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SOME REMARKS ON A DISTRIBUTION OCCURRING IN NEURAL STUDIES

by

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Introduction. Suppose that X is a non-negative random variable (which does not vanish with probability one) such that $\mathcal{E} \log(1 + X)$ is finite and, for real positive s, put $\varphi(s) = \mathcal{E} e^{-sX}$. For any $c > 0$ we shall show there exists a distribution function $G(x)$, of a non-negative random variable, such that

$$(1) \qquad G^*(s) = \int_0^\infty e^{-sx} \, dG(x) = e^{-c\int_0^s \frac{1-\varphi(z)}{z} dz} .$$

This distribution function $G(x)$ arises in a variety of contexts. The author obtained it many years ago in some unpublished work on the initiation of nerve pulses. It has also arisen in studies of a certain recording apparatus (Takacs, 1955) and of the "present value" of a renewal process (Dall'Aglio, 1963). More recently it was derived in a colloquium at University College, London, by Dr. J. Keilson, who raised the question of whether $G(x)$ is absolutely continuous and, if so, of how the corresponding probability density function behaves near the origin. It is the object of the present paper to prove the following theorem of several parts.

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Theorem 1. If $F(x)$ is the distribution function of X and we assume that

(2) $$\int_0^\infty \log(1+x)\, dF(x) < \infty ,$$

then: (1.1) Equation (1) defines an absolutely continuous distribution function $G(x)$ with a probability density function $g(x)$, say, which is continuous on the open interval $(0, \infty)$. (1.2) There is a strictly decreasing function $D_1(x)$ such that $G(x) = x^c D_1(x)$, and $D_1(0+)$ is finite if and only if (in addition to (2))

(3) $$\int_0^1 \frac{F(x)}{x}\, dx < \infty ,$$

in which case

$$D_1(0+) = \frac{1}{\Gamma(1+c)}\, e^{-c \int_0^\infty \frac{1 - e^{-x} - F(x)}{x} dx} .$$

Furthermore, if $F(\tau) = 0$ for some $\tau > 0$ then $D_1(x)$ is constant in $(0, \tau)$.

(1.3) There is a strictly decreasing convex function $D_2(x)$ such that $[1 - G(x)] = x^c D_2(x)$. If, for some $0 \leq \gamma < 1$,

(4) $$\int_0^x [1 - F(y)]\, dy \sim x^\gamma L(x) , \qquad \text{as } x \rightarrow \infty ,$$

where $L(x)$ is a function of slow growth, then

(5) $$[1 - G(x)] \sim \frac{c\, \gamma\, L(x)}{(1-\gamma)\, x^{(1-\gamma)}} , \quad \text{as } x \rightarrow \infty .$$

If, however,

(6) $$\int_0^x [1 - F(y)]\, dy \sim x\, L(x) , \qquad \text{as } x \rightarrow \infty ,$$

then

(7) $$[1 - G(x)] \sim c\, M(x) \quad , \qquad \text{as } x \longrightarrow \infty ,$$

where

$$M(x) = \int_x^\infty \frac{L(z)}{z}\, dz$$

and $M(x)$ is also a function of slow growth.

(1.4) The continuous probability density function $g(x)$ is such that $x^{(1-c)} g(x) = d(x)$, say, a strictly decreasing function of x, and $x\, g(x)$ is a function of bounded variation. Moreover $d(0+)$ is finite if and only if (3) holds, in which case $d(0+) = c\, D_1(0+)$. If (4) should hold, then

(8) $$g(x) \sim \frac{c\, \gamma\, L(x)}{x^{2-\gamma}} \quad , \qquad \text{as } x \longrightarrow \infty ,$$

while, if (6) should hold, then

(9) $$g(x) \sim \frac{c\, L(x)}{x} \quad , \qquad \text{as } x \longrightarrow \infty .$$

(1.5) If, for some $A > 0$, $\lambda \geq 0$, $\nu \geq 0$, and for all sufficiently large x,

$$1 - F(x) \leq \frac{A\, e^{-\lambda x}\, x^{\nu}}{\Gamma(\nu+1)}$$

then, as $x \longrightarrow \infty$,

$$g(x) = O\left\{ \frac{\exp\{-\lambda x + \frac{\nu+1}{\nu} (A\, c)^{\frac{1}{\nu+1}}\, x^{\frac{\nu}{\nu+1}} \}}{x^{\frac{1}{2}(\frac{\nu+1}{\nu+2})}} \right\} , \qquad \nu > 0 ,$$

$$= O\left(e^{-\lambda x}\, x^{Ac-1}\right) \quad , \qquad \nu = 0 .$$

To prove Theorem 1 we find it necessary to establish the following three theorems concerning a more general class of density functions.

Theorem 2. If $a(x) \geq 0$ and

$$\int_0^1 \frac{a(x)}{x}\, dx = \infty, \quad \int_0^\infty \frac{a(x)}{1+x}\, dx < \infty,$$

and if we write, for $\mathcal{R}\, s \geq 0$,

$$a^0(s) = \int_0^\infty e^{-sx}\, a(x)\, dx,$$

then there is a probability density function $\Delta_a(x)$, say, on $(0, \infty)$ such that

$$\Delta_a^0(s) = \int_0^\infty e^{-sx}\, \Delta_a(x)\, dx = e^{-\int_0^s a^0(z)\, dz},$$

where the contour integral in the exponent is taken along a straight line.

Theorem 3. In the notation of Theorem 2, if $a(x)$ is continuous and of bounded variation then we may take $\Delta_a(x)$ as continuous and of bounded variation in any interval not containing the origin. Moreover, if $a(x) < A\, e^{-\eta x}$ for some $A > 0$, $\eta > 0$, then $\Delta_a(x) = O(e^{-\eta x}\, x^{A-1})$.

Theorem 4. If $a_1(x)$ and $a_2(x)$ both satisfy the conditions of Theorem 2 and if $a_1(x) \geq a_2(x)$ for all x and

$$\int_0^1 \frac{a_1(x) - a_2(x)}{x}\, dx < \infty,$$

then, in an obvious extension of notation,

$$\int_x^\infty \Delta_{a1}(y)\, dy \geq \int_x^\infty \Delta_{a2}(y)\, dy$$

for all $x \geq 0$, and

$$e^{\int_0^\infty \frac{a_1(x) - a_2(x)}{x} dx} \quad \Delta_{a1}(x) \geq \Delta_{a2}(x)$$

for almost all x.

In part of our argument we make use of the continuity theorem for Laplace-Stieltjes transforms. There does not seem to be any convenient reference for this useful theorem (although its use occurs in the literature from time to time). We therefore append a short proof in an appendix.

Proof of Theorem 2. Write, for fixed $\delta > 0$,

$$I_\delta = \int_\delta^\infty \frac{a(x)}{x} dx$$

and define

$$g_\delta(x) = 0 \quad , \text{ for } x < \delta ,$$
$$= \frac{a(x)}{x I_\delta} \quad , \text{ for } x \geq \delta .$$

Then $g_\delta(x)$ is a probability density function. Suppose that $Z_1, Z_2, Z_3, \ldots$ is an infinite sequence of independent random variables, each governed by the density function $g_\delta(x)$. Suppose M is an integer-valued random variable, independent of the $\{Z_n\}$, such that for $r = 0, 1, 2, \ldots$

$$P\{M = r\} = \frac{e^{-I_\delta}(I_\delta)^r}{r!} .$$

Define a random variable $Y = 0$ if $M = 0$, and $Y = Z_1 + Z_2 + \ldots + Z_M$ otherwise. Then it is an easy matter to see that Y has a distribution function $G_\delta(x)$, say, where

$$G_\delta^*(s) = \int_0^\infty e^{-sx} d\, G_\delta(x) = e^{-I_\delta + I_\delta g_\delta^{\circ}(s)}$$

(in the notation already suggested in the enunciation of Theorem 2 we have written $g_\delta^{\circ}(s)$ for the ordinary Laplace transform of $g_\delta(x)$). Thus we have

$$(10) \qquad \log G^*_\delta(s) = -\int_\delta^\infty \frac{(1-e^{-sx})}{x} a(x)\, dx \quad .$$

As δ decreases to zero we see from (10) and Beppo Levi's theorem that $G^*_\delta(s) \longrightarrow G^*_o(s)$, say, where

$$(11) \qquad \log G^*_o(s) = -\int_o^\infty \frac{(1-e^{-sx})}{x} a(x)\, dx \quad .$$

In view of our hypothesis about $a(x)$ it is clear that the integral on the right of (11) is absolutely convergent. Also, from Lebesgue's theorem on dominated convergence, we can deduce that $G^*_o(s) \longrightarrow 1$ as s decreases through real values to zero. It follows therefore, from the continuity theorem for Laplace-Stieltjes transforms, that there is a distribution function $G_o(x)$ over $[0, \infty)$ such that

$$G^*_o(s) = \int_o^\infty e^{-sx}\, d\, G_o(x) \quad .$$

Furthermore, by Fubini's Theorem ,

$$\int_o^\infty \frac{(1-e^{-sx})}{x} a(x)\, dx = \int_o^\infty \int_o^s e^{-zx} a(x)\, dz\, dx$$

$$= \int_o^s a^o(z)\, dz \quad .$$

Thus the theorem will be proved if we show that $G_o(x)$ is absolutely continuous. To this end, we differentiate (11) and find that

$$(12) \qquad -\frac{d}{ds} G^*_o(s) = a^o(s)\, G^*_o(s) \quad .$$

Hence, if $\ell(x)$ is defined by

$$\ell(x) = \int_0^x a(x - z)\, d\, G_0(z)$$

it is a consequence of (12) that

$$(13) \qquad \int_0^\infty e^{-sx}\, x\, d\, G_0(x) = \int_0^\infty e^{-sx}\, \ell(x)\, dx \quad .$$

From (13) we infer that, except for a possible discontinuity at the origin, $G_0(x)$ is absolutely continuous with a density function

$$\Delta_a(x) = \frac{d}{dx}\, G_0(x) = \frac{\ell(x)}{x} \quad .$$

Finally, we rule out the possibility of a point mass of probability at the origin by observing that its weight must equal (taking the limit through real values)

$$\lim_{s \to \infty} e^{-\int_0^s a^0(z)\, dz} = e^{-\int_0^\infty \frac{a(x)}{x}\, dx} \quad ,$$

which is zero, by our hypothesis that the integral

$$\int_0^1 \frac{a(x)}{x}\, dx$$

diverges. Thus the theorem is proved.

<u>Proof of Theorem 3</u>. The continuity and bounded variation properties claimed for $\Delta_a(x)$ are easy consequences of the representation

$$\Delta_a(x) = \frac{1}{x} \int_0^x a(x - z)\, d\, G_0(z) \quad .$$

From this equation we also see that if $a(x) < A\, e^{-\eta x}$ then

$$e^{\eta x}\, \Delta_a(x) \; < \; \frac{A}{x} \int_0^x e^{\eta z}\, \Delta_a(z)\, dz \quad . \tag{14}$$

Therefore

$$\frac{d}{dx} \log \int_0^x e^{\eta z}\, \Delta_a(z)\, dz \; < \; \frac{A}{x}$$

and so

$$\log \left\{ \frac{\int_0^x e^{\eta z}\, \Delta_a(z)\, dz}{\int_0^1 e^{\eta z}\, \Delta_a(z)\, dz} \right\} < \qquad A \log x \quad .$$

Thus

$$\int_0^x e^{\eta z}\, \Delta_a(z)\, dz \; < \; x^A \int_0^1 e^{\eta z}\, \Delta_a(z)\, dz$$

and hence, from (14) again, we have

$$e^{\eta x}\, \Delta_a(x) \; < \; A\, x^{A-1} \int_0^1 e^{\eta z}\, \Delta_a(z)\, dz \quad ,$$

which completes the proof of the theorem.

<u>Proof of Theorem 4.</u> We extend the notation used in the earlier proofs, with the aid of suffices, in an obvious way. Except for the convergence of the integral

$$\int_0^1 \frac{a_1(x) \; - \; a_2(x)}{x} \; dx \tag{15}$$

the non-negative function $a_1(x) - a_2(x)$ satisfies all the conditions imposed upon $a(x)$ in Theorem 2. Thus we can say there is a distribution function $H(x)$, say, of a non-negative random variable, such that

$$(16) \qquad e^{-\int_0^s \{a_1^o(z) - a_2^o(z)\}\, dz} = H^*(s) \quad .$$

Because of the convergence of (15) it will be seen from the proof of Theorem 2 that the function $H(x)$ will have a discontinuity at the origin, but will otherwise be absolutely continuous. From (16) it follows that

$$G_{o1}(x) = \int_0^x G_{o2}(x - z)\, d\, H(z)$$

$$\leq G_{o2}(x) \quad ,$$

and therefore

$$\int_x^\infty \Delta_{a1}(y)\, dy \geq \int_x^\infty \Delta_{a2}(y)\, dy$$

as claimed.

Let us now write

$$g_\delta^{[2]}(x) = \int_0^x g_\delta(x-z)\, g_\delta(z)\, dz$$

and, for $n > 2$,

$$g_\delta^{[n]}(x) = \int_0^x g_\delta^{[n-1]}(x-z)\, g_\delta(z)\, dz \quad .$$

Then, from the proof of Theorem 2, it is evident that $G_{\delta 1}(x)$ has a jump at the origin of amount $e^{-I_{\delta 1}}$, but is absolutely continuous otherwise, and, for almost all $x > 0$,

$$\frac{d}{dx}\, G_{\delta 1}(x) = \sum_{n=1}^{\infty} \frac{e^{-I_{\delta 1}}\, (I_{\delta 1})^n}{n!}\, g_{\delta 1}^{[n]}(x) \quad .$$

But, by our hypothesis,

$$(I_{\delta 1})^n \; g_{\delta 1}^{[n]}(x) \geq (I_{\delta 2})^n \; g_{\delta 2}^{[n]}(x)$$

for all x. Hence

$$\frac{d}{dx} G_{\delta 1}(x) \geq e^{-\{I_{\delta 1} - I_{\delta 2}\}} \frac{d}{dx} G_{\delta 2}(x) \quad .$$

Thus, if $0 < \alpha < \beta$,

$$G_{\delta 1}(\beta) - G_{\delta 1}(\alpha) \geq e^{-\{I_{\delta 1} - I_{\delta 2}\}}\{G_{\delta 2}(\beta) - G_{\delta 1}(\alpha)\} .$$

If we now let δ decrease to zero we find that

$$G_{o1}(\beta) - G_{o1}(\alpha) \geq e^{-\{I_{\delta 1} - I_{\delta 2}\}}\{G_{o2}(\beta) - G_{o2}(\alpha)\} \quad ,$$

that is

$$\int_{\alpha}^{\beta} \Delta_{a1}(x)\, dx \geq e^{-\int_0^{\infty} \frac{a_1(x) - a_2(x)}{x} dx} \int_{\alpha}^{\beta} \Delta_{a2}(x) \, dx \; .$$

Since the last inequality holds for arbitrary α and β (> 0), the final contention of Theorem 4 is proved.

<u>Proof of Theorem 1, Part (1.1).</u> It is clear that the integral

$$\int_0^1 \frac{1 - F(x)}{x} \, dx$$

diverges, and an integration by parts will show that

$$\int_0^{\infty} \log\,(1 + x)\, dF(x) = \int_0^{\infty} \frac{1 - F(x)}{1 + x} \, dx \quad .$$

Thus, if we put $a(x) = c\{1 - F(x)\}$ then this function satisfies all the conditions of Theorem 2. Upon noting that

$$\int_0^\infty e^{-zx} \{1 - F(x)\} \, dx = \frac{1 - \varphi(z)}{z}$$

we can therefore infer that

$$e^{-c \int_0^\delta \frac{1 - \varphi(z)}{z} \, dz}$$

is, indeed, the Laplace-Stieltjes transform of an absolutely continuous distribution function $G(x)$. Furthermore, if we write $g(x)$ for a density function corresponding to $G(x)$ then we may put

$$g(x) = \frac{c}{x} \int_0^x \{1 - F(x - z)\} \, g(z) \, dz$$

(17) $$= \frac{c \, G(x)}{x} - \frac{c}{x} \int_0^x G(x - z) \, dF(z) \quad .$$

The distribution function $G(x)$ is continuous and therefore

$$\int_0^x G(x - z) \, dF(z)$$

is also a continuous function of x. Equation (17) therefore shows $g(x)$ to be continuous as claimed.

<u>Proof of Part (1.2)</u>. From (17) we see that

$$\frac{d}{dx} G(x) = \frac{c}{x} G(x) - \frac{c}{x} \int_0^x G(x-z) \, dF(z)$$

so that

(18) $$\frac{d}{dx} [x^{-c} G(x)] = - \frac{c}{x^{(1+c)}} \int_0^x G(x-z) \, dF(z) \quad .$$

The right-hand side of (18) is negative; therefore $x^{-c}\,G(x) = D_1(x)$, say, is a decreasing function as was to be proved.

Suppose that $D_1(x)$ increases to a finite limit A, say, as x decreases to zero. Then, by a familiar Abelian theorem for Laplace-Stieltjes transforms (Widder, 1941, p. 181) $s^c\,G(s) \longrightarrow A\,\Gamma(1 + c)$ as $s \longrightarrow +\infty$ (through real values). Therefore

$$c \log s - c \int_0^s \frac{1 - \varphi(z)}{z}\, dz \longrightarrow \log [A\,\Gamma(1 + c)] ,$$

that is

$$c \int_1^s \frac{\varphi(z)}{z}\, dz - c \int_0^1 \frac{1 - \varphi(z)}{z}\, dz \longrightarrow \log [A\,\Gamma(1 + c)]$$

as $s \longrightarrow \infty$. But, by Fubini's theorem,

$$\int_1^s \frac{\varphi(z)}{z}\, dz = \int_1^s \int_0^\infty e^{-zx} F(x)\, dx\, dz$$

$$= \int_0^\infty \frac{e^{-x} - e^{-sx}}{x} F(x)\, dx .$$

We can thence deduce from Beppo Levi's theorem that

$$\int_1^\infty \frac{\varphi(z)}{z}\, dz = \int_0^\infty \frac{e^{-x} F(x)}{x}\, dx \leq \infty .$$

From all this we may conclude that A is finite if and only if $F(x)/x$ belongs to $L_1(0, 1)$, as is claimed in this part of the theorem. When A happens to be finite we see that

$$\log [A\,\Gamma(1 + c)] = c \int_0^\infty \frac{e^{-x} F(x)}{x}\, dx - c \int_0^1 \frac{1 - \varphi(z)}{z}\, dz .$$

But

$$\int_0^1 \frac{1 - \varphi(z)}{z} \, dz = \int_0^\infty \frac{(1 - e^{-x})}{x} \, [1 - F(x)] \, dx \quad ,$$

and so

$$\log [A \, \Gamma(1 + c)] = - c \int_0^\infty \frac{1 - e^{-x} - F(x)}{x} \, dx \quad ,$$

which proves the value for $D_1(0+)$. We also note that should $F(x) = 0$ for all $x < \tau$ then, by (18), $x^{-c} \, G(x)$ is constant for all $x < \tau$. This completes the proof of Part (1.2).

Proof of Part (1.3). From (17) we have, for $x > 0$,

$$(19) \qquad \frac{d}{dx} \, [x^{-c} \, \{1 - G(x)\}] = - \frac{c}{x^{(1+c)}} \, \{1 - K(x)\}$$

where $K(x)$ is the absolutely continuous distribution function

$$(20) \qquad K(x) = \int_0^x F(x - z) \, dG(z) \quad .$$

From (19) it is apparent that $x^{-c} \, \{1 - G(x)\} = D_2(x)$, say, is a decreasing function with an increasing derivative; in particular, $D_2(x)$ is convex.

Now suppose that for some $0 \leq \gamma \leq 1$ and some function of slow growth $L(x)$

$$\int_0^x [1 - F(y)] dy \sim x^\gamma \, L(x) \quad , \qquad \text{as } x \longrightarrow \infty \ .$$

Then, by a slightly more complicated Abelian theorem than the one we have already used (Doetsch, 1950, p.   ), we have that

$$\frac{1 - \varphi(s)}{s} \sim \frac{\Gamma(1+\gamma)\, L(\frac{1}{s})}{s^{\gamma}}$$

as $s \longrightarrow 0+$ through real values. However, we can discover from (1) that as $s \longrightarrow 0+$

$$\frac{1 - G^{*}(s)}{s} \sim \frac{c}{s} \int_{0}^{s} \frac{1 - \varphi(z)}{z}\, dz$$

$$\sim \frac{c}{s}\, \Gamma(1+\gamma) \int_{0}^{s} \frac{L(\frac{1}{z})}{z^{\gamma}}\, dz \, .$$

Before we can proceed we must discover the asymptotic behavior of the integral on the right. By an obvious change of variable we have

$$\frac{1}{s} \int_{0}^{s} \frac{L(z^{-1})}{z^{\gamma}}\, dz = \frac{1}{s^{\gamma}} \int_{1}^{\infty} \frac{L(u/s)}{u^{2-\gamma}}\, du \, .$$

Now Karamata (1930) has shown that for a given function of slow growth $L(x)$ there is necessarily a function $\rho(x)$ such that $\rho(x) \longrightarrow 1$ as $x \longrightarrow \infty$ and

$$L(x) = \frac{\rho(x)}{x}\, e^{\int_{1}^{x} \frac{\rho(v)}{v}\, dv} \, .$$

From this fact it is an easy deduction that for arbitrary $\epsilon > 0$

$$0 < \frac{L(u/s)}{L(s^{-1})} < (1+\epsilon)\, u^{\epsilon}$$

for all sufficiently large u and all sufficiently small s. Therefore, if $\gamma < 1$, we can appeal to dominated convergence to infer that

$$\lim_{s \longrightarrow 0+} \int_{1}^{\infty} \frac{L(u/s)}{L(s^{-1})} \frac{du}{u^{2-\gamma}} = \frac{1}{(1-\gamma)}$$

and hence that

$$\frac{1}{s} \int_0^s \frac{L(z^{-1})}{z^\gamma} dz \sim \frac{L(s^{-1})}{s^\gamma(1-\gamma)} .$$

Hence

$$\frac{1 - G^*(s)}{s} \sim \frac{c\,\Gamma(1+\gamma)\,L(s^{-1})}{(1-\gamma)\,s^\gamma} ,$$

as $s \to 0+$. From a Tauberian theorem for Laplace transforms (Doetsch, 1950, p. 511) we can then deduce that

$$\int_0^x \{1 - G(y)\}dy \sim \frac{c\,L(x)\,x^\gamma}{(1-\gamma)} , \quad \text{as} \quad x \to \infty .$$

Furthermore, from (20) ,

$$K^*(s) = \varphi(s)\,G^*(s) ,$$

so that, as $s \to 0+$,

$$\frac{1 - K^*(s)}{s} \sim \frac{1 - \varphi(s)}{s} + \frac{1 - G^*(s)}{s}$$

$$\sim \frac{\Gamma(1+\gamma)\,L(s^{-1})}{s^\gamma} + \frac{c\,\Gamma(1+\gamma)\,L(s^{-1})}{(1-\gamma)s^\gamma} .$$

Hence, by another Tauberian argument ,

$$\int_0^x \{1 - K(y)\}dy \sim \frac{(1-\gamma+c)L(x)\,x^\gamma}{(1-\gamma)} , \quad \text{as} \quad x \to \infty .$$

If we multiply (19) by $x^{(1+c)}$ and integrate by parts we find that

(21) $$x\{1 - G(x)\} = (1+c)\int_0^x \{1 - G(y)\}dy - c\int_0^x \{1 - K(y)\}\,dy .$$

From the asymptotic results we have obtained it follows from (21) that

$$1 - G(x) \sim \frac{c\,\gamma\, L(x)}{(1-\gamma)\, x^{1-\gamma}}\,, \qquad \text{as} \quad x \longrightarrow \infty\,.$$

In the case $\gamma = 1$ we cannot employ the dominated convergence argument and the results come out somewhat differently. Let us define

$$M(x) = \int_x^\infty \frac{L(z)}{z}\, dz\,.$$

Then for any fixed $\alpha > 0$

$$M(\alpha x) = \int_x^\infty \frac{L(\alpha z)}{z}\, dz$$

and hence, for an arbitrary $\epsilon > 0$ and all sufficiently large x,

$$(1-\epsilon)\int_x^\infty \frac{L(z)}{z}\, dz < M(\alpha\, x) < (1+\epsilon)\int_x^\infty \frac{L(z)}{z}\, dz\,.$$

It is obvious therefore that $M(\alpha\, x) \sim M(x)$ as $x \longrightarrow \infty$ and that $M(x)$ is consequently a function of slow growth. We thus obtain for this case

$$\frac{1 - G^*(s)}{s} \sim \frac{c\, M(s^{-1})}{s}\,, \quad \text{as} \quad s \longrightarrow 0+\,,$$

and so, via the Tauberian theorem,

$$\int_0^x \{1 - G(y)\}dy \sim c\, x\, M(x)\,, \qquad \text{as} \quad x \longrightarrow \infty\,.$$

We shall show in a moment that $L(x)/M(x) \longrightarrow 0$ as $x \longrightarrow \infty$. It then follows, as before, that

$$\frac{1 - K^*(s)}{s} \sim \frac{L(s^{-1})}{s} + \frac{c\, M(s^{-1})}{s}$$

$$\sim \frac{c\, M(s^{-1})}{s}$$

and so ,

$$\int_0^x \{1 - K(y)\}dy \sim c\, x\, M(x) \quad .$$

From (21) we can then deduce that

$$\{1 - G(x)\} \sim c\, M(x) \quad , \quad \text{as} \quad x \longrightarrow \infty \quad ,$$

which was to be proved.

To see that $L(x)/M(x) \longrightarrow 0$ as $x \longrightarrow \infty$ we observe that for Δ arbitrarily large and positive

$$M(x) > \int_x^{x\Delta} \frac{L(z)}{z}\, dz$$

$$= \int_1^{\Delta} \frac{L(ux)}{u}\, du$$

Thus

$$\frac{M(x)}{L(x)} > \int_1^{\Delta} \left\{\frac{L(ux)}{L(u)}\right\} \frac{du}{u}$$

and so, by a dominated convergence which can be justified much as before ,

$$\liminf_{x \longrightarrow \infty} \frac{M(x)}{L(x)} \geq \int_1^{\Delta} \frac{du}{u} \quad .$$

This establishes the correctness of our assertion.

<u>Proof of Part (1.4).</u> By (17) and (20) we have

(22) $$x\, g(x) = c\, G(x) - c\, K(x) \quad ,$$

so that $x\, g(x)$ is of bounded variation as claimed. If we differentiate this last equation (and write $k(x)$ for the, necessarily continuous, density function associated with $K(x)$) we find that

$$x\, g'(x) + (1 - c)\, g(x) = -c\, k(x)$$

which implies that

$$\frac{d}{dx}\, [x^{(1-c)}\, g(x)] = - \frac{c\, k(x)}{x^c} \quad .$$

Therefore $x^{(1-c)}\, g(x) = d(x)$, say, where $d(x)$ is a strictly decreasing function. If $d(0+) = \infty$ then, given any large Δ we have $g(x) > \Delta\, x^{-(1-c)}$ and therefore $G(x) > x^c \Delta / c$, for all sufficiently small x. Hence $d(0+) = \infty$ only if $D_1(0+) = \infty$. On the other hand, if $d(0+)$ is finite it is clear that, for small x, $G(x) \sim d(0+)\, x^c/c$, so that $d(0+) = c\, D_1(0+)$ and, incidentally, $D_1(0+)$ is seen to be finite.

To complete the proof of this part we need the following

Lemma 1. If $\int_0^x \{1 - F(y)\} dy \sim x^\gamma L(x)$ as $x \longrightarrow \infty$, where $0 \leq \gamma \leq 1$ and $L(x)$ is a function of slow growth, then

$$- \frac{s^\gamma\, \varphi'(s)}{L(s^{-1})} \longrightarrow (1 - \gamma)\, \Gamma\, (1 + \gamma), \quad \text{as} \quad s \longrightarrow 0+ \quad .$$

Proof. We note that, as $x \longrightarrow \infty$,

$$\frac{1}{x^{1+\gamma}} \int_0^x \left\{ \int_0^y [1 - F(z)] dz \right\} dy \sim \frac{1}{x^{1+\gamma}} \int_0^x y^\gamma\, L(y)\, dy$$

$$\sim \frac{L(x)}{\gamma + 1} \quad ,$$

by Theorème 1 of Karamata (1930, p. 40). But an integration by parts shows

$$\frac{1}{x^{1+\gamma}} \int_0^x y\{1 - F(y)\}\, dy$$

$$= \frac{1}{x^\gamma} \int_0^x \{1 - F(y)\} dy - \frac{1}{x^{1+\gamma}} \int_0^x \{ \int_0^y [1-f(z)] dz \} dy$$

and hence we have

$$\int_0^x y\{1 - F(y)\}dy \sim \frac{\gamma\, x^{\gamma+1}\, L(x)}{\gamma+1}\ , \qquad \text{as} \quad x \longrightarrow \infty\ .$$

The Laplace transform of $x\{1 - F(x)\}$ is

$$\frac{1 - \varphi(s)}{s^2} + \frac{\varphi'(s)}{s}$$

and so, by the Abelian theorem for Laplace transforms, as $s \longrightarrow 0+$,

$$\frac{1 - \varphi(s)}{s^2} + \frac{\varphi'(s)}{s} \sim \frac{\gamma\, \Gamma(\gamma + 2)\, L(s^{-1})}{(\gamma + 1)\, s^{\gamma + 1}}\ .$$

But

$$\frac{1 - \varphi(s)}{s} \sim \frac{\Gamma(1 + \gamma)\, L(s^{-1})}{s^{\gamma}}\ ,$$

from the hypothesis $\int_0^x \{1 - F(y)\}dy \sim x^{\gamma}\, L(x)$. Thus

$$-\ \frac{s^{\gamma}\, \varphi'(s)}{L(s^{-1})} \longrightarrow (1 - \gamma)\, \Gamma(1 + \gamma)$$

as claimed.

Returning to the proof of Part (1.4), let us define

$$r(x) = \int_0^x g(x - z)\, z\, d\, F(z)\ .$$

Then

$$r^{o}(s) = \int_0^{\infty} e^{-sx}\ r(x)\, dx = -\ \varphi'(s)\, G^{*}(s)$$

and so, under the conditions of Lemma 1 ,

$$\frac{s^{\gamma}\, r^{o}(s)}{L(s^{-1})} \longrightarrow (1 - \gamma)\, \Gamma\, (1 + \gamma)\ , \qquad \text{as} \quad s \longrightarrow 0+$$

Therefore, by the Tauberian theorem we have been using ,

$$\frac{1}{x^{\gamma} L(x)} \int_0^x r(y)\, dy \longrightarrow (1 - \gamma), \qquad \text{as} \quad x \longrightarrow \infty .$$

If we convolute both sides of (22) with $F(x)$ we find

$$(23) \qquad x\, k(x) - r(x) = c\, K(x) - c\, H(x)$$

where $H(x)$ is the distribution function

$$H(x) = \int_0^x K(x - z)\, dF(z) .$$

On integrating (23) we obtain

$$(24) \qquad -x\{1 - K(x)\} + \int_0^x \{1 - K(y)\}dy - \int_0^x r(y)\, dy$$

$$= c \int_0^x \{K(y) - H(y)\}\, dy .$$

From (21), (22), and (24) we then find

$$(25) \qquad \frac{x^2 g(x)}{c} = (1 + c) \int_0^x \{G(y) - K(y)\}\, dy$$

$$- \int_0^x r(y)\, dy$$

$$- c \int_0^x \{K(y) - H(y)\}\, dy .$$

The function $\{G(y) - K(y)\}$ is non-negative and its Laplace transform is easily seen to be

$$\frac{1 - \varphi(s)}{s}\, G^*(s)$$

$$\sim \frac{\Gamma(1 + \gamma)\, L(s^{-1})}{s^{\gamma}} \qquad \text{as} \quad s \longrightarrow 0+ .$$

Thus

$$\int_0^x \{G(y) - K(y)\}dy \sim x^\gamma L(x) \quad , \quad \text{as} \quad x \longrightarrow \infty .$$

Similarly, the non-negative function $\{K(y) - H(y)\}$ has Laplace transform

$$\frac{1 - \varphi(s)}{s} \quad \varphi(s) \quad G^*(s)$$

and so

$$\int_0^x \{K(y) - H(y)\} \, dy \sim x^\gamma L(x) , \quad \text{as} \quad x \longrightarrow \infty ,$$

also.

We now have enough asymptotic results to deduce from (25) that

$$\frac{x^2 \, g(x)}{c} \sim \gamma \quad x^\gamma L(x)$$

i.e.

$$g(x) \sim \frac{c \; \gamma \; L(x)}{x^{2-\gamma}} \quad , \quad \text{as} \quad x \longrightarrow \infty \quad .$$

This completes the proof of this part.

Proof of Part 1.5. We begin first with

Lemma 2. *If $\ell_1(x)$, $\ell_2(x)$, $\ell_3(x)$ are bounded integrable functions such that*

$$\ell_1(x) = \int_0^x \ell_2(x - z) \, \ell_3(z) \, dz$$

and if, for $x > 0$,

$$\ell_2(x) = O \left\{ \frac{e^{-\lambda x + Ax^\alpha}}{x^\beta} \right\} ,$$

$$\ell_3(x) = O \left(e^{-\mu x} \right) ,$$

<u>for some $\mu > \lambda > 0$, $A > 0$, $\beta \geq 0$, $1 > \alpha > 0$, then</u>

$$\ell_1(x) = O\left\{ \frac{e^{-\lambda x + A x^{\alpha}}}{x^{\beta}} \right\} .$$

<u>Proof.</u> For any $m > 0$

$$\frac{d}{dx}\left\{ \frac{e^{Ax^{\alpha}}}{(m + x)^{\beta}} \right\} = \frac{e^{Ax^{\alpha}}}{(m + x)^{\beta}} \left\{ \frac{A\,\alpha}{x^{1 - \alpha}} - \frac{\beta}{(m + x)} \right\}$$

so that we can always choose m large enough to make

$$f(x) = \frac{e^{Ax^{\alpha}}}{(m + x)^{\beta}}$$

an increasing function of $x > 0$. Having chosen m we can then find constants N_1, N_2, such that

$$\ell_2(x) \leq N_1 \frac{e^{-\lambda x + Ax^{\alpha}}}{(m + x)^{\beta}} ,$$

$$\ell_3(x) \leq N_2 \, e^{-\mu x} ,$$

for all x. Thus

$$\ell_1(x) \leq N_1 N_2 \int_0^x \frac{e^{-\mu x + \mu z - \lambda z + Az^{\alpha}}}{(m + z)^{\beta}}$$

$$\leq \frac{N_1 N_2 \, e^{-\lambda x + Ax^{\alpha}}}{(m + x)^{\beta}} \int_0^x e^{-(\mu - \lambda)(x - z)} \, dz ,$$

in view of the fact that $f(x)$ increases. Thus the lemma is proved.

That part of (1.5) concerning the case $\nu = 0$ is already covered by Theorem 3. We shall therefore assume from here on that $\nu > 0$, and suppose there are constants $C > 0$, $\Delta > 0$ such that

$$1 - F(x) < \frac{C\, e^{-\lambda x}\, x^{\nu}}{\Gamma(\nu+1)}, \qquad x \geq \Delta .$$

Define

$$\theta(x) = c \text{ Max } \left\{ 1 - F(x), \quad \frac{C\, e^{-\mu x}\, x^{\nu}}{\Gamma(\nu+1)} \right\} .$$

Then $\theta(x) \geq c\{1 - F(x)\}$ for all x and $\theta(x) = c\{1 - F(x)\}$ for all sufficiently small x. Therefore

$$\int_0^1 \frac{\theta(x) - c\{1 - F(x)\}}{x}\, dx < \infty$$

and we can deduce from Theorem 4 that

$$(26) \qquad g(x) = O(\Delta_\theta(x)) ,$$

for almost all x.

Define

$$\sigma(x) = \frac{C\, e^{-\lambda x}\, x^{\nu}}{\Gamma(\nu+1)}$$

and

$$\tau(x) = \theta(x) - \sigma(x) .$$

Then $\tau(x) \geq 0$ and $\tau(x) = 0$ for all $x > \Delta$. Hence $\Delta_\tau(x)$ is defined. Moreover, since $\tau(x) = O(e^{-\eta x})$ for arbitrarily large η, it follows from Theorem 3 that

(27) $$\Delta_\tau(x) = O(e^{-\eta x})$$

for η arbitrarily large. We also note that

(28) $$\Delta_\theta(x) = \int_0^x \Delta_\tau(x - z)\, \Delta_\sigma(z)\, dz \quad .$$

For typographic ease, let us write

$$\psi(x) = \frac{\exp\{-\lambda x + \frac{\nu+1}{\nu}(Cc)^{\frac{1}{\nu+1}} x^{\frac{\nu}{\nu+1}}\}}{x^{\frac{1}{2}(\frac{\nu+1}{\nu+2})}} \quad .$$

Then, in view of (26), (27), (28), and Lemma 2, we shall have proved Theorem (1.5) if we show that $\Delta_\sigma(x) = O(\psi(x))$. Our task thus becomes one of estimating $\Delta_\sigma(x)$. From all that we have proved so far we can say that $\Delta_\sigma(x)$ is continuous and locally of bounded variation; also, from Theorem 3, $\Delta_\sigma(x) = O(e^{-\gamma x})$ for any $\gamma < \lambda$. Thus we can deduce from Theorem 7.3 of Widder (1941, p. 66) that

(29) $$\Delta_\sigma(x) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{-\gamma - iT}^{-\gamma + iT} e^{sx + \frac{Cc}{\nu(s+\lambda)^\nu}}\, ds \quad .$$

Let us put

$$h(s) = sx + \frac{Cc}{\nu(s+\lambda)^\nu} \quad .$$

Then

$$h'(s) = x - \frac{Cc}{(s+\lambda)^{\nu+1}}$$

so that $h'(s) = 0$ where

$$s = -\lambda + \left(\frac{Cc}{x}\right)^{\frac{1}{\nu+1}}$$

$$= -\lambda + \delta(x) \ , \quad \text{say} \ ,$$

and this is a point on the real axis a little to the right of the point $s = -\lambda$.

Choose an arbitrarily small $\epsilon > 0$.

As the real parameter t runs from $-\epsilon\,\delta(x)$ to $+\epsilon\,\delta(x)$ the point

$$s = -\lambda + \delta(x) - t^2 + it$$

runs along a small parabolic arc $\mathcal{C}$, say. Now

$$h''(s) = \frac{Cc(\nu+1)}{(s+\lambda)^{(\nu+2)}}$$

$$h'''(s) = -\frac{Cc(\nu+1)(\nu+2)}{(s+\lambda)^{(\nu+3)}} \quad .$$

Thus, for all s on $\mathcal{C}$ we have

$$|h'''(s)| < \frac{K_1}{\delta^{(\nu+3)}} \quad ,$$

where K_1 is some constant which does not depend on δ and ϵ, provided they are both small. Therefore, if s is any point on $\mathcal{C}$,

$$h''(s) = \frac{Cc\,(\nu+1)}{\delta^{(\nu+2)}} - (t^2 - it)\, h'''(s^*)$$

where s^* is some point on $\mathcal{C}$ between s and $-\lambda + \delta$. Hence, on $\mathcal{C}$,

$$h''(s) = \frac{Cc(\nu+1)}{\delta^{(\nu+2)}} \{1 + \rho_1(t)\}$$

where $|\rho_1(t)| < K_2 \epsilon \delta$, K_2 being some further constant which does not depend on ϵ or δ. On $\mathcal{C}$ we thus have

$$\Re\, \alpha(s) = -\lambda x + x\delta + \frac{Cc}{\nu\delta^{\nu}} + \Re\left\{ \frac{1}{2}(-t^2 - 2it^3 + t^4)\, \frac{Cc(\nu+1)}{\delta^{(\nu+2)}}\, [1+\rho_1(t)] \right\}$$

$$= -\lambda x + x\delta + \frac{Cc}{\nu\,\delta^{\nu}} - \frac{Cc(\nu+1)t^2}{2\,\delta^{(\nu+2)}}\ \{1 + \rho_2(t)\}$$

where $|\rho_2(t)| < K_3 \epsilon \delta$, for some constant K_3 not depending on ϵ or δ. Hence, noting that $ds = (i - 2t)\,dt$, we have

$$\left| \int_{\mathcal{C}} e^{h(s)} ds \right| \leq (1 + 2\epsilon\delta)\, e^{-\lambda x + x\delta + \frac{Cc}{\nu\delta^{\nu}}} \int_{-\epsilon\delta}^{+\epsilon\delta} e^{-\frac{Cc(\nu+1)\{1-K_3\epsilon\delta\}}{2\,\delta^{(\nu+2)}}\, t^2}\, dt$$

$$\leq \frac{(1 + 2\epsilon\delta)\,(2\pi)^{\frac{1}{2}}\, e^{-\lambda x + x\delta + \frac{Cc}{\nu\delta^{\nu}}}}{\sqrt{\left\{ \frac{Cc(\nu+1)\,\{1 - K_3\,\epsilon\,\delta\}}{\delta^{(\nu+2)}} \right\}}} .$$

If we substitute for δ in terms of x in the last inequality, we discover

(30) $$\int_{\mathcal{C}} e^{h(s)} ds = O(\psi(x)), \qquad \text{as } x \longrightarrow \infty .$$

Let T be a large positive number, η a small one, and let $\mathcal{L}(T)$ be the line mapped out by

$$s = (-\lambda + \delta - \epsilon^2\,\delta^2 - \eta t) + i(\epsilon\,\delta + t)$$

as t runs from 0 to T. Notice that $\mathcal{L}(T)$ is a straight line segment sloping away from the imaginary axis and linking up with one end of $\mathcal{C}$.

On the line $\mathcal{L}(T)$

$$\Re\, h(s) < -\lambda x + x\delta - \epsilon^2 \delta^2 x - \eta x t + \frac{Cc}{\nu r^\nu}$$

where

$$r^2 = \delta^2\{(1 - \epsilon^2 \delta^2)^2 + \epsilon^2\} \quad .$$

Thus

$$\Re\, h(s) < -\lambda x + \frac{\nu+1}{\nu} x^{\frac{\nu}{\nu+1}} (Cc)^{\frac{1}{\nu+1}} - \eta x t + v \text{ , say } ,$$

where

$$v = x\delta - \epsilon^2 \delta^2 x + \frac{Cc}{\nu r^\nu} - \frac{\nu+1}{\nu} x^{\frac{\nu}{\nu+1}} (Cc)^{\frac{1}{\nu+1}} \quad .$$

On substituting for δ in terms of x we find

$$v = (Cc)^{\frac{1}{(\nu+1)}} x^{\frac{\nu}{(\nu+1)}} w \text{ , say } ,$$

where

$$w = 1 - \epsilon^2\delta + \frac{1}{\nu}\{(1 - \epsilon^2\delta^2)^2 + \epsilon^2\}^{-\frac{1}{2}\nu} - \frac{\nu+1}{\nu}$$

$$= -(\tfrac{1}{2} + \delta - \delta^2)\,\epsilon^2 + O(\epsilon^4) \quad .$$

Hence there is a $\kappa > 0$ such that $w < -\kappa\,\epsilon^2$ and we see that on $\mathcal{L}(T)$

$$\Re\, h(s) < -\lambda x + \left(\frac{\nu+1}{\nu}\right) x^{\frac{\nu}{\nu+1}} (Cc)^{\frac{1}{\nu+1}} \left(1 - \frac{\kappa\nu\epsilon^2}{\nu+1}\right) - \eta x t \quad .$$

Thus, noting that $ds = (i - \eta)dt$ on $\mathcal{L}(T)$, and that

$$\int_0^T e^{-\eta x t}\, dt < \frac{1}{\eta x} \quad ,$$

we have

$$\left|\int_{\mathcal{L}} e^{h(s)} ds\right| < \frac{(1+\eta)}{\eta x} \exp\left\{-\lambda x + \frac{\nu+1}{\nu} x^{\frac{\nu}{\nu+1}} (Cc)^{\frac{1}{\nu+1}} \left(1 - \frac{\kappa\nu\epsilon^2}{\nu+1}\right) \right\} .$$

Hence,

(31) $$\int_{\mathcal{C}} e^{h(s)}ds = o(\psi(x)) \quad , \quad \text{as} \quad x \longrightarrow \infty ,$$

and this result is uniform in T.

Lastly, consider the straight line segment $\mathcal{J}(T)$, say, which is parallel to the real axis and mapped out by

$$s = (-\lambda + \delta - \epsilon^2 \delta^2 - t) + i(\epsilon \delta + T)$$

as t runs from 0 to ηT. On $\mathcal{J}(T)$ we have

$$|e^{h(s)}| < K_4 \; e^{-\lambda x + x\delta - x \epsilon^2 \delta^2 - tx}$$

for some K_4 which is independent of T provided it is sufficiently large, and of ϵ and δ provided they are both small. Thus

$$\left| \int_{\mathcal{J}(T)} e^{h(s)}ds \right| < \frac{K_4}{x} \; e^{-\lambda x + x\delta - \epsilon^2 \delta^2 x}$$
$$= o(\psi(x)) \quad , \quad \text{as} \quad x \longrightarrow \infty ,$$

uniformly in T.

In (29) we may suppose that $\gamma = \lambda + \epsilon^2 \delta^2 - \delta$, for then $\gamma < \lambda$ correctly if ϵ is small enough. Combining (29), (30), and (31), and noting especially the uniformity of (30) and (31) with respect to T, we can now easily prove

$$\Delta_\sigma (x) = O(\psi(x)) \; , \qquad \text{as} \quad x \longrightarrow \infty .$$

This completes the proof of the theorem.

Appendix

Let $\{F_n(x)\}$ be an infinite sequence of distribution functions of non-negative random variables and, for real $s \geq 0$, let

$$F_n^*(s) = \int_{0-}^{\infty} e^{-sx} \, dF_n(x) \qquad , \qquad n = 1, 2, \ldots$$

be the corresponding Laplace-Stieltjes transforms. Suppose $F(x)$ is a further distribution function of a non-negative random variable and that $F_n(x) \longrightarrow F(x)$ as $n \longrightarrow \infty$, at every continuity point of $F(x)$. Then, by dominated convergence,

$$\int_0^{\infty} e^{-sx} \, F_n(x) \, dx \longrightarrow \int_0^{\infty} e^{-sx} \, F(x) \, dx$$

as $n \longrightarrow \infty$, for every fixed real $s > 0$. Hence $F_n^*(s) \longrightarrow F^*(s)$ as $n \longrightarrow \infty$, for every $s \geq 0$ (for $F_n^*(0) = F^*(0) = 1$ for all n).

On the other hand, suppose $F_n^*(s) \longrightarrow \Phi(s)$, for every real $s \geq 0$, as $n \longrightarrow \infty$; suppose further that $\Phi(s)$ is continuous to the right at the origin. By the usual Helly-Bray compactness argument there is a bounded non-decreasing function $M(x)$, say, and a subsequence $\{F_{n_m}(x)\}$ such that $F_{n_m}(x) \longrightarrow M(x)$ at every continuity point of $M(x)$. Moreover, we can take $M(x) = 0$ for $x < 0$. By the dominated convergence argument already used we see $F_{n_m}^*(s) \longrightarrow M^*(s)$ and so $M^*(s) = \Phi^*(s)$ for all real $s > 0$. But $F_n^*(0) = 1$ for all n and so $\Phi^*(0) = 1$. However, $\Phi^*(s)$ is continuous to the right at the origin and hence $M^*(0+)=1$. This proves that $M(x)$ is a distribution function, and indeed, the unique distribution with Laplace-Stieltjes transform $M^*(s) = \Phi(s)$. By familiar reasoning it now follows that $F_n(x) \longrightarrow M(x)$ at every continuity point of $M(x)$.

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