \quad p^2_{12} = wm_1 \quad , \quad u + w = d \; .$$

In a non-GD scheme p_{12}^1 and p_{12}^2 are positive; then u and w are positive and $d \geq 2$.

COROLLARY. In a two-class association scheme not of GD type, v cannot be of the form p+1, p a prime.

THEOREM 4. If p is an odd prime, there are exactly two GD association schemes with v = 2p, but no other two-class schemes unless p is of the form

$$p = 2s^2 + 2s + 1 \quad , \tag{2.6}$$

in which case the only possible parameters are given by

$$p_{12}^1 = p_{12}^2 = s(s+1) \quad , \tag{2.7}$$

$$n_1 = s(2s+1), \qquad n_2 = (s+1)(2s+1) \quad . \tag{2.8}$$

PROOF. The assertion about GD schemes is proved by noting that 2 and p are the only proper divisors of v. Now assume the scheme not of GD type. From Theorem 3, n_1 and n_2 are not relatively prime and must be distinct from p and p-1. Then p but not p^2 is a factor of $vn_1n_2 = \Delta\,\alpha_1\alpha_2$. Since $v \not\equiv 1 \pmod 4$, it follows from (1.15) that the integer Δ is a perfect square; not being divisible by p^2, it is not divisible by p. Therefore the product of α_1 and α_2 is divisible by p. Neither of α_1, α_2 is as large as 2p, since their sum is v-1 = 2p - 1; hence one of them is equal to p. We choose notation so that

$$\alpha_1 = p \quad , \qquad \alpha_2 = p - 1 \quad . \tag{2.9}$$

Then (2.3) reduces to

$$2n_1n_2 = \Delta\,(p-1) \quad . \tag{2.10}$$

The parameters s and t are integers. From (1.9) ,

$$p\,\Delta^{\frac{1}{2}} = s(n_1 + n_2) + n_2 \quad ,$$

(2.11)

$$(p-1)\Delta^{\frac{1}{2}} = t(n_1 + n_2) + n_1 \quad .$$

Subtracting,

(2.12) $$\Delta^{\frac{1}{2}} = (s - t)(n_1 + n_2) + n_2 - n_1 \quad .$$

The integer $s-t$ is non-negative since $\Delta^{\frac{1}{2}}$ is positive. If $s-t$ is positive, then (2.12) shows $\Delta^{\frac{1}{2}} \geq 2n_2$ and $\Delta \geq 4(n_2)^2$. Then

$$4(n_2)^2\,(p-1) \leq \Delta\,(p-1) = 2n_1n_2 = 2(2p - 1 - n_2)n_2 \quad ,$$

reducing to $n_2 \leq 1$, which is impossible for a non-GD association scheme. Therefore $s-t$ is non-positive, we have $s=t$, and (2.2) gives (2.7). Simplifying (2.12) and using (1.1) we have

$$n_2 - n_1 = \Delta^{\frac{1}{2}} = s + t + 1 = 2s + 1 \quad ,$$

$$n_2 + n_1 = v - 1 = 2p - 1 \quad ,$$

which can be solved to give

$$n_1 = p - s - 1, \qquad n_2 = p + s \quad .$$

Using this in (2.10), we obtain this quadratic equation in p.

$$2(p - s - 1)(p + s) = (2s + 1)^2\,(p-1) \quad .$$

The solutions are $p = \frac{1}{2}$, extraneous to this problem, and $p = 2s^2 + 2s + 1$, proving (2.6) and leading to (2.8) to complete the proof.

Theorem 4 excludes two-class schemes not of GD type and with $v = 2p$ for many primes, including those of the form $4m + 3$. The only primes less than 300 of the form (2.6) are 5, 13, 41, 61, 113, 181. Association schemes of the family specified by (2.7) and (2.8) have the special property that $v = \Delta + 1$ and are known for many values of s [6], including some in which $2s^2 + 2s + 1$ is composite; in the latter case, however, other non-GD schemes for the same v may be possible. For example, $s=3$ gives $2s^2 + 2s + 1 = 25$, and two non-GD schemes are known with $v = 50$, one in the present family with parameters $n_1 = 21$, $n_2 = 28$, $p^1_{12} = p^2_{12} = 12$, and another [5] with parameters $n_1 = 7$, $n_2 = 42$, $p^1_{12} = p^2_{12} = 6$.

THEOREM 5. In a two-class association scheme not of GD type, $n_i \geq (v - 1)^{\frac{1}{2}}$, $i = 1, 2$.

PROOF. The following proof for n_1 uses (1.1) - (1.4); it follows from (1.14) and (1.4) that we may assume $p^2_{11} \geq 1$. Interchanging the indices 1 and 2 gives a proof for n_2.

$$n_1 = p^1_{11} + p^1_{12} + 1 ,$$

$$\begin{aligned}(n_1)^2 &= n_1 p^1_{11} + n_1 p^1_{12} + n_1 \\ &= n_1 + n_2 + n_1 p^1_{11} + n_2(p^2_{11} - 1) \geq n_1 + n_2 = v - 1 .\end{aligned}$$

The inequality of Theorem 5 need not hold for GD schemes and is the best possible for other schemes, as shown for example by the triangular scheme with $v = 10$, $n_1 = 3$ and by the above parameters with $v = 50$, $n_1 = 7$. On the other hand, this equality is possible only in isolated cases. To see this, assume $(n_1)^2 = v - 1$. It follows from the above proof that $p^1_{11} = 0$ and $p^2_{11} = 1$, which is enough to determine the parameters with $p^1_{12} = p^2_{12} = n_1 - 1 = s^2 + s$. Then, using (1.9) ,

$$\alpha_1 = [s\ (s^2 + s + 1) + (s + 1)(s^2 + s + 1)(s^2 + s)]/(2s + 1)$$

$$= 2s + 5s^3 - 7s^4 + 15s^5/(2s + 1) \quad ,$$

which is integral only if $2s + 1$ is a divisor of 15. The three possibilities $s = 1, 2, 7$ lead to the examples just mentioned and to one more with $v = 3250$.

3. Acknowledgements. Theorems 1 (with a different proof), 2 and 3 appeared in the author's doctoral dissertation at Michigan State University. Equation (2.3) was subsequently called to the author's attention by J. S. Frame.

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