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DISTRIBUTION OF THE LARGEST OR THE SMALLEST CHARACTERISTIC ROOT UNDER NULL HYPOTHESIS CONCERNING COMPLEX MULTIVARIATE NORMAL POPULATIONS

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1. Introduction:

It has been pointed out by the author [1] that one can handle all the classical problems of point estimation and testing hypothesis concerning the parameters of complex multivariate normal populations in a similar manner as one handles those for multivariate normal populations in real variates. In [1, 2], the author has derived an asymptotic formulae for certain likelihood test-procedures and in [2], the author has mentioned the maximum characteristic root statistic for testing the reality of a covariance matrix. The distribution of the characteristic roots under null hypothesis established in those two papers can be written in a general form as

$$(1) \qquad c_1\left\{\prod_{j=1}^{q}\omega_j^m (1-\omega_j)^n\right\}\left\{\prod_{j=1}^{q-1}\prod_{k=j+1}^{q}(\omega_j-\omega_k)^2\right\} d\omega_1 \ldots d\omega_q$$

where $c_1 = \prod_{i=1}^{q} \Gamma(n+m+q+j)/\{\Gamma(n+j)\Gamma(m+j)\ \Gamma(j)\}$ and $0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_q \leq 1$.

We may also note that when n is large, the joint distribution of $n\omega_j = f_j$ $(j = 1, 2, \ldots, q)$, $0 \leq f_1 \leq \cdots \leq f_q < \infty$, can be written as

$$(2) \qquad c_2\left(\prod_{j=1}^{q} f_j^m\right)\exp\left(-\sum_{j=1}^{q} f_j\right)\left\{\prod_{j=1}^{q-1}\prod_{k=j+1}^{q}(f_j-f_k)^2\right\} df_1\ldots df_k$$

where $c_2 = 1/\left\{\prod_{j=1}^{q}[\Gamma(m+j)\Gamma(j)]\right\}$.

[1] This research was supported by the Mathematics Division of the Air Force Office of Scientific Research.

In this paper, we derive the distribution of ω_q (or f_q) and ω_1 (or f_1). The percentage points will be given and some applications will be discussed in another paper.

2. Distribution of ω_q or ω_1.

For the distribution of ω_q, we shall require the following two lemmas:

Lemma 1:

$$\Sigma \int_{\mathscr{D}} \prod_{j=1}^{s} \left[x_j^{m'_j} (1 - x_j)^{n'_j} \, dx_j \right] = \prod_{j=1}^{s} \left[\int_0^x x_j^{m_j} (1 - x_j)^{n_j} \, dx_j \right]$$

where $\mathscr{D}$: $(0 \leq x_1 \leq \dots \leq x_s \leq x)$, $(x \leq 1)$; and on the left hand side $(m'_s, n'_s), \dots, (m'_1, n'_1)$ is any permutation of $(m_s, n_s), \dots, (m_1, n_1)$ and the summation is taken over all such permutations.

For proof, one may refer to Roy [3, (A.9.3), p. 203].

Lemma 2:

$$\prod_{j=1}^{q-1} \prod_{k=j+1}^{q} (\omega_j - \omega_k)^2 = \Sigma \begin{vmatrix} \omega_{j_1}^{2q-2} & \omega_{j_2}^{2q-3} & \dots & \omega_{j_q}^{q-1} \\ \omega_{j_1}^{2q-3} & \omega_{j_2}^{2q-4} & \dots & \omega_{j_q}^{q-2} \\ \cdot & \cdot & \dots & \cdot \\ \omega_{j_1}^{q-1} & \omega_{j_2}^{q-2} & \dots & \omega_{j_q}^{0} \end{vmatrix}$$

where Σ means the summation over $(j_1, j_2, \dots, j_q)$, the permutation of $(1, 2, \dots, q)$, and $|A|$ means the determinant of A.

Proof: It is well known that a Vandermonde determinant

$$\begin{vmatrix} \omega_1^{q-1} & \omega_2^{q-1} & \dots & \omega_q^{q-1} \\ \omega_1^{q-2} & \omega_2^{q-2} & \dots & \omega_q^{q-2} \\ \cdot & \cdot & \dots & \cdot \\ \omega_1 & \omega_2 & \dots & \omega_q \\ 1 & 1 & \dots & 1 \end{vmatrix}^2 = \left[\prod_{j=1}^{q-1} \prod_{k=j+1}^{q} (\omega_j - \omega_q) \right]^2 = \alpha, \quad \text{(say)}.$$

Then, the above expression can be written as

$$\alpha = \begin{vmatrix} \sum_{j=1}^{q} \omega_j^{2q-2} & \sum_{j=1}^{q} \omega_j^{2q-3} & \cdots & \sum_{j=1}^{q} \omega_j^{q-1} \\ \sum_{j=1}^{q} \omega_j^{2q-3} & \sum_{j=1}^{q} \omega_j^{2q-4} & \cdots & \sum_{j=1}^{q} \omega_j^{q-2} \\ \cdot & \cdot & \cdots & \cdot \\ \sum_{j=1}^{q} \omega_j^{q-1} & \sum_{j=1}^{q} \omega_j^{q-2} & \cdots & q \end{vmatrix}$$

$$= \sum_{j_1, j_2, \ldots, j_q} \begin{vmatrix} \omega_{j_1}^{2q-2} & \omega_{j_2}^{2q-3} & \cdots & \omega_{j_q}^{q-1} \\ \omega_{j_1}^{2q-3} & \omega_{j_2}^{2q-4} & \cdots & \omega_{j_q}^{q-2} \\ \cdot & \cdot & \cdots & \cdot \\ \omega_{j_1}^{q-1} & \omega_{j_2}^{q-2} & \cdots & 1 \end{vmatrix}$$

If in the right hand side, any two j_i and j_t are equal, then the value of the determinant is zero. Hence the summation over the right hand side over $(j_1, j_2, \ldots, j_q)$ reduces to the permutations of $(1, 2, \ldots, q)$, which establishes the lemma 2.

Now we shall prove the following theorem:

Theorem 1: If the joint distribution of $\omega_1, \omega_2, \ldots, \omega_q$ is given by (1), then

$$(3) \qquad \Pr(\omega_q \leq x) = c_1 \begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_{q-1} \\ \beta_1 & \beta_2 & \cdots & \beta_q \\ \cdot & \cdot & \cdots & \cdot \\ \beta_{q-1} & \beta_q & \cdots & \beta_{2q-2} \end{vmatrix} = c_1 |(\beta_{i+j-2})|$$

where c_1 is defined in (2), $\beta_{i+j-2} = \int_0^x \omega^{m+i+j-2}(1-\omega)^n \, d\omega$

for $i, j = 1, 2, \ldots q$ and (β_{i+j-2}) is a $q \times q$ matrix.

<u>Proof</u>: By definition, we have

$$Pr(\omega_q \leq x) = Pr(0 \leq \omega_1 \leq \cdots \leq \omega_q \leq x)$$

$$= c_1 \int_{\mathcal{D}} \prod_{j=1}^{m} \left[\omega_j^m (1-\omega_j)^n \right] \left[\prod_{j=1}^{q-1} \prod_{k=j+1}^{q} (\omega_j - \omega_k)^2 \right] \prod_{j=1}^{q} d\omega_j ,$$

where D: $(0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_q \leq x, \; x \leq 1)$.

Using lemma 2, the above expression can be written as

$$(4) \quad Pr(\omega_q \leq x) = c_1 \Sigma \int_{\mathcal{D}} \begin{vmatrix} \omega_{j_1}^{2q-2} & \omega_{j_2}^{2q-3} & \cdots & \omega_{j_q}^{q-1} \\ \omega_{j_1}^{2q-3} & \omega_{j_2}^{2q-4} & \cdots & \omega_{j_q}^{q-2} \\ \cdot & \cdot & \cdots & \cdot \\ \omega_{j_1}^{q-1} & \omega_{j_2}^{q-2} & \cdots & \omega_{j_q}^{0} \end{vmatrix} \prod_{j=1}^{q} \left[\omega_j^m (1-\omega_j)^n \, d\omega_j \right],$$

where Σ means the summation over $(j_1, \ldots, j_q)$, the permutation of $(1, 2, \ldots, q)$. Now the determinant in the integral sign of (4), can be written as

$$\Sigma_1 \, \text{sign} \, (t_1, \ldots, t_q) \; \omega_{j_1}^{q-1+t_1} \; \omega_{j_2}^{q-2+t_2} \; \cdots \; \omega_{j_q}^{t_q}$$

where $(t_1, \ldots, t_q)$ is a permutation of $(0, 1, \ldots, q-1)$, sign $(t_1, \ldots, t_q)$ is positive if the permutation is even and negative if the permutation is odd, and Σ_1 means the summation over all such permutations. Then (4) becomes

$$\Pr(\omega_q \leq x) = c_1 \; \Sigma \; \Sigma_1 \int_{\mathcal{D}} \text{sign}(t_1, \ldots, t_q)(\omega_{j_1}^{q-1+t_1} \ldots \omega_{j_q}^{t_q}) \cdot \prod_{j=1}^{q} \left[\omega_j^m (1-\omega_j)^n \; d\omega_j \right] .$$

First taking summation over $(j_1, j_2, \ldots, j_q)$, the permutation of $(1, 2, \ldots, q)$ and applying the lemma 1, we get

$$\Pr(\omega_q \leq x) = c_1 \Sigma_1 \; \text{sign}\,(t_1, \ldots, t_q) \; \beta_{q-1+t_1} \; \beta_{q-2+t_2} \ldots \beta_{t_q} = c_1 \, |(\beta_{i+j-2})|$$

which proves the theorem 1.

It may be noted here that

$$\Pr(\omega_1 \leq x) = 1 - \Pr(\omega_1 \geq x) = 1 - \Pr(x \leq \omega_1 \leq \; \ldots \; \leq \omega_q \leq 1) .$$

Going back to the c.d.f. of $(\omega_1, \ldots, \omega_q)$ and using the transformation $\omega_j = 1 - z_j \; (j = 1,2,\ldots,q)$, we have

$$(5) \quad \Pr(\omega_1 \leq x) = 1 = \Pr(x \leq \omega_1 \leq \ldots \leq \omega_q \leq 1) = 1 - c_1 \; |(\delta_{i+j-2})|$$

where $\delta_{i+j-2} = \int_0^{1-x} z^{n+i+j-2} (1 - z)^m \, dz$ and (δ_{i+j-2}) is a qxq matrix.

<u>Theorem 2</u>: If the distribution of $f_1, \ldots, f_q$ is given by (2) then

$$(6) \qquad \Pr(f_q \leq x) = c_2 \; |(\gamma_{i+j-2})|$$

where $\gamma_{i+j-2} = \int_0^x \omega^{m+i+j-2} \exp(-\omega) \, d\omega$, (γ_{i+j-2}) is a qxq matrix and c_2 is defined in (2).

Proof is similar to that of theorem 1.

3. The author thanks Professor S. N. Roy for discussion and help.

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