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CUMULATIVE SUM CONTROL CHARTS AND THE WEIBULL DISTRIBUTION

by

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A method for construction of cumulative sum control charts for controlling the mean of a Weibull distribution is described. As a special case, charts appropriate to exponentially distributed variables can be constructed. There is some investigation of the results of using such charts when a non-exponential Weibull distribution would be more appropriate. The paper concludes with a discussion of certain formulas in the related analysis of sequential probability ratio tests for Weibull distributions.

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CUMULATIVE SUM CONTROL CHARTS AND THE WEIBULL DISTRIBUTION [1]

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1. INTRODUCTION

Cumulative sum control charts (CSCC) have found an increasing variety of applications since their introduction about ten years ago. These charts differ from the standard ('Shewhart') type control charts in that cumulative results are plotted, rather than separate results of measurements on distinct samples. Various methods of applying control limits to the plotted points have been described. In the present paper the method described in [2] and applied in [3] will be used. This method makes use of approximate formulas, derived from those appropriate to a Wald sequential probability ratio test (SPRT) discriminating between two simple hypotheses H_o and H_1. The hypothesis H_o is chosen to correspond to the desired "state of control"; H_1 is chosen to correspond to an amount of departure which should be detected in a reasonably short period of time. The nominal probability (α_1) of failing to detect that H_1 (rather than H_o) is valid is made very small; the nominal probability (α_o) of 'detecting' H_1 when there is really no departure from control (i.e., H_o is valid) can be chosen to have an arbitrary (usually rather small) value

Comparison of the Shewhart charts and CSCC will be largely based on the 'average run length' (ARL) needed for an indication of lack of control to appear. When H_o is valid, it is advantageous for the ARL to be large; when H_1 is valid, on the other hand, the ARL should be as small as possible. In this paper we will calculate ARL's, when H_1 is valid, for charts having the same value of α_o. We will thus be comparing the average numbers of observations needed to detect a real departure (of specified amount) from control. When comparing Shewhart charts and

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CSCC's in this way, it should be borne in mind that, for Shewhart charts, α_o is used to denote the probability of a single observation falling outside the control limit, while for CSCC it represents the probability that a sequence of plotted points will fall outside the control limit, without previously falling below another (probably unused) limit. This means that comparisons of the kind used in this paper - based on ARL when there is lack of control - are rather unfair to CSCC's. If we consider the situation where the process remains in control, the ARL for the Shewhart chart would be α_o^{-1}, while the ARL for the CSCC would be theoretically infinite, in practice very large.

In the present paper attention is concentrated on the construction of CSCC's for controlling the mean of sequences of independent variables each having the same Weibull distribution. The exponential distribution is a special form of the Weibull distribution, and is often used to represent observed values in 'life-testing' types of situation. In such situations, however, it has been found that a Weibull distribution (for which the c-th power of the variable is exponentially distributed) often gives a markedly more accurate representation. If c is known, then a CSCC can be constructed; a method of construction is described. If, as may well happen, c is not known with sufficient accuracy, an assumed value, c_o, may be used. Methods of investigating the likely effects of choice of an incorrect value for c is discussed. Particular attention is given to the case when c is taken equal to 1. This situation is of special interest, because a CSCC appropriate to an exponential distribution may be used, in ignorance that a Weibull distribution (with $c \neq 1$) would be more appropriate. Certain results of a general nature are obtained, indicating under what circumstances, and in what respects, such a CSCC can be expected to operate satisfactorily.

A final section contains some results helping to assess the performance of SPRT's comparing values of the mean, constructed on the assumption of exponentially distributed variation, when a Weibull distribution (with $c \neq 1$) of variation would be appropriate.

2. CONSTRUCTION OF CSCC FOR THE MEAN OF A WEIBULL DISTRIBUTION

Since the exponential distribution is a special form of Weibull distribution (obtained by putting $c = 1$), application of the method described in [2] to the construction of a CSCC for the mean of a Weibull distribution will also cover the case of the exponential distribution.

The successive observations will be represented (in order) by independent random variables $x_1, x_2, x_3, \ldots$, each having the same Weibull probability density function

$$(1) \qquad p(x_i \mid \theta) = \theta^{-1} c x_i^{c-1} \exp(-x_i^c/\theta) \qquad (x_i > 0;\ \theta > 0) .$$

This implies

$$\Pr[x_i \leq X] = \exp(-X^c/\theta) \qquad (X > 0)$$

The mean of this distribution is

$$\xi = \theta^{c^{-1}} \Gamma(c^{-1} + 1)$$

so that

$$\theta = [\xi / \Gamma(c^{-1} + 1)]^c \quad .$$

Suppose that it is desired to control the mean value of x at ξ_o (so that H_o is defined by $\xi = \xi_o$, c being supposed known) and to use the hypothesis (H_1) that $\xi = \xi_1$ in constructing the CSCC. The likelihood ratios to be used in the corresponding SPRT's are based on the observations in reverse order, starting from the last observed value, represented by x_m, and are given by the formula

$$\prod_{j=0}^{\ell-1} \left[\frac{p(x_{m-j} \mid H_1)}{p(x_{m-j} \mid H_0)} \right] = \left(\frac{\theta_0}{\theta_1} \right)^{\ell} \exp\left[(\theta_0^{-1} - \theta_1^{-1}) \prod_{j=0}^{\ell-1} x_{m-j}^c \right]$$

where

$$\theta_s = \frac{\xi_s^c}{[\Gamma(c^{-1} + 1)]^c} \qquad (s = 0, 1) \ .$$

'Lack of control' is indicated if

$$(\theta_0^{-1} - \theta_1^{-1}) \sum_{j=0}^{\ell-1} x_{m-j}^c > -\log \alpha_0 + \ell \log (\theta_1/\theta_0)$$

If $\xi_1 > \xi_0$ this inequality can be written in the form

$$\text{(2)} \quad [\Gamma(c^{-1}+1)]^c \sum_{j=0}^{\ell-1} x_{m-j}^c > [-\log \alpha_0 + \ell c \log (\xi_1/\xi_0)](\xi_0^{-c} - \xi_1^{-c})^{-1}$$

(If $\xi_0 < \xi_1$, the inequality (2) is reversed.)

The CSCC is constructed by plotting points with co-ordinates $(m, \sum_{i=1}^{m} x_i^c)$. Control limits are applied as shown in Figure 1, where A is the last plotted point. In this diagram (which corresponds to the case $\xi_1 > \xi_0$)

$$A\,P = (-\log \alpha_0)/[c \log (\xi_1/\xi_0)]$$

and

$$\tan A\hat{P}Q = \frac{c \log (\xi_1/\xi_0)}{(\xi_0^{-c} - \xi_1^{-c})\,[\Gamma(c^{-1} + 1)]^c}$$

Any plotted point below the line PQ is regarded as evidence of lack of control.

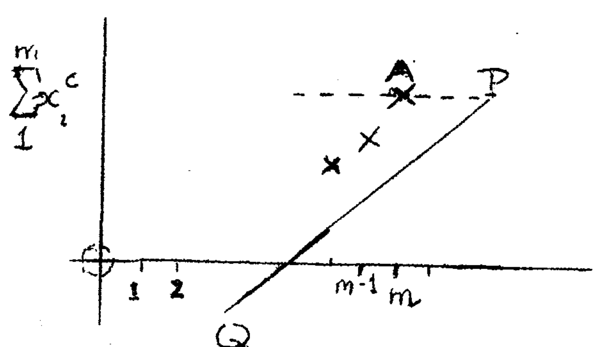

Figure 1.

3. ARL FOR EXPONENTIALLY DISTRIBUTED VARIABLES

The exponential probability density function is obtained by putting c equal to 1 in (1). In this case the CSCC is constructed by plotting points with co-ordinates $(m, \sum_{i=1}^{m} x_i)$ and (if $\xi_1 > \xi_0$) making $AP = (-\log \alpha_0)/[\log(\xi_1/\xi_0)]$ and

$$\tan APQ = \frac{\log(\xi_1/\xi_0)}{\xi_0^{-1} - \xi_1^{-1}}$$ in Figure 1.

If the true mean is ξ then, provided $\xi >$ tan APQ the ARL is approximately

$$\frac{-\log \alpha_0}{(\xi_0^{-1} - \xi_1^{-1})\xi - \log(\xi_1/\xi_0)}$$

If H_1 is valid (that is, there is the specified amount of departure from control) the ARL is approximately

$$(3) \qquad \frac{-\log \alpha_0}{(\xi_1/\xi_0) - 1 - \log(\xi_1/\xi_0)}$$

In a Shewhart control chart for controlling the mean of an exponential distribution, the upper 100 α_0% control limit would be $-\xi_0 \log \alpha_0$. When H_1 is valid the probability of an individual observation exceeding this value is

$$(4) \qquad \xi_1^{-1} \int_{-\xi \log \alpha_0}^{\infty} \exp(-x/\xi_1)dx = \alpha_0^{\xi_0/\xi_1}$$

The ARL is therefore $\alpha^{-\xi_0/\xi_1}$.

In the foregoing discussion it has been assumed that we are interested only in departures from control resulting in an <u>increased</u> mean value. Similar analysis applies when we wish to detect only a <u>decrease</u> in the mean value. Then ξ_1 is taken to be less than ξ_o and the inequality which would lead to an inference of 'lack of control' is

$$\sum_{j=o}^{m-1} x_{m-j} < [\log \alpha_o + \ell \log (\xi_o / \xi_1)] (\xi_1^{-1} - \xi_o^{-1})^{-1}$$

A CSCC, with control limit appropriate to the case $\xi_1 < \xi_o$ is shown in Figure 2.

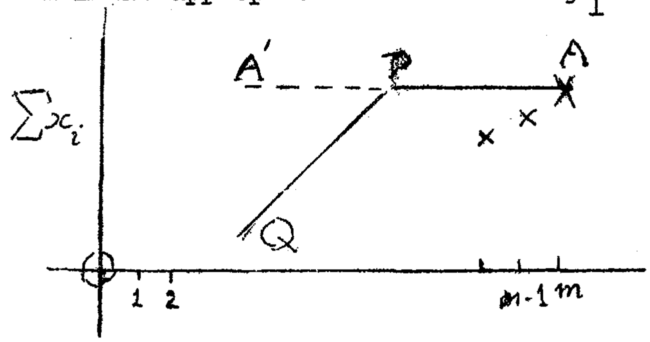

Figure 2.

In this diagram $AP = (- \log \alpha_o)/[\log(\xi_o / \xi_1)]$

and $\tan \widehat{A'PQ} = [\log(\xi_o / \xi_1)]/[\xi_1^{-1} - \xi_o^{-1}]$.

Plotted points above the line PQ are taken as indication of lack of control. The approximate ARL is still given by formula (2).

The lower 100 α_o % control limit on the Shewhart chart is $-\xi_o \log(1-\alpha_o)$. The ARL when H_1 is valid is $[1 - (1 -\alpha)^{\xi_o/\xi_1}]^{-1}$ in this case .

Table 1 presents some values of ARL for CSCC and Shewhart charts, calculated from the approximate formulas described in this section. Comparisons are similar, in general nature, to those found in [3] - in particular, the CSCC shows to better advantage when wider limits (0.001 or "3σ" as opposed to 0.025 or "2σ") are used. Also the CSCC has a greater relative advantage when $\xi_1 < \xi_o$, a case likely to be encountered frequently.

TABLE 1: Average Run Lengths for Exponential Distribution Control Charts

	CSCC				SHEWHART			
ξ_1/ξ_0	α_0 = 0.025	0.01	0.005	0.001	0.025	0.01	0.005	0.001
0.5	19.1	23.8	27.4	35.8	20.3	50.3	100	500
0.6	33.3	41.6	47.8	62.3	24.8	60.2	120	600
0.75	77.4	96.6	111	145	30.1	75.2	150	750
1.25	137	171	197	257	19.1	39.8	69.3	250
1.5	39.0	48.7	56.0	73.1	11.7	21.5	34.2	100
1.75	19.4	24.2	27.8	36.3	8.2	13.9	20.6	51.8
2.0	12.0	15.0	17.3	22.5	6.3	10.0	14.1	31.6
2.25	8.4	10.5	12.1	15.7	5.2	7.7	10.5	21.5
2.5	6.3	7.9	9.1	11.8	4.4	6.3	8.3	15.8

4. EXPONENTIAL CSCC'S WITH VARIABLES HAVING WEIBULL DISTRIBUTIONS

The CSCC for the exponential distribution is rather simpler than the CSCC's for Weibull distributions with c not equal to 1. In default of clear information to the contrary a CSCC based on the assumption of an exponential distribution might be used, it being hoped that if there is a departure (not affecting the mean value) from this form of distribution it will not have a seriously adverse effect on the operation of the control chart. The likely behavior of such a CSCC when each of the random variables representing observations have the same Weibull distribution (or, indeed, any other distribution) can be studied by methods developed in the study of SPRT's (see [4], for example). Some technical details of the application of these methods to the present problems will be given in section 5. Certain results of a general nature, relevant to practical considerations, are presented here.

Provided the true mean (ξ) is

(i) greater than $[\log(\xi_1/\xi_0)]/(\xi_0^{-1} - \xi_1^{-1})$ if $\xi_1 > \xi_0$ or

(ii) less than $[\log(\xi_1/\xi_0)]/(\xi_0^{-1} - \xi_1^{-1})$ if $\xi_1 < \xi_0$

the ARL is approximately

$$(5) \qquad \frac{-\log \alpha_0}{(\xi_0^{-1} - \xi_1^{-1})\xi - \log(\xi_1/\xi_0)}$$

whatever be the value of c. This is a very helpful result. It implies that the figures given in Table 1 for CSCC's constructed on the assumption of an exponential form of distribution can be used with some confidence that they will indicate the ARL when $\xi = \xi_1$, even when there is some doubt about the actual form of the distribution. Indeed, formula (5) can be used even when the true distribution is not of Weibull form.

For the Shewhart chart the ARL when $\xi = \xi_1$ is (assuming the true distribution of each x_i to be of the same Weibull form):

(i) $\exp[\{-(\xi_0/\xi_1)\Gamma(c^{-1}+1)\log\alpha_0\}^c]$ if $\xi_1 > \xi_0$

(ii) $[1 - \exp[-\{-(\xi_0/\xi_1)\Gamma(c^{-1}+1)\log(1-\alpha_0)\}^c]]^{-1}$ if $\xi_1 < \xi_0$.

If $\xi_1 > \xi_0$, the ARL is greater than the value given in Table 1 if

$$(-\log \alpha_0)^{c-1}\left(\frac{\xi_0}{\xi_1}\right)^{c-1} > [\Gamma(c^{-1}+1)]^{-c} \quad ;$$

and conversely.

If $\xi_1 < \xi_0$, the ARL is greater than the value given in Table 1 if

$$[-\log(1-\alpha_0)]^{c-1}\left(\frac{\xi_0}{\xi_1}\right)^{c-1} < [\Gamma(c^{-1}+1)]^{-c}$$

and conversely.

Some values of $[\Gamma(c^{-1}+1)]^c$ are given in Table 2.

TABLE 2. VALUES OF $[\Gamma(c^{-1}+1)]^c$

c	$[\Gamma(c^{-1}+1)]^c$	c	$[\Gamma(c^{-1}+1)]^c$	c	$[\Gamma(c^{-1}+1)]^c$
0.5	1.414	1.0	1.000	1.6	0.840
0.6	1.281	1.1	0.963	1.7	0.824
0.7	1.180	1.2	0.929	1.8	0.810
0.8	1.105	1.3	0.902	1.9	0.797
0.9	1.047	1.4	0.878	2.0	0.785
		1.5	0.858		

Usually (and for all cases in Table 1)

$$-(\xi_o/\xi_1)\log\alpha_o \gg 1 \quad \text{for} \quad \xi_1 > \xi_o$$

and

$$-(\xi_o/\xi_1)\log(1-\alpha_o) \ll 1 \quad \text{for} \quad \xi_1 < \xi_o .$$

So the ARL (when $\xi = \xi_1$) can be expected to be increased (as compared with the corresponding value in Table 1) if $c > 1$, decreased if $c < 1$, whether ξ_1 is greater or less than ξ_o.

When the process is in control (so that $\xi = \xi_o$), the ARL for the Shewhart chart is

(i) $\exp[\{-\Gamma(c^{-1}+1)\log\alpha_o\}^c]$ if $\xi_1 > \xi_o$,

(ii) $[1 - \exp[-\{-\Gamma(c^{-1}+1)\log(1-\alpha_o)\}^c]]^{-1}$ if $\xi_1 < \xi_o$.

Just as when the mean is ξ_1, the ARL is increased if $c > 1$, decreased if $c < 1$. The desirability of such changes is, however, reversed since a longer ARL is desirable when $\xi = \xi_o$, while a shorter ARL is desirable when $\xi = \xi_1$.

A similar analysis can be carried out to investigate the performance of a CSCC based on a Weibull distribution, when the use of such a distribution is, indeed, justified but an incorrect value, c', of c has been chosen. In such a situation, the ARL when ξ is the true mean value is approximately

$$(6) \qquad \frac{-\log \alpha_o}{(\xi_o^{-c'} - \xi_1^{-c'})\xi^{c'} \dfrac{[\Gamma(c'^{-1}+1)]^{c'}}{[\Gamma(c^{-1}+1)]^{c}} - c' \log(\xi_1/\xi_o)}$$

when $\xi_1 > \xi_o$ and $\xi > \dfrac{\log(\xi_1/\xi_o)}{\xi_o^{-1} - \xi_1^{-1}}$. Similar formulas can be obtained for the other cases studied above.

5. ANALYTICAL DISCUSSION

This final section contains a discussion of some points connected with the analysis of properties of SPRT's discriminating between hypotheses H_o and H_1 (as defined above) with specified nominal error probabilities α_o and α_1 (α_1 is not now assumed to be very small.) A standard approximate formula for the probability of rejecting H_o (or 'accepting H_1'), using such a test, supposing a hypothesis H is to be valid, is

$$(7) \qquad [1 - (\frac{\alpha_1}{1-\alpha_o})^h] \, [(\frac{1-\alpha_1}{\alpha_o})^h - (\frac{\alpha_1}{1-\alpha_o})^h]^{-1}$$

where h is the non-zero root (if such exists) of the equation

(8) "Expected value of $\left[\frac{p(x \mid H_1)}{p(x \mid H_0)} \right]^h$, assuming H to be valid, is equal to 1".

In the construction of CSCC's, α_1 is taken to be very small. Taking limiting values in (7) we obtain probabilities approximately equal to

$$\alpha^h \quad \text{if } h > 0$$

$$1 \quad \text{if } h < 0 .$$

Formula (5) above is obtained by using this result and noting that h is usually negative when $\xi = \xi_1$.

If we take

$$p(x|H_j) = \xi_j^{-1} \exp[-x/\xi_j] \qquad (x > 0;\ j = 0, 1)$$

and

$$p(x|H) = c\,\theta^{-1} x^{c-1} \exp[-x^c/\theta] \qquad (x > 0;\ \theta = \xi^c[\Gamma(c^{-1}+1)]^{-c})$$

we have the situation discussed in section 4 - that of a procedure based on the assumption that c equals 1 (exponential distribution) being used when a Weibull distribution with c not equal to 1 provides an accurate representation. Equation (8) can now be written

$$(9) \qquad c\,\theta^{-1}(\xi_0/\xi_1)^h \int_0^\infty x^{c-1} \exp[hx(\xi_0^{-1} - \xi_1^{-1}) - x^c/\theta]dx = 1$$

(where, of course, $\theta = \xi^c[\Gamma(c^{-1}+1)]^{-c}$).

We first note an interesting circumstance where c is less than 1. Consider, for definiteness, the case $\xi_1 > \xi_0$. Then the integral in the left-hand-side of equation (9) does not converge if h is positive. (Similarly, the integral does not converge if $\xi_1 < \xi_0$ and h is negative.)

The value of the derivative of the left-hand-side of (9) with respect to h (if it exists) evaluated at $h = 0$ is

$$\xi(\xi_0^{-1} - \xi_1^{-1}) - \log(\xi_1/\xi_0)$$

Hence, there can be a positive root, h, only if

$$\xi(\xi_o^{-1} - \xi_1^{-1}) - \log(\xi_1 | \xi_o) < 0$$

and conversely.

So if $c < 1$ and $\xi_1 > \xi_o$, equation (9) can have a non-zero root for h only if

$$\xi > \frac{\log(\xi_1 / \xi_o)}{\xi_o^{-1} - \xi_1^{-1}}$$

while if $\xi_1 < \xi_o$, there can be a non-zero root for h only if

$$\xi > \frac{\log(\xi_1 / \xi_o)}{\xi_o^{-1} - \xi_1^{-1}} \quad \text{in this case also.}$$

Explicit formulas for the integral can be obtained for the cases $c = 2$, $c = 1$, and (provided the integral converges) for $c = \frac{1}{2}$. These lead to the following forms for equation (9):

(10a) <u>$c = 2$</u>

$$\left(\frac{\xi_1}{\xi_o}\right)^{\left(\frac{\xi}{\xi_o} - \frac{\xi}{\xi_1}\right)^{-1}} = \left[1 + H e^{\frac{1}{2}H^2} \int_{\infty}^{H} e^{-\frac{1}{2}u^2}\, du\right]^{\frac{1}{H}\sqrt{\frac{2}{\pi}}}$$

where $H = h\,\xi(\xi_o^{-1} - \xi_1^{-1})\sqrt{\frac{2}{\pi}}$

(10b) <u>$c = 1$</u>

$$\left(\frac{\xi_1}{\xi_o}\right)^{\left(\frac{\xi}{\xi_o} - \frac{\xi}{\xi_1}\right)^{-1}} = (1 + H)^{\frac{1}{H}}$$

where $H = -h\,\xi(\xi_o^{-1} - \xi_1^{-1})$.

(10c) $c = \frac{1}{2}$

$$\left(\frac{\xi_1}{\xi_0}\right)^{\left(\frac{\xi}{\xi_0} - \frac{\xi}{\xi_1}\right)^{-1}} = \left[H\, e^{\frac{1}{2}H^2} \int_H^{\infty} e^{-\frac{1}{2}u^2}\, du\right]^{-H^2}$$

$$\text{where} \quad H = [-h\,\xi(\xi_0^{-1} - \xi_1^{-1})]^{-\frac{1}{2}}$$

Table 3 presents values of h which solve these three formulas, taking ξ equal to ξ_1, for a few values of the ratio ξ_1/ξ_0. When $c=1$, the conditions assumed in constructing the SPRT are satisfied and $h = -1$ for all values of ξ_1/ξ_0, so the approximate probability of rejecting H_0 calculated from (7) is $(1-\alpha_1)$, the nominal value.

TABLE 3: VALUES OF h WHEN THE MEAN IS EQUAL TO ξ_1

ξ_1/ξ_0	$c = 2$	$c = 1$	$c = \frac{1}{2}$
0.5	-4.48	-1.00	*
0.75	-3.96	-1.00	*
1.25	-3.46	-1.00	-0.213
1.5	-3.31	-1.00	-0.223
1.75	-3.20	-1.00	-0.231
2.0	-3.12	-1.00	-0.238

(* No value possible)

used in constructing the SPRT's.

If c is equal to 2, h is considerably less than -1. (In fact $h < -2$ and tends to -2 as ξ_1/ξ_0 increases without limit.) The probability of rejection of H_o will be greater than $(1 - \alpha_1)$; very nearly 1, in fact. On the other hand, if c is equal to $\frac{1}{2}$, h is between -0.2 $(= \lim_{\xi_1/\xi_o \to \infty} h)$ and -0.5 $(= \lim_{\xi_1/\xi_o \to 1+} h)$. The probability of rejection of H_o will be less than $(1 - \alpha_1)$.

In the limiting case of very small α_1, corresponding to the CSCC situation, the average sample number (or ARL for the control chart) is not affected by these results. In the standard SPRT situation, however, where α_1, though small, is of the same order as α_o, these results imply that the formulas for average sample number based on the exponential distribution (on which the SPRT is also based) will overestimate the average sample number if c equals 2; and underestimate it if c equals $\frac{1}{2}$.

REFERENCES

[1] Page, E. S., "Continuous inspection schemes," Biometrika, Vol. 41 (1954), pp. 100-115.

[2] Johnson, N. L., "A simple theoretical approach to cumulative sum control charts," J. Amer. Statist. Assoc., Vol. 56 (1961), pp. 835-840.

[3] Johnson, N. L. and Leone, F. C., "Cumulative sum control charts: Mathematical principles applied to their construction and use," Indust. Qual. Control,

I. Vol. 18 (12) (1962) pp. 15-21

II. Vol. 19 (1) (1962) pp. 29-36

III. Vol. 19 (2) (1962) pp. 22-28

[4] Ewens, W. J., "Departures from assumption in sequential analysis," Biometrika, Vol. 48 (1961), pp. 206-211.