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1. Introduction and summary.

The usual MANOVA model [11, p. 83] was generalised by Potthoff and Roy [8], keeping in view its importance to growth curve problems and they discuss some applications of the generalised MANOVA model. In a more special case of one population, Rao [9] has solved the problem from a different view point and considered some applications to growth curve. We give below the generalised MANOVA model and hypothesis as studied by Potthoff and Roy [8].

Let $\underset{\sim}{X}$: pxn be a random matrix such that

$$E(\underset{\sim}{X}) = \underset{\sim}{B}\,\underset{\sim}{\xi}\,\underset{\sim}{A} \tag{1}$$

and the columns of $\underset{\sim}{X}$ are independent multivariate normals with unknown covariance matrix $\underset{\sim}{\Sigma}$: pxp. The matrices $\underset{\sim}{B}$: pxq and $\underset{\sim}{A}$: mxn are assumed to be known and further, for our purpose and without any loss of theoretical generality, they are assumed to have ranks q and m respectively. Suppose the ranks of $\underset{\sim}{B}$ and $\underset{\sim}{A}$ are equal to $r(\leq q)$ and $s(\leq m)$ respectively. Then we can always find the basis for $\underset{\sim}{B}$ and for $\underset{\sim}{A}$, and we can write, without any loss of generality,

$$\underset{\sim}{B} = (\underset{\sim}{B}_3 \quad \underset{\sim}{B}_2) \qquad \text{and} \qquad \underset{\sim}{A}' = (\underset{\sim}{A}_1' \quad \underset{\sim}{A}_2')\ , \tag{2}$$

where $\underset{\sim}{B}_2$: $px(q-r) = \underset{\sim}{B}_3\,\underset{\sim}{L}_1$, rank of $\underset{\sim}{B}_3$: pxr is r, $\underset{\sim}{A}_2'$: $nx(m-s) = \underset{\sim}{A}_1'\,\underset{\sim}{L}_2$ and the rank of $\underset{\sim}{A}_1'$: nxs is s. With this, we can write

$$\underset{\sim}{B}\,\underset{\sim}{\xi}\,\underset{\sim}{A} = \underset{\sim}{B}_3\,\underset{\sim}{\xi}_3\,\underset{\sim}{A}_1, \qquad \underset{\sim}{\xi}_3 = (\underset{\sim}{I}_r \quad \underset{\sim}{L}_1)\,\underset{\sim}{\xi}\,(\underset{\sim}{I}_s \quad \underset{\sim}{L}_2)'. \tag{3}$$

From this it is easy to see that the unknown parameters qm of $\underset{\sim}{\xi}$: qxm are determined in terms of rs parameters of $\underset{\sim}{\xi}_3$: rxs. Hence, if we know the estimate of $\underset{\sim}{\xi}_3$, the estimate of ξ satisfies the relation (3). From this consideration, it is evident that the function $\underset{\sim}{C}\,\underset{\sim}{\xi}\,\underset{\sim}{V}$ will be estimable if and only if

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(4) $\quad \underset{\sim}{C}_2 = \underset{\sim}{C}_1 \underset{\sim}{L}_1$, $\underset{\sim}{C} = (\underset{\sim}{C}_1 \quad \underset{\sim}{C}_2)$, $\underset{\sim}{V}'_2 = \underset{\sim}{V}'_1 \underset{\sim}{L}_2$ and $\underset{\sim}{V}' = (\underset{\sim}{V}'_1 \quad \underset{\sim}{V}'_2)$,

where $\underset{\sim}{C}_1$: cxr, $\underset{\sim}{C}_2$: cx(p-r), $\underset{\sim}{V}_1$: vxs, $\underset{\sim}{V}_2$:vx(n-s), rank of $\underset{\sim}{C}$ is c and the rank of $\underset{\sim}{V}$ is v. (4) is true even when the ranks of $\underset{\sim}{C}$ and $\underset{\sim}{V}$ are less than c and v, but, for the analysis, the dependent rows of $\underset{\sim}{C}$ and the dependent columns of $\underset{\sim}{V}$ are redundant. Thus,, if we assume that the function $\underset{\sim}{C}\,\underset{\sim}{\xi}\,\underset{\sim}{V}$ is estimable (which we shall always assume), then we can, without any loss of generality, assume that the ranks of $\underset{\sim}{B}$: pxq and $\underset{\sim}{A}$: mxn are $q(\leq p)$ and $m(< n)$ respectively.

With these remarks, under model (1), we are interested in testing H_o given by

(5) $\quad H_o(\underset{\sim}{C}\ \underset{\sim}{\xi}\ \underset{\sim}{V} = \underset{\sim}{0})$ against $H(\underset{\sim}{C}\,\underset{\sim}{\xi}\,\underset{\sim}{V} \neq \underset{\sim}{0})$,

where $\underset{\sim}{C}$: cxq and $\underset{\sim}{V}$: mxv are of ranks c and v respectively. The hypothesis H_o was tested by Potthoff and Roy [8] by using the transformation

(6) $\quad \underset{\sim}{X}_o = \underset{\sim}{C}(\underset{\sim}{B}'\,\underset{\sim}{G}\,\underset{\sim}{B})^{-1}\,\underset{\sim}{B}'\,\underset{\sim}{G}\,\underset{\sim}{X}$,

where $\underset{\sim}{G}$: pxp is any arbitrary non-singular real matrix such that $(\underset{\sim}{B}'\underset{\sim}{G}\,\underset{\sim}{B})$ is non-singular. Then the results of Roy [11, p. 83] were used. They give some suggestions as to the choice of the arbitrary matrix $\underset{\sim}{G}$. When m=v=c=1 and $\underset{\sim}{A}$: 1xn has all unit elements, Rao [9] solved the problem by using the least squares estimates obtained by replacing the unknown $\underset{\sim}{\Sigma}$ by its estimate based on $\underset{\sim}{S} = \underset{\sim}{X}[\underset{\sim}{I} - \underset{\sim}{A}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{A}]\,\underset{\sim}{X}'$, which is independently distributed of $\underset{\sim}{Z} = \underset{\sim}{X}\,\underset{\sim}{A}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}$. In section (2.1), we use the likelihood ratio method for H_o under model (1) and give two other associated test procedures on trace and maximum root~~s~~.

In section (2.2), it is shown that the test procedures derived, in section (2.1), are applicable to testing H'_o defined by (8) when the model matrix $\underset{\sim}{B}$ is completed as

(7) $\quad E(\underset{\sim}{X}) = \underset{\sim}{B}\,\underset{\sim}{\xi}\,\underset{\sim}{A} + \underset{\sim}{B}_1\,\underset{\sim}{\xi}_1\,\underset{\sim}{A}$,

where $\underset{\sim}{B}$, $\underset{\sim}{\xi}$, $\underset{\sim}{A}$ are the same as defined in (1), $\underset{\sim}{\xi}_1$: (p-q)xp is an unknown matrix and $\underset{\sim}{B}_1$: px(p-q) is called a completion matrix such that $(\underset{\sim}{B} \quad \underset{\sim}{B}_1)$ is non-singular. The completion can be done in many ways which we shall not discuss here.

In each case of completion, these exists a symmetric matrix $\underset{\sim}{G}$: pxp such that $\underset{\sim}{B}_1' \underset{\sim}{G}\, \underset{\sim}{B} = \underset{\sim}{0}$ and $(\underset{\sim}{B}\ \underset{\sim}{B}_1)'\ \underset{\sim}{G}\ (\underset{\sim}{B}\ \underset{\sim}{B}_1)$ is non-singular. The hypothesis H_0' is given by

(8) $\quad H_0'(\underset{\sim}{C}\ \underset{\sim}{\eta}\ \underset{\sim}{V} = \underset{\sim}{0})$ against $H'(\underset{\sim}{C}\ \underset{\sim}{\eta}\ \underset{\sim}{V} \neq \underset{\sim}{0})$,

where $\underset{\sim}{\eta} = \underset{\sim}{\xi} + (\underset{\sim}{B}'\ \ \underset{\sim}{\Sigma}^{-1}\underset{\sim}{B})^{-1}\ (\underset{\sim}{B}'\ \ \underset{\sim}{\Sigma}^{-1}\ \underset{\sim}{B}_1)\ \underset{\sim}{\xi}_1$. When $\underset{\sim}{\xi}_1 = \underset{\sim}{0}$, then (8) is exactly equal to (5). The hypothesis (8) can be looked upon as the general linear hypothesis due to multicollinearity. The test procedures for H_0' are derived by applying the union intersection principle, and they turn out to be exactly the same as those derived for (5) under model (1) by using the likelihood ratio method.

In section (2.3), it is pointed out that $H_0^* = (H_0$, model (1)) is the intersection of two hypotheses H_0' and $H_0^{(3)}(\underset{\sim}{\xi}_1 = \underset{\sim}{0})$ under model (7), and the step-down procedures for testing H_0^* are suggested instead of those derived in section (2.1). The following consideration brings out the deeper physical implications of the step-down procedure. Unlike $\underset{\sim}{A}$, we could not be sure, whether $\underset{\sim}{\xi}$ and $\underset{\sim}{B}$ are, in fact, "incomplete" as in model (1) rather than "complete" as in model (7). Thus, it would be reasonable and safe to test first the adequacy of model (1) itself by $H_0^{(3)}$, and then go on to test the hypothesis H_0. Moreover, the main purpose of changing model (1) to model (7) is that if we are uncertain about the order of $\underset{\sim}{\xi}$ and $\underset{\sim}{B}$ (because the maximum value of q can be p), then the test procedures carried out in section (2.1) will still hold for H_0' and not for H_0. If we want to test H_0 under model (7), then the test procedures are identical with those given by Potthoff and Roy [8], when $\underset{\sim}{G}$ is defined as in the model (7). Hence, we have an entirely different interpretation of an arbitrary matrix $\underset{\sim}{G}$ introduced by them, and, if the model (1) is not true, then the tests with this arbitrary $\underset{\sim}{G}$ will not be natural and proper unless $\underset{\sim}{G} = \underset{\sim}{I}$ or $\underset{\sim}{G}^{-1}$ is the covariance matrix based on the similar previous data. This is because H_0 has different implications in two different models.

In section 3, the distribution and the property of monotonicity of the test procedures derived in section 2 are established under model (7) (and so they are also true under model (1)). The simultaneous confidence bounds on the parametric function of $\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V}$ under model (7) or $\underset{\sim}{C}\,\underset{\sim}{\xi}\,\underset{\sim}{V}$ under model (1) are given in section 4, and in section 5 a numerical example illustrates the types of computations involved in the test procedures.

2. Derivation of the test procedures:

(2.1) Likelihood ratio method.

For this purpose, we shall consider model (1) and the hypothesis H_o given by (5). We give below two lemmas which are used in the derivation of the test statistic.

Lemma 1: Let $\underset{\sim}{B}$: pxq and $\underset{\sim}{B}^*_1$: px(p-q) be of ranks q and (p-q) such that $\underset{\sim}{B}^{*\prime}_1 \underset{\sim}{B} = \underset{\sim}{0}$. Then if $\underset{\sim}{S}$: pxp is a symmetric positive definite (p.d.) matrix, then

$$\underset{\sim}{S}^{-1} - \underset{\sim}{S}^{-1}\underset{\sim}{B}(\underset{\sim}{B}'\, \underset{\sim}{S}^{-1}\, \underset{\sim}{B})^{-1}\underset{\sim}{B}'\, \underset{\sim}{S}^{-1} = \underset{\sim}{B}^*_1(\underset{\sim}{B}^{*\prime}_1\, \underset{\sim}{S}\, \underset{\sim}{B}^*_1)^{-1}\, \underset{\sim}{B}^{*\prime}_1 \quad .$$

Proof: If a matrix $\underset{\sim}{T}$ is symmetric and p.d., we shall write it as $\underset{\sim}{T} = (\underset{\sim}{T}^{\frac{1}{2}})^2$ where $\underset{\sim}{T}^{\frac{1}{2}}$ is a symmetric matrix. With this notation it is easy to verify that if

$$\underset{\sim}{\Delta} = [\underset{\sim}{S}^{-\frac{1}{2}}\underset{\sim}{B}(\underset{\sim}{B}'\, \underset{\sim}{S}^{-1}\, \underset{\sim}{B})^{-\frac{1}{2}} \quad \underset{\sim}{S}^{\frac{1}{2}}\underset{\sim}{B}^*_1(\underset{\sim}{B}^{*\prime}_1\, \underset{\sim}{S}\, \underset{\sim}{B}^*_1)^{-\frac{1}{2}}] \quad ,$$

then $\underset{\sim}{\Delta}'\underset{\sim}{\Delta} = \underset{\sim}{I}$ and so $\underset{\sim}{I} = \underset{\sim}{\Delta}\,\underset{\sim}{\Delta}' = \underset{\sim}{S}^{-\frac{1}{2}}\underset{\sim}{B}(\underset{\sim}{B}'\, \underset{\sim}{S}^{-1}\, \underset{\sim}{B})^{-1}\, \underset{\sim}{B}'\, \underset{\sim}{S}^{-\frac{1}{2}} + \underset{\sim}{S}^{\frac{1}{2}}\underset{\sim}{B}^*_1(\underset{\sim}{B}^{*\prime}_1\underset{\sim}{S}\underset{\sim}{B}^*_1)^{-1}\underset{\sim}{B}^{*\prime}_1\underset{\sim}{S}^{\frac{1}{2}}$. From this, the lemma 2 is obvious.

Lemma 2: Denoting $\delta(\underset{\sim}{Y})$ a matrix whose elements are the differentials of the corresponding elements of $\underset{\sim}{Y}$ matrix, we have

(i) $\delta(\underset{\sim}{Y}_1\, \underset{\sim}{Y}_2) = \delta(\underset{\sim}{Y}_1)\underset{\sim}{Y}_2 + \underset{\sim}{Y}_1\, \delta(\underset{\sim}{Y}_2)$ and (ii) $\delta(\mathrm{Log}\,|\underset{\sim}{Y}_3|) = \mathrm{tr}\, \underset{\sim}{Y}_3^{-1}\delta(\underset{\sim}{Y}_3)$, if $|\underset{\sim}{Y}_3| > 0$.

Lemma 2 can be verified directly from the definition.

For the likelihood ratio statistic, we are required to obtain $\underset{H}{\max}\, L$ (i.e. maximum value of L under H) and $\underset{H_o}{\max}\, L$ where

(9) $L = N(\utilde{X};\ \utilde{B}\utilde{\xi}\utilde{A}, \utilde{\Sigma}) = (2\pi)^{-\frac{1}{2}pn}\ |\utilde{\Sigma}|^{-\frac{1}{2}n}\exp[-\ \frac{1}{2}\mathrm{tr}\ \utilde{\Sigma}^{-1}(\utilde{X}-\utilde{B}\utilde{\xi}\utilde{A})(\utilde{X}-\utilde{B}\utilde{\xi}\utilde{A})']$.

First, we shall derive $(\overset{\max}{H}\ L)$. Let us maximize L with respect to $\utilde{\Sigma}$ (assuming $\utilde{\xi}$ to be known). We find that

(10) $(\underset{\utilde{\Sigma}}{\max}\ L) = (2\pi n)^{-\frac{1}{2}pn}\ |\utilde{S}|^{-\frac{1}{2}n}\ |\utilde{\phi}||^{-\frac{1}{2}n}\exp(-\ \frac{1}{2}pn)$

where

(11) $\utilde{\phi} = \utilde{I}_m + \utilde{A}\utilde{A}'\ (\utilde{Z} - \utilde{B}\utilde{\xi})'\ \utilde{S}^{-1}\ (\utilde{Z}-\utilde{B}\utilde{\xi}),\ \utilde{Z} = \utilde{X}\ \utilde{A}'(\utilde{A}\utilde{A}')^{-1}$,

and $\utilde{S} = \utilde{X}\ \{\utilde{I}_n - \utilde{A}'(\utilde{A}\utilde{A}')^{-1}\utilde{A}\ \}\ \utilde{X}'$ Now it is apparent that $(\overset{\max}{H}\ L)$ will be obtained if we minimize $|\utilde{\phi}|$ with respect to $\utilde{\xi}$. Taking differential of (Log $|\utilde{\phi}|$), we get after some adjustments as

$$\delta(\mathrm{Log}\ \ |\utilde{\phi}|) = -\ 2\ \mathrm{tr}\ [\ \utilde{\phi}^{-1}\ \utilde{A}\utilde{A}'(\utilde{Z}-\utilde{B}\ \utilde{\xi})'\ \utilde{S}^{-1}\utilde{B}\ \delta(\ \utilde{\xi})]\ ,$$

and so $\partial\ |\utilde{\phi}| / \partial\ \xi_{ij} = 0$ ($i = 1,2,\ldots,q$ and $j = 1,2,\ldots,m$) give us the maximum likelihood estimate $\hat{\utilde{\xi}}$ of $\utilde{\xi}$ as

(12) $(\utilde{Z} - \utilde{B}\hat{\utilde{\xi}})'\ \utilde{S}^{-1}\utilde{B} = \utilde{0}$ or $\hat{\utilde{\xi}} = (\utilde{B}'\ \utilde{S}^{-1}\ \utilde{B})^{-1}\ \utilde{B}'\ \utilde{S}^{-1}\ \utilde{Z}$.

Putting (12) in (11) and using the lemma 1, we get the minimum value of $|\utilde{\phi}|$ over $\utilde{\xi}$ as

(13) $(\underset{\utilde{\xi}}{\min}\ |\utilde{\phi}|) = |\ \utilde{I}_m + \utilde{A}\utilde{A}'\ \utilde{Z}'\ \utilde{B}^*_1(\utilde{B}^{*\prime}_1\ \utilde{S}\ \utilde{B}^*_1)^{-1}\ \utilde{B}^{*\prime}_1\ \utilde{Z}\ |$,

where $\utilde{B}^{*\prime}_1\ \utilde{B} = \utilde{0}$ and $\utilde{B}^*_1$: px(p-q) is of rank (p-q). Hence, we get

(14) $(\overset{\max}{H}\ L) = (2\pi\ n)^{-\frac{1}{2}pn}\ |\utilde{S}|^{-\frac{1}{2}n}\ (\underset{\utilde{\xi}}{\min}\ |\utilde{\phi}|\ )^{-\frac{1}{2}n}\exp(-\ \frac{1}{2}pn)$.

Now, to obtain $(\overset{\max}{H_0}\ L)$, we have to minimize $|\utilde{\phi}|$ subject to $\utilde{C}\utilde{\xi}\utilde{V} = \utilde{0}$. For this purpose, let $\utilde{C}_1$:(q-c)xq and $\utilde{V}_1$:m x (m-v) be of ranks (q-c) and (m-v) respectively such that $\utilde{C}\ \utilde{C}'_1 = \utilde{0}$ and $\utilde{V}'_1\ \utilde{V} = \utilde{0}$. Then, if $\utilde{C}_1\ \utilde{\xi} = \utilde{\eta}_1$:(q-c)xm and $\utilde{C}\ \utilde{\xi}\ \utilde{V}_1 = \utilde{\eta}_2$:cx(m-v), we get from $\utilde{C}\ \utilde{\xi}\ \utilde{V} = \utilde{0}$,

(15) $\utilde{\xi} = \utilde{C}'_1(\utilde{C}_1\ \utilde{C}'_1)^{-1}\ \ \utilde{\eta}_1 + \utilde{C}'(\utilde{C}\ \utilde{C}')^{-1}\ \ \utilde{\eta}_2(\utilde{V}'_1\ \ \utilde{V}_1)^{-1}\ \utilde{V}'_1$.

Putting this value of ξ in (10), we find that we have to minimize $|\phi|$ with respect to η_1 and η_2. First, we shall minimize $|\phi|$ with respect to η_1; and similar to (12), the equation for η_1 for given η_2 is

$$(16)\quad (C_1\ C_1')^{-1}\ \eta_1 = (C_1 B'\ S^{-1} B C_1')^{-1}\ C_1\ B'\ S^{-1}[Z-B\ C'(C\ C')^{-1}\eta_2(V_1'\ V_1)^{-1}\ V_1'].$$

Now letting $T_1 : p\text{x}(p-q+c)$, a matrix of rank $(p-q+c)$ such that $T_1'\ B\ C_1' = 0$, we get similar to (13)

$$(17)\quad (\min_{\eta_1} |\phi|) = |I_m + AA'[Z-BC'(CC')^{-1}\ \eta_2(V_1'V_1)^{-1}V_1']'T_1(T_1'ST_1)^{-1}T_1'[Z-BC'(CC')^{-1}\ \eta_2(V_1'V_1)^{-1}V_1']|.$$

By lemma 1, $T_1(T_1'\ S\ T_1)^{-1}\ T_1'$ remains the same for any $T_1 : p\text{x}(p-q+c)$ of rank $(p-q+c)$ satisfying $T_1'\ B\ C_1' = 0$. Hence, let us take $T_1 = [B_1^*\ \ S^{-1}B(B'\ S^{-1}B)^{-1}C']$. Then, we find that

$$[Z-BC'(CC')^{-1}\ \eta_2(V_1'V_1)^{-1}V_1']'\ \ T_1 = [Z'\ B_1^*\ \ \hat{\xi}'\ C'-V_1(V_1'\ V_1)^{-1}\ \eta_2'],\quad \text{and}$$

$$(T_1'\ S\ T_1)^{-1} = \begin{pmatrix} (B_1^{*\prime}\ S\ B_1^*)^{-1} & 0 \\ 0 & [C(B'S^{-1}B)^{-1}C']^{-1} \end{pmatrix},$$

where $\hat{\xi}$ is defined by (12). Let $R = (AA')^{-1} + Z'B_1^*(B_1^{*\prime}\ S\ B_1^*)^{-1}B_1^{*\prime}Z$, $Z_1 = C\ \hat{\xi}\ R^{-\frac{1}{2}}$, $Q = C(B'\ S^{-1}B)^{-1}C'$ and $A_3 = (V_1'\ V_1)^{-1}V_1'\ R^{-\frac{1}{2}}$; then (17) can be written as

$$(18)\quad (\min_{\eta_1}|\phi|) = |\ AA'\ |\ \ |R|\ \ |I_c + Q^{-1}(Z_1 - \eta_2A_3)(Z_1 - \eta_2A_3)'\ |\ .$$

Now the minimum value of (18) over η_2 will be obtained by using

$$(19)\qquad \eta_2 = Z_1\ A_3'\ (A_3\ A_3')^{-1}\ .$$

With the help of (15), (16), (19) and the lemma 1, the value of ξ which minimizes $|\phi|$ under $C\ \xi\ V = 0$ is

(20) $$\underset{\sim}{\xi}_o = \hat{\underset{\sim}{\xi}} - (\underset{\sim}{B}' \underset{\sim}{S}^{-1} \underset{\sim}{B})^{-1} \underset{\sim}{C}' \underset{\sim}{Q}^{-1} \underset{\sim}{C} \hat{\underset{\sim}{\xi}} \underset{\sim}{V}(\underset{\sim}{V}' \underset{\sim}{R} \underset{\sim}{V})^{-1} \underset{\sim}{V}' \underset{\sim}{R} \ ,$$

and

(21) $$(\min_{\underset{\sim}{\eta}_1, \underset{\sim}{\eta}_2} |\underset{\sim}{\phi}|) = |\underset{\sim}{A}\underset{\sim}{A}'| \ |\underset{\sim}{R}| \ |\underset{\sim}{I}_c + \underset{\sim}{Q}^{-1} \underset{\sim}{P}| \ ,$$

where $\underset{\sim}{P} = (\underset{\sim}{C}\hat{\underset{\sim}{\xi}} \underset{\sim}{V})(\underset{\sim}{V}' \underset{\sim}{R} \underset{\sim}{V})^{-1} (\underset{\sim}{C}\hat{\underset{\sim}{\xi}} \underset{\sim}{V})'$. Hence the maximum value of L under H_o is

(22) $$(\max_{H_o} L) = (2\pi n)^{-\frac{1}{2} np} |\underset{\sim}{S}|^{-\frac{1}{2} n} (\min_{\underset{\sim}{\eta}_1, \underset{\sim}{\eta}_2} |\underset{\sim}{\phi}|)^{-\frac{1}{2} n} \exp(- \tfrac{1}{2} pn) \ .$$

Using (14) and (22), we get the likelihood ratio statistic

(23) $$\Lambda = | \underset{\sim}{I}_c + \underset{\sim}{Q}^{-1} \underset{\sim}{P} |^{-1}$$

where $\underset{\sim}{Q} = \underset{\sim}{C}(\underset{\sim}{B}' \underset{\sim}{S}^{-1} \underset{\sim}{B})^{-1} \underset{\sim}{C}'$, $\underset{\sim}{P} = (\underset{\sim}{C}\hat{\underset{\sim}{\xi}} \underset{\sim}{V})(\underset{\sim}{V}' \underset{\sim}{R} \underset{\sim}{V})^{-1} (\underset{\sim}{C}\hat{\underset{\sim}{\xi}} \underset{\sim}{V})'$, $\underset{\sim}{Z} = \underset{\sim}{X} \underset{\sim}{A}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}$, $\hat{\underset{\sim}{\xi}} = (\underset{\sim}{B}' \underset{\sim}{S}^{-1} \underset{\sim}{B})^{-1} \underset{\sim}{B}' \underset{\sim}{S}^{-1} \underset{\sim}{Z}$, $\underset{\sim}{R} = (\underset{\sim}{A}\underset{\sim}{A}')^{-1} + \underset{\sim}{Z}' \underset{\sim}{B}_1^*(\underset{\sim}{B}_1^{*\prime} \underset{\sim}{S} \underset{\sim}{B}_1^*)^{-1} \underset{\sim}{B}_1^{*\prime} \underset{\sim}{Z}$, and $\underset{\sim}{B}_1^*(\underset{\sim}{B}_1^{*\prime} \underset{\sim}{S} \underset{\sim}{B}_1^*)^{-1} \underset{\sim}{B}_1^{*\prime} = \underset{\sim}{S}^{-1} - \underset{\sim}{S}^{-1}\underset{\sim}{B}(\underset{\sim}{B}' \underset{\sim}{S}^{-1} \underset{\sim}{B})^{-1} \underset{\sim}{B}' \underset{\sim}{S}^{-1}$.

Now with the help of (23), we suggest the following three criteria for testing H_o against H under model (1):

(24) Reject H_o if $\Lambda \leq \lambda_1$, or

(25) reject H_o if $\text{tr}\, \underset{\sim}{P} \underset{\sim}{Q}^{-1} \geq \lambda_2$, or

(26) reject H_o if $\text{ch}_1 \underset{\sim}{P} \underset{\sim}{Q}^{-1} \geq \lambda_3$

where $\text{ch}_j(\cdot)$ = j-th maximum characteristic root of $(\cdot)$, and

$$\Pr(\Lambda \leq \lambda_1 | H_o) = \Pr(\text{tr}\, \underset{\sim}{P} \underset{\sim}{Q}^{-1} \geq \lambda_2 \| H_o) = \Pr(\text{ch}_1 \underset{\sim}{P} \underset{\sim}{Q}^{-1} \geq \lambda_3 | H_o) = \alpha \ .$$

Now, when $m = v = c = 1$, it is easy to verify that $\Lambda = \text{tr}\, \underset{\sim}{P} \underset{\sim}{Q}^{-1} = \text{ch}_1 \underset{\sim}{P} \underset{\sim}{Q}^{-1}$ reduces to the test procedure given by Rao [9] and so it is a likelihood ratio test procedure. In the next section, we shall derive the test procedures for H_o' under model (7) and they are shown to be (24) or (25) or (26) by union-intersection principle.

(2.2) Union-intersection principle for testing H_o' under model (7):

Lemma 3: Let $M:(r+s)x(r+s)$ be a symmetric positive definite matrix and $N = M + YY'$, where $Y: (r+s)xv$. Let them be partitioned as

$$M = \begin{array}{cc} \left(\begin{array}{cc} M_{11} & M_{12} \\ M_{12}' & M_{22} \end{array}\right) & \begin{array}{c} r \\ s \end{array} \\ \begin{array}{cc} r & s \end{array} & \end{array}, \quad N = \begin{array}{cc} \left(\begin{array}{cc} N_{11} & N_{12} \\ N_{12}' & N_{22} \end{array}\right) & \begin{array}{c} r \\ s \end{array} \\ \begin{array}{cc} r & s \end{array} & \end{array} \text{ and } Y = \begin{array}{cc} \left(\begin{array}{c} Y_1 \\ Y_2 \end{array}\right) & \begin{array}{c} r \\ s \end{array} \\ v & \end{array} .$$

Then

$$(N_{11} - N_{12} N_{22}^{-1} N_{12}') - (M_{11} - M_{12} M_{22}^{-1} M_{12}')$$

$$= (Y_1 - M_{12}M_{22}^{-1}Y_2)(I_v + Y_2'M_{22}^{-1} Y_2)^{-1}(Y_1 - M_{12}M_{22}^{-1} Y_2)' \ \{\text{See } [3]\} .$$

Proof: We note that

$$N_{22}^{-1} = (M_{22} + Y_2Y_2')^{-1} = M_{22}^{-1} - M_{22}^{-1} Y_2(I_v + Y_2' M_{22}^{-1} Y_2)^{-1} Y_2' M_{22}^{-1} ,$$

$$N_{12}N_{22}^{-1} = M_{12}M_{22}^{-1} + (Y_1 - M_{12} M_{22}^{-1} Y_2)(I_v + Y_2'M_{22}^{-1} Y_2)^{-1} Y_2' M_{22}^{-1} , \text{ and so,}$$

$$N_{12}N_{22}^{-1} N_{12}' = M_{12} M_{22}^{-1} M_{12}' + Y_1Y_1' - (Y_1-M_{12}M_{22}^{-1}Y_2)(I_v+Y_2' M_{22}^{-1}Y_2)^{-1}(Y_1 - M_{12}M_{22}^{-1} Y_2)',$$

From this the lemma 3 is obvious.

With the help of model (7), it is easy to verify that the joint distribution of $D_1 = C(B'G B)^{-1}B'G X$ and $D_2 = (B_1' G B_1)^{-1}B_1' G X$ is normal with respective means $C \xi A$ and $\xi_1 A$, and covariance matrix

$$\Sigma_o = \left(\begin{array}{c} C(B'GB)^{-1}B'G \\ (B_1'GB_1)^{-1}B_1'G \end{array}\right) \Sigma \left[GB(B'GB)^{-1}C' \quad GB_1(B_1'GB_1)^{-1}\right]$$

$$= \begin{array}{cc} \left(\begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{array}\right) & \begin{array}{c} c \\ p-q \end{array} \\ \begin{array}{cc} c & p-q \end{array} & \end{array} , \text{ (say)}.$$

Then noting $\Sigma_{12}\ \Sigma_{22}^{-1} = -(B'\ \Sigma^{-1}B)^{-1}(B'\ \Sigma^{-1}B_1)$ and $\Sigma_{11} - \Sigma_{12}\ \Sigma_{22}^{-1}\ \Sigma_{12}' = C(B'\ \Sigma^{-1}B)^{-1}\ C'$, it is easy to verify that the conditional distribution of D_1, when D_2 is fixed, is multivariate normal with mean $C\,\eta\,A - (B'\ \Sigma^{-1}B)^{-1}(B'\ \Sigma^{-1}B_1)D_2$ and covariance matrix $C(B'\ \Sigma^{-1}B)^{-1}C'$. Now let a: cxl be any non-null vector. Then the conditional distribution of $a'D_1$, when D_2 is fixed, is normal with mean $\alpha'A + \beta'\ D_2$ and variance $\sigma^2 = a'C(B'\ \Sigma^{-1}B)^{-1}\ C'\ a$ where $\alpha' = a'\ C\eta$ and $\beta' = -\,a'(B'\ \Sigma^{-1}B)^{-1}(B'\ \Sigma^{-1}B_1)$. Now, the problem of testing $H_o'(C\ \eta\ V = 0)$ given by (8) is equivalent to testing $\bigcap_a\ H_{oa}'(a'\ C\ \eta\ V = \alpha'\ V = 0)$.

For this purpose, it is easy to verify that the estimate of σ^2 under $H_a'(\alpha'\ V \neq 0)$ and $H_{oa}(\alpha'\ V = 0)$ are respectively given by

$$(27)\qquad \hat{\sigma}^2_{H_a'} = a'(M_{11} - M_{12}\ M_{22}^{-1}\ M_{12}')a \quad \text{and} \quad \hat{\sigma}^2_{H_{oa}'} = a'(N_{11}-N_{12}N_{22}^{-1}N_{12}')\ a$$

where

$$(28)\qquad M = \begin{pmatrix} C(B'GB)^{-1}B'G \\ (B_1'GB_1)^{-1}B_1'G \end{pmatrix}\ S\left[GB(B'GB)^{-1}C' \quad GB_1(B_1'GB_1)^{-1}\right] = \begin{array}{cc} \begin{pmatrix} M_{11} & M_{12} \\ M_{12}' & M_{22} \end{pmatrix} & \begin{array}{c} c \\ p-q \end{array} \\ \begin{array}{cc} c & \quad p-q \end{array} & \end{array}$$

and

$$(29)\qquad N - M = YY' = \begin{pmatrix} C(B'GB)^{-1}B'G \\ (B_1'GB_1)^{-1}B_1'G \end{pmatrix}\ ZV[V'(AA')^{-1}V]^{-1}\ V'Z'\left[GB(B'GB)^{-1}C'\ GB_1(B_1'GB_1)^{-1}\right].$$

It is easy to verify that $\hat{\sigma}^2_{H_a'} \mid \sigma^2$ and $(\hat{\sigma}^2_{H_{oa}'} - \hat{\sigma}^2_{H_a'}) \mid \sigma^2$ under $H_{oa}'(\alpha'V = 0)$ are independently distributed as χ^2 with n-m-p+q and v degrees of freedom. Hence, the test procedure for testing H_{oa} is

$$(30)\qquad \text{to reject } H_{oa} \text{ if } f = \left(\hat{\sigma}^2_{H_{oa}'} - \hat{\sigma}^2_{H_a'}\right) \Big/ \hat{\sigma}^2_{H_a'} \geq f_\alpha$$

where $\Pr\{f \geq f_\alpha \mid H_{oa}'\} = \alpha$. With the help of (28) and (29), it is easy to verify that

$$(31)\qquad M_{11} - M_{12}M_{22}^{-1}\ M_{12}' = C(B'\ S^{-1}\ B)^{-1}\ C' = Q\ ,$$

(32) $$\underset{\sim}{D}_1 - \underset{\sim}{M}_{12}\underset{\sim}{M}_{22}^{-1}\underset{\sim}{D}_2 = \underset{\sim}{C}(\underset{\sim}{B}'\underset{\sim}{G}\underset{\sim}{B})^{-1}\underset{\sim}{B}'\underset{\sim}{G}[\underset{\sim}{I}_p - \underset{\sim}{S}\underset{\sim}{G}\underset{\sim}{B}_1(\underset{\sim}{B}_1'\underset{\sim}{G}\underset{\sim}{S}\underset{\sim}{G}\underset{\sim}{B}_1)^{-1}\underset{\sim}{B}_1'\underset{\sim}{G}]\underset{\sim}{X}$$
$$= \underset{\sim}{C}(\underset{\sim}{B}'\underset{\sim}{S}^{-1}\underset{\sim}{B})^{-1}\underset{\sim}{B}'\underset{\sim}{S}^{-1}\underset{\sim}{X} \quad ,$$

(33) $$\underset{\sim}{Y}_1 - \underset{\sim}{M}_{12}\underset{\sim}{M}_{22}^{-1}\underset{\sim}{Y}_2 = \underset{\sim}{C}(\underset{\sim}{B}'\underset{\sim}{S}^{-1}\underset{\sim}{B})^{-1}\underset{\sim}{B}'\underset{\sim}{S}^{-1}\underset{\sim}{Z}\ \underset{\sim}{V}[\underset{\sim}{V}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{V}]^{-\frac{1}{2}}$$

(34) $$\underset{\sim}{Y}_2'\ \underset{\sim}{M}_{22}^{-1}\ \underset{\sim}{Y}_2 = [\underset{\sim}{V}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{V}]^{-\frac{1}{2}}\ \underset{\sim}{V}'\underset{\sim}{Z}'\underset{\sim}{G}\underset{\sim}{B}_1(\underset{\sim}{B}_1'\underset{\sim}{G}\underset{\sim}{S}\underset{\sim}{G}\underset{\sim}{B}_1)^{-1}\underset{\sim}{B}_1'\underset{\sim}{G}\underset{\sim}{Z}\underset{\sim}{V}[\underset{\sim}{V}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{V}]^{-\frac{1}{2}} \quad ,$$

and on account of lemma 1, $\underset{\sim}{G}\underset{\sim}{B}_1(\underset{\sim}{B}_1'\underset{\sim}{G}\underset{\sim}{S}\underset{\sim}{B}_1)^{-1}\underset{\sim}{B}_1'\underset{\sim}{G} = \underset{\sim}{B}_1^*(\underset{\sim}{B}_1^{*\prime}\ \underset{\sim}{S}\ \underset{\sim}{B}_1^*)^{-1}\underset{\sim}{B}_1^{*\prime}$ where $\underset{\sim}{B}_1^*$ is the same as defined in (23). Hence using (31) , (32), (33) and (34) in (35), we have

(35) $$f = (\underset{\sim}{a}'\ \underset{\sim}{P}\ \underset{\sim}{a})/(\underset{\sim}{a}'\ \underset{\sim}{Q}\ \underset{\sim}{a})$$

where $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ are the same as defined in (23). The matrices $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ can be interpreted as s.p. (sum of squares and product) matrices due to H_o' and H' under model (7). Now for testing $H_o' = \bigcap_{\underset{\sim}{a}} H_{o\underset{\sim}{a}}'$, we get the test procedure by using union-intersection procedure~~s~~ to (30) with the help of (35) as follows:

(36) $$\text{Reject } H_o \text{ if } ch_1\ \underset{\sim}{P}\ \underset{\sim}{Q}^{-1} \geq \lambda_3$$

where $\Pr[ch_1\ \underset{\sim}{P}\ \underset{\sim}{Q}^{-1} \geq \lambda_3 \mid H_o'] = \alpha$. Note that (36) is the same as (26) suggested in (2.1) and we shall see in section 3, that their distributions under the null hypotheses are identical and so the same constant is used. It may now be noted that (26) or (36) implies (24) and (25), and hence we have the same three test procedures for testing H_o' under model (7).

(2.3) Connection between H_o and H_o'

It is easy to see that ,

(37) $$H_o^* = [H_o(\underset{\sim}{C}\ \underset{\sim}{\xi}\ \underset{\sim}{V} = \underset{\sim}{0}),\ \text{model (1)}] = H_o''(\ \underset{\sim}{\xi}_1 = \underset{\sim}{0},\ \underset{\sim}{C}\ \underset{\sim}{\xi}\ \underset{\sim}{V} = \underset{\sim}{0}) \quad \text{under model (7)}$$
$$= H_o^{(3)}\ (\ \underset{\sim}{\xi}_1 = \underset{\sim}{0}) \bigcap\ H_o'(\underset{\sim}{C}\ \underset{\sim}{\eta}\ \underset{\sim}{V} = \underset{\sim}{0}) \qquad \text{under model (7)}$$

Thus, the hypothesis H_o under model (1) is in fact the intersection of two hypotheses $H_o^{(3)}$ and H_o' under wider model (7). We have seen in section (2.2) the test procedures for H_o'. Now, the test procedures for testing $H_o^{(3)}$ under model (7) can be obtained as particular cases of (24), (25) or (26) replacing B by $(B \;\; B_1)$, $V = I_n$ and $C = (0 \;\; I_{p-q})$, we get ,

(38) Reject $H_o^{(3)}$ if $\Lambda_1 = |I_{p-q} + Q_1^{-1} P_1|^{-1} \leq \lambda_1'$, or

(39) reject $H_o^{(3)}$ if $\mathrm{tr}\,(Q_1^{-1} P_1) \geq \lambda_2'$, or

(40) reject $H_o^{(3)}$ if $\mathrm{ch}_1(Q_1^{-1} P_1) \geq \lambda_3'$

where nonzero $\mathrm{ch}_j(Q_1^{-1} P_1)$ = nonzero $\mathrm{ch}_j[AA' Z' B_1^*(B_1^{*\prime} S B^*)^{-1} B_1^{*\prime} Z]$ $(j=1,2,\ldots)$,

(41) $P_1 = B_1' G X A'(AA')^{-1} A X' G B_1$ and $Q_1 = B_1' G SGB_1$, and

$$\Pr(\Lambda_1 \leq \lambda_1' \mid H_o^{(3)}) = \Pr[\mathrm{tr}(Q_1^{-1} P_1) \geq \lambda_2' \mid H_o^{(3)}] =$$

$$= \Pr[\mathrm{ch}_1(Q_1^{-1} P_1) \geq \lambda_3' \,|H_o^{(3)}] = \alpha \; .$$

Hence the step down procedures according to Roy and Bargman [12] or J. Roy [10] for testing H_o^* in the sense of (37) are suggested as

(42) Accept H_o under model (1) if $(\Lambda \leq \lambda_4 \cap \Lambda_1 \leq \lambda_4)$, or

(43) accept H_o under model (1) if $(\mathrm{tr}\; P Q^{-1} \geq \lambda_5 \cap \mathrm{tr}(P_1 Q_1^{-1}) \geq \lambda_5)$ or

(44) accept H_o under model (1) if $(\mathrm{ch}_1\; PQ^{-1} \geq \lambda_6 \cap \mathrm{ch}_1(P_1 Q_1^{-1}) \geq \lambda_6)$

where λ_4, λ_5, λ_6 satisfy

(45) $\Pr(\Lambda \geq \lambda_4 \,|H_o') \Pr(\Lambda_1 \geq \lambda_4 \mid H_o^{(3)}) = \Pr(\mathrm{tr}\; P\, Q^{-1} \leq \lambda_5 |H_o')\Pr(\mathrm{tr} P_1 Q_1^{-1} \cancel{\geq}$

$\leq \lambda_5 \,|H_o^{(3)}) = \Pr(\mathrm{ch}_1 PQ^{-1} \leq \lambda_6 |H_o')\Pr(\mathrm{ch}_1 P_1 Q_1^{-1}$

$\leq \lambda_6 \,|H_o^{(3)}) = 1 - \alpha$, for the statistics are independent under H_o with model (1). The test procedures suggested in (42), (43) and (44) in place of (24), (25) and (26) reveal one fact that if $H_o^{(3)}$ is rejected, then (24), (25) and (26) are no longer valid for H_o under model (1), but they are only valid for H_o' under model (7). Hence, to carry out the test procedure for H_o under model (7), we obtain, with necessary changes in (24), (25), (26), the same test procedures as considered by Potthoff and Roy [8] when $\underset{\sim}{G}$ is any non-singular symmetric matrix as defined in the model (7). Hence we have a different interpretation of an arbitrary $\underset{\sim}{G}$ matrix introduced by them, and if model (1) is true, then test procedures with this arbitrary $\underset{\sim}{G}$ will not be natural and proper unless $\underset{\sim}{G} = \underset{\sim}{I}$ or $\underset{\sim}{G}^{-1}$ is the covariance matrix based on the similar previous data. This is because H_o has different implications $\overset{\#}{\wedge}$ in two different models.

It is of particular interest to note that the procedure in sections 2.1, 2.2 and 2.3 turn out eventually to be invariant under the choice of $\underset{\sim}{B}_1$ for completion of $\underset{\sim}{B}$ into a non-singular matrix, or, in other words, independent of $\underset{\sim}{G}$ in the sense in which $\underset{\sim}{G}$ occurs up to equation (41). However, as has been remarked in the previous paragraph in a slightly different language, the choice of $\underset{\sim}{B}_1$ affects the test procedures of Potthoff and Roy [8], if a test in that set up were sought to be obtained from the complete model.

3. Distribution and monotonicity of the test procedures under model (7):-

Let $\underset{\sim}{\Gamma}_1 = \underset{\sim}{A}'(\underset{\sim}{A}\,\underset{\sim}{A}')^{-1}\underset{\sim}{V}\,[\underset{\sim}{V}'(\underset{\sim}{A}\,\underset{\sim}{A}')^{-1}\underset{\sim}{V}]^{-\frac{1}{2}}$: n x v and $\Gamma_2 = \underset{\sim}{A}'\underset{\sim}{V}_1[\underset{\sim}{V}_1'\underset{\sim}{A}\,\underset{\sim}{A}'\,\underset{\sim}{V}_1']^{-\frac{1}{2}}$:nx(m-v), Then $\underset{\sim}{I}_n - (\underset{\sim}{\Gamma}_1\underset{\sim}{\Gamma}_1' + \underset{\sim}{\Gamma}_2\underset{\sim}{\Gamma}_2') = \underset{\sim}{I}-\underset{\sim}{A}'(\underset{\sim}{A}\,\underset{\sim}{A}')^{-1}\underset{\sim}{A}$ $= \underset{\sim}{\Gamma}_3\underset{\sim}{\Gamma}_3'$, where $\underset{\sim}{\Gamma}_3'$: (n-m)xn is an orthonormal matrix, i.e. $\underset{\sim}{\Gamma}_3'\;\underset{\sim}{\Gamma}_3 = \underset{\sim}{I}_{n-m}$. Hence $\underset{\sim}{\Gamma} = (\underset{\sim}{\Gamma}_1\;\underset{\sim}{\Gamma}_2\;\underset{\sim}{\Gamma}_3)$: nxn is an orthogonal matrix. Moreover, let $\underset{\sim}{\Delta} = [\,\underset{\sim}{\Sigma}^{-\frac{1}{2}}\underset{\sim}{B}(\underset{\sim}{B}'\;\underset{\sim}{\Sigma}^{-1}\,\underset{\sim}{B})^{-\frac{1}{2}}\quad \underset{\sim}{\Sigma}^{+\frac{1}{2}}\underset{\sim}{B}_1^*(\underset{\sim}{B}_1^{*\prime}\;\underset{\sim}{\Sigma}\;\underset{\sim}{B}_1^*)^{-\frac{1}{2}}\,]$: pxp be an orthogonal matrix

Now, we use the transformation

$$(46) \qquad \Delta' \; \Sigma^{-\frac{1}{2}} X \; \Gamma = Y : p \times n = \begin{array}{c} \left(\begin{array}{ccc} Y_1 & Y_2 & Y_3 \\ Y_4 & Y_5 & Y_6 \end{array} \right) \\ \begin{array}{ccc} v & m-v & n-m \end{array} \end{array} \begin{array}{c} q \\ p-q \end{array} \quad .$$

The jacobian of the transformation is $|\Sigma|^{\frac{1}{2}n}$, and with the help of (46),

$$(47) \qquad (B' \; S^{-1} \; B)^{-1} = (B' \; \Sigma^{-1} \; B)^{-\frac{1}{2}} \; [Y_3 \, Y_3' - Y_3 \, Y_6' \, (Y_6 \, Y_6')^{-1} Y_6 Y_3'](B' \; \Sigma^{-1} \; B)^{-\frac{1}{2}} ,$$

$$(48) \qquad \hat{\xi} \, V[V'(AA')^{-1}V]^{-\frac{1}{2}} = (B' \; \Sigma^{-1} B)^{-\frac{1}{2}} \, [Y_1 - Y_3 \, Y_6'(Y_6 \, Y_6')^{-1} \, Y_4] \quad ,$$

$$(49) \qquad [V'(AA')^{-1} \, V]^{-\frac{1}{2}} \, V' \; Z' \; B_1^* \; (B_1^{*\prime} \; S \; B_1^*)^{-1} \; B_1^{*\prime} \; Z \; V[V'(AA')^{-1}V]^{-\frac{1}{2}} = Y_4'(Y_6 Y_6')^{-1} Y_4$$

and

(50) the nonzero ch. roots of $P_1 Q_1^{-1}$ are the nonzero ch. roots of

$$(Y_6 \, Y_6')^{-1} \; [Y_4 \, Y_4' + Y_5 \, Y_5'] \; .$$

Moreover, since the ch. roots of $P_1 Q_1^{-1}$ and the matrices

$$P = C_3[Y_1 - Y_3 Y_6'(Y_6 Y_6')^{-1} Y_4][I_v + Y_4'(Y_6 \, Y_6')^{-1} Y_4]^{-1} \; [Y_1 - Y_3 Y_6'(Y_6 Y_6')^{-1} Y_4]' \; C_3'$$

and

$$Q = C_3[Y_3 \, Y_3' - Y_3 Y_6'(Y_6 \, Y_6')^{-1} \, Y_6 \, Y_3'] \, C_3' \quad ,$$

where $C_3 = C(B' \; \Sigma^{-1} \; B)^{-\frac{1}{2}}$, do not contain Y_2 , we may integrate out Y_2. Then, the joint density function of Y_1, Y_3, Y_4, Y_5 and Y_6 is given by

$$(51) \qquad N[Y_1; \; \xi_2, \; I_q] \; N \; [(Y_4 \; Y_5); \; \xi_4, \; I_{p-q}] \; N[Y_3; \; 0, \; I_q] \; N \; [Y_6; \; 0, \; I_{p-q}]$$

where

$$\xi_4 = (B_1' \, G \, \Sigma \; G \, B_1)^{-\frac{1}{2}} \, B_1' \, G \, B_1 \; \xi_1 \; \{ V[V'(AA')^{-1}V]^{-\frac{1}{2}} \; AA'V_1(V_1' \, AA'V_1)^{-\frac{1}{2}} \} \text{ and}$$

$$\xi_2 = (B' \; \Sigma^{-1} \; B)^{\frac{1}{2}} \; \eta \; V \; [V'(AA')^{-1}V]^{-\frac{1}{2}} , \quad \eta = \xi + (B' \; \Sigma^{-1} B)^{-1} (B' \; \Sigma^{-1} \; B_1) \; \xi_1 .$$

Let us consider the orthogonal matrices

$$(52)\qquad \underset{\sim}{\Delta}_1 = [(\underset{\sim}{B}'\ \underset{\sim}{\Sigma}^{-1}\underset{\sim}{B})^{-\frac{1}{2}}\ \underset{\sim}{C}'\ \{\ \underset{\sim}{C}(\underset{\sim}{B}'\ \underset{\sim}{\Sigma}^{-1}\ \underset{\sim}{B})^{-1}\ \underset{\sim}{C}'\}^{-\frac{1}{2}}\ \ \underset{\sim}{\Delta}_2]:\ q\mathrm{x}q$$

and

$$(53)\qquad \underset{\sim}{\Delta}_3 = [\underset{\sim}{Y}_6'\ (\underset{\sim}{Y}_6\ \underset{\sim}{Y}_6')^{-\frac{1}{2}}\ \ \underset{\sim}{\Delta}_4]:\ (n-m)\ \mathrm{x}\ (n-m)\quad .$$

Now we use the orthogonal transformations

$$(54)\qquad \underset{\sim}{\Delta}_1'\underset{\sim}{Y}_1 = \begin{array}{cc} \begin{pmatrix} \underset{\sim}{W}_1 \\ \underset{\sim}{W}_2 \end{pmatrix} & \begin{array}{c} c \\ q-c \end{array} \\ v & \end{array} \quad \text{and} \quad \underset{\sim}{\Delta}_1'\ \ \underset{\sim}{Y}_3\ \underset{\sim}{\Delta}_3 = \begin{array}{cc} \begin{pmatrix} \underset{\sim}{W}_3 & \underset{\sim}{W}_4 \\ \underset{\sim}{W}_5 & \underset{\sim}{W}_6 \end{pmatrix} & \begin{array}{c} c \\ q-c \end{array} \\ \begin{array}{cc} p-q & \quad n-m-p+q \end{array} & \end{array} \quad .$$

Then, it is easy to verify that the nonzero ch. roots of $\underset{\sim}{P}\underset{\sim}{Q}^{-1}$ are the nonzero ch. roots of

$$(55)\qquad (\underset{\sim}{W}_4\ \underset{\sim}{W}_4')^{-1}\ [\underset{\sim}{W}_1 - \underset{\sim}{W}_3(\underset{\sim}{Y}_6\underset{\sim}{Y}_6')^{-\frac{1}{2}}\ \underset{\sim}{Y}_4][\underset{\sim}{I}_v + \underset{\sim}{Y}_4'(\underset{\sim}{Y}_6\underset{\sim}{Y}_6')^{-1}\underset{\sim}{Y}_4]^{-1}[\underset{\sim}{W}_1-\underset{\sim}{W}_3(\underset{\sim}{Y}_6\underset{\sim}{Y}_6')^{-\frac{1}{2}}\ \underset{\sim}{Y}_4]'\ .$$

Since (55) does not contain $\underset{\sim}{W}_2$, $\underset{\sim}{W}_5$ and $\underset{\sim}{W}_6$, we can integrate out $\underset{\sim}{W}_2$, $\underset{\sim}{W}_5$ and $\underset{\sim}{W}_6$, and obtain the joint density function of $\underset{\sim}{Y}_4$, $\underset{\sim}{Y}_5$, $\underset{\sim}{Y}_6$, $\underset{\sim}{W}_1$, $\underset{\sim}{W}_3$ and $\underset{\sim}{W}_4$ as

$$(56)\qquad N[\underset{\sim}{W}_1;\ \underset{\sim}{\xi}_5,\ \underset{\sim}{I}_c]\ N\ [\underset{\sim}{W}_3;\ \underset{\sim}{0},\ \underset{\sim}{I}_c]\ N\ [\underset{\sim}{W}_4;\ \underset{\sim}{0},\ \underset{\sim}{I}_c]\ N\ [(\underset{\sim}{Y}_4,\ \underset{\sim}{Y}_5);\ \underset{\sim}{\xi}_4,\ \underset{\sim}{I}_{p-q}]N[\underset{\sim}{Y}_6;\underset{\sim}{0},\underset{\sim}{I}_{p-q}],$$

where

$$\underset{\sim}{\xi}_5 = [\underset{\sim}{C}(\underset{\sim}{B}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{B})^{-1}\underset{\sim}{C}']^{-\frac{1}{2}}\ (\underset{\sim}{C}\ \underset{\sim}{\eta}\ \underset{\sim}{V})[\underset{\sim}{V}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\ \underset{\sim}{y}]^{-\frac{1}{2}}\ :c\mathrm{x}v\ .$$

Now there exist two orthogonal matrices $\underset{\sim}{\Delta}_5 : c\times c$ and $\underset{\sim}{\Delta}_6 : v\times v$ such that

$$(57)\qquad \underset{\sim}{\xi}_5 = \underset{\sim}{\Delta}_5\ \underset{\sim}{\theta}\ \underset{\sim}{\Delta}_6\quad ,$$

where $\underset{\sim}{\theta} = (\ \theta_{ij}):\ c\mathrm{x}v$, $\theta_{ij} = 0$ for $i \neq j$ and θ_{ii}^2 $(i = 1,2,\ldots.)$ are the possibly ch. roots of $\underset{\sim}{\xi}_5\ \underset{\sim}{\xi}_5'$ or $[\underset{\sim}{C}(\underset{\sim}{B}'\ \underset{\sim}{\Sigma}^{-1}\ \underset{\sim}{B})^{-1}\underset{\sim}{C}']^{-1}(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})[\underset{\sim}{V}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{V}]^{-1}(\underset{\sim}{C}\underset{\sim}{\eta}\ \underset{\sim}{V})'$.

Let us use the orthogonal transformation

$$(58)\qquad \underset{\sim}{W}_1 = \underset{\sim}{\Delta}_5\ \underset{\sim}{W}_7\ \underset{\sim}{\Delta}_6,\quad \underset{\sim}{W}_4 = \underset{\sim}{\Delta}_5\ \ \underset{\sim}{U}_3\quad \text{and}\quad \underset{\sim}{W}_3 = \underset{\sim}{\Delta}_5\ \underset{\sim}{W}_8\quad .$$

Then the nonzero ch. roots of $\underset{\sim}{P}\ \underset{\sim}{Q}^{-1}$ are the nonzero ch. roots of

$$(59)\qquad (\underset{\sim}{U}_3\underset{\sim}{U}_3')^{-1}[\underset{\sim}{W}_7-\underset{\sim}{W}_8(\underset{\sim}{Y}_6\underset{\sim}{Y}_6')^{-\frac{1}{2}}\ \underset{\sim}{Y}_4\ \underset{\sim}{\Delta}_6'][\underset{\sim}{I}_v+\ \underset{\sim}{\Delta}_6\underset{\sim}{Y}_4'(\underset{\sim}{Y}_6\underset{\sim}{Y}_6')^{-1}\underset{\sim}{Y}_4\ \underset{\sim}{\Delta}_6']^{-1}[\underset{\sim}{W}_7-\underset{\sim}{W}_8(\underset{\sim}{Y}_6\underset{\sim}{Y}_6')^{-\frac{1}{2}}\underset{\sim}{Y}_4\ \underset{\sim}{\Delta}_6']'\ ,$$

and the joint density function of $\underset{\sim}{W}_7, \underset{\sim}{W}_8, \underset{\sim}{U}_3, \underset{\sim}{Y}_4, \underset{\sim}{Y}_5$ and $\underset{\sim}{Y}_6$ are given by

(60) $N[\underset{\sim}{W}_7; \underset{\sim}{\theta}, \underset{\sim}{I}_c]N[\underset{\sim}{W}_8; \underset{\sim}{0}, \underset{\sim}{I}_c]N[\underset{\sim}{U}_3; \underset{\sim}{0}, \underset{\sim}{I}_c]N[(\underset{\sim}{Y}_4,\underset{\sim}{Y}_5); \underset{\sim}{\xi}_4, \underset{\sim}{I}_{p-q}]N[\underset{\sim}{Y}_6; \underset{\sim}{0}, \underset{\sim}{I}_{p-q}]$.

Now, since $\underset{\sim}{W}_7$ and $\underset{\sim}{W}_8$ are independent normal variates, then the distribution of $\underset{\sim}{H} = \underset{\sim}{W}_7 - \underset{\sim}{W}_8(\underset{\sim}{Y}_6 \underset{\sim}{Y}_6')^{-\frac{1}{2}} \underset{\sim}{Y}_4 \underset{\sim}{\Delta}_6'$ is normal when $\underset{\sim}{Y}_6$ and $\underset{\sim}{Y}_4$ are fixed. Note that if $\underset{\sim}{h}_i$ $(i = 1,2,\ldots,c)$ is the i-th row of $\underset{\sim}{H}$, the covariance matrix of $\underset{\sim}{h}_i$ $(i = 1,2,\ldots,c)$ is $[\underset{\sim}{I}_v + \underset{\sim}{\Delta}_6 \underset{\sim}{Y}_4' (\underset{\sim}{Y}_6 \underset{\sim}{Y}_6')^{-1} \underset{\sim}{Y}_4 \underset{\sim}{\Delta}_6']$ and cov $(\underset{\sim}{h}_i, \underset{\sim}{h}_j) = \underset{\sim}{0}$ for $i \neq j$. Moreover $E(\underset{\sim}{H}) = \underset{\sim}{\theta}$. Now find $\underset{\sim}{\Delta}_7$: cxc and $\underset{\sim}{\Delta}_8$: vxv such that $\underset{\sim}{\theta} [\underset{\sim}{I}_v + \underset{\sim}{\Delta}_6 \underset{\sim}{Y}_4'(\underset{\sim}{Y}_6 \underset{\sim}{Y}_6')^{-1} \underset{\sim}{Y}_4 \underset{\sim}{\Delta}_6']^{-\frac{1}{2}} = \underset{\sim}{\Delta}_7 \underset{\sim}{\nu} \underset{\sim}{\Delta}_8$ where $\underset{\sim}{\nu} = (r_{ij})$, $r_{ij} = 0$ for $i \neq j$ and r_{ii}^2 $(i = 1,2,\ldots)$ are possibly the ch. roots of $\underset{\sim}{\theta}'\underset{\sim}{\theta} [\underset{\sim}{I}_v + \underset{\sim}{\Delta}_6 \underset{\sim}{Y}_4'(\underset{\sim}{Y}_6 \underset{\sim}{Y}_6')^{-1} \underset{\sim}{Y}_6 \underset{\sim}{\Delta}_6']^{-1}$. Making use of the transformation

(61) $\underset{\sim}{U}_4 = \underset{\sim}{H}[\underset{\sim}{I}_v + \underset{\sim}{\Delta}_6 \underset{\sim}{Y}_4'(\underset{\sim}{Y}_6 \underset{\sim}{Y}_6')^{-1} \underset{\sim}{Y}_4 \underset{\sim}{\Delta}_6']^{-\frac{1}{2}} = \underset{\sim}{\Delta}_7 \underset{\sim}{U}_2 \underset{\sim}{\Delta}_8', \underset{\sim}{U}_3 = \underset{\sim}{\Delta}_7 \underset{\sim}{U}_1$,

we get the joint density function of $\underset{\sim}{Y}_4$: (p-q) x v, $\underset{\sim}{Y}_5$: (p-q) x (m-v), $\underset{\sim}{Y}_6$: (p-q) x (n-m), $\underset{\sim}{U}_1$: c x(n-m-p+q) and $\underset{\sim}{U}_2$: cxv as

(62) $N[\underset{\sim}{U}_2; \underset{\sim}{\nu}, \underset{\sim}{I}_c]N[\underset{\sim}{U}_1; \underset{\sim}{0}, \underset{\sim}{I}_c]N[(\underset{\sim}{Y}_4 \underset{\sim}{Y}_5); \underset{\sim}{\xi}_4, \underset{\sim}{I}_{p-q}]$ $N[\underset{\sim}{Y}_6; \underset{\sim}{0}, \underset{\sim}{I}_{p-q}]$;

the nonzero ch. roots of $\underset{\sim}{P}\underset{\sim}{Q}^{-1}$ are the nonzero ch. roots of $(\underset{\sim}{U}_1 \underset{\sim}{U}_1')^{-1}(\underset{\sim}{U}_2 \underset{\sim}{U}_2')$, and the nonzero ch. roots of $\underset{\sim}{P}_1\underset{\sim}{Q}_1^{-1}$ are the nonzero ch. roots of $(\underset{\sim}{Y}_6 \underset{\sim}{Y}_6')^{-1} (\underset{\sim}{Y}_4\underset{\sim}{Y}_4'+\underset{\sim}{Y}_5\underset{\sim}{Y}_5')$.

We may note that when $\underset{\sim}{C} \underset{\sim}{\eta} \underset{\sim}{V} = \underset{\sim}{0}$ i.e. $\underset{\sim}{\nu} = \underset{\sim}{0}$, the ch. roots of $\underset{\sim}{P}\underset{\sim}{Q}^{-1}$ and the ch. roots of $\underset{\sim}{P}_1\underset{\sim}{Q}_1^{-1}$ are independently distributed whatever $\underset{\sim}{\xi}_1$ may be. Hence the distributions of Λ , tr $\underset{\sim}{P}\underset{\sim}{Q}^{-1}$ and $ch_1(\underset{\sim}{P}\underset{\sim}{Q}^{-1})$ can be obtained respectively by using the methods given in Anderson [1], Pillai [7] or Minoro Siotani [14] or Koichi Ito [5, 6], and Roy [11, Ch. 8] or Heck [2]. In Heck's notation (taking s*, m* and n* instead of s, m, and n). The parameters for the distribution of $[ch_1 \underset{\sim}{P}\underset{\sim}{Q}^{-1}/ \{ 1 + ch_1 \underset{\sim}{P}\underset{\sim}{Q}^{-1} \}] = w$ under H_o' [or under H_o under model (1)] will be

(63) $s^* = \min(c, v)$, $m^* = \frac{1}{2} (|c-v| - 1)$ and

$n^* = \frac{1}{2} (n - m - p + q - c - 1)$, while for the distribution of $w_1 = ch_1(\underset{\sim}{P}_1\underset{\sim}{Q}_1^{-1})]$ under $H_o^{(3)}(\underset{\sim}{\xi}_1 = \underset{\sim}{0})$, the parameters are

(64) $s^* = \min(p-q, m)$, $m^* = \frac{1}{2}(|p-q-m|-1)$ and $n^* = \frac{1}{2}(n-m-p+q-1)$.

Now with the help of (62), the results on the monotonicity or restricted monotonicity are the same as given by Khatri [4] and hence they are not repeated here. Moreover, instead of model (7) and the hypothesis H_o', if we consider model (1) and the hypothesis H_o, we get the same results as mentioned above, by taking $\underset{\sim}{\xi}_1 = \underset{\sim}{0}$.

4. Simultaneous confidence bounds on $\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V}$ [or $\underset{\sim}{C}\,\underset{\sim}{\xi}\,\underset{\sim}{V}$]:

For this purpose, we may note that if we consider $\underset{\sim}{P}^*$ and $\underset{\sim}{Q}^*$ given by

(65) $\underset{\sim}{P}^* = (\underset{\sim}{C}\,\hat{\underset{\sim}{\xi}}\,\underset{\sim}{V} - \underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})(\underset{\sim}{V}'\,\underset{\sim}{R}\,\underset{\sim}{V})^{-1}(\underset{\sim}{C}\,\hat{\underset{\sim}{\xi}}\,\underset{\sim}{V} - \underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})'$, and $\underset{\sim}{Q}^* = \underset{\sim}{Q}$ where ,

$\underset{\sim}{Q}$ and $\underset{\sim}{R}$ are the same as defined in (23). Then using the same method employed in section (3), it is easy to show that the distribution of the nonzero ch. roots of $\underset{\sim}{P}^*\underset{\sim}{Q}^{*-1}$ under model (7) is the same as that of the nonzero ch. roots of $\underset{\sim}{P}\underset{\sim}{Q}^{-1}$ under H_o'. Hence, we can find λ such that

(66) $$\Pr[ch_1(\underset{\sim}{P}^*\,\underset{\sim}{Q}^{*-1}) \leq \lambda] = 1 - \alpha ,$$

where λ will depend on c, v, and n-m-p+q. Hence (66) implies with probability equal to $(1-\alpha)$

(67) $$\underset{\sim}{a}'(\underset{\sim}{C}\,\hat{\underset{\sim}{\xi}}\,\underset{\sim}{V})\underset{\sim}{b} - \{\lambda(\underset{\sim}{a}'\underset{\sim}{Q}\,\underset{\sim}{a})(\underset{\sim}{b}'\,\underset{\sim}{V}'\,\underset{\sim}{R}\,\underset{\sim}{V}\,\underset{\sim}{b})\}^{\frac{1}{2}} \leq \underset{\sim}{a}'(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})\underset{\sim}{b}$$
$$< \underset{\sim}{a}'(\underset{\sim}{C}\,\hat{\underset{\sim}{\xi}}\,\underset{\sim}{V})\underset{\sim}{b} + \{\lambda(\underset{\sim}{a}'\underset{\sim}{Q}\,\underset{\sim}{a})(\underset{\sim}{b}'\,\underset{\sim}{V}'\,\underset{\sim}{R}\,\underset{\sim}{V}\,\underset{\sim}{b})\}^{\frac{1}{2}}$$

for all non-null vectors $\underset{\sim}{a}$: cxl and $\underset{\sim}{b}$: vxl. We may note that (67) will be the same for the simultaneous confidence bounds on $\underset{\sim}{C}\,\underset{\sim}{\xi}\,\underset{\sim}{V}$ if the model (1) is correct (i.e. $\underset{\sim}{\xi}_1 = \underset{\sim}{0}$). Hence, we shall not treat the case of $\underset{\sim}{C}\,\underset{\sim}{\xi}\,\underset{\sim}{V}$ separately.

If we put some elements of $\underset{\sim}{a}$ or $\underset{\sim}{b}$ as zero, we get the simultaneous confidence bounds on the partials of $\underset{\sim}{C}\,\underset{\sim}{\xi}\,\underset{\sim}{V}$ with probability greater than or equal to $(1-\alpha)$.

Now, let us consider the left hand side of (67). We rewrite it for the maximization over $\underset{\sim}{b}$ as

$$(68)(\quad [\underset{\sim}{a}'(\underset{\sim}{C}\hat{\underset{\sim}{\xi}}\underset{\sim}{V})\underset{\sim}{b}][\underset{\sim}{b}'\underset{\sim}{V}'\underset{\sim}{R}\,\underset{\sim}{V}\,\underset{\sim}{b}]^{-\frac{1}{2}} - [\lambda(\underset{\sim}{a}'\underset{\sim}{Q}\,\underset{\sim}{a})]^{-\frac{1}{2}} \leq (\underset{\sim}{a}'\,\underset{\sim}{C}\underset{\sim}{\eta}\,\underset{\sim}{V}\,\underset{\sim}{b})[\underset{\sim}{b}'\underset{\sim}{V}'\,\underset{\sim}{R}\,\underset{\sim}{V}\,\underset{\sim}{b}]^{-\frac{1}{2}}\ ,$$

and, it is easy to see that maximizing left of (68) with respect to $\underset{\sim}{b}$ implies for all non-null vector a: cxl,

$$(69)\qquad \begin{aligned}(\underset{\sim}{a}'\,\underset{\sim}{P}\,\underset{\sim}{a})^{\frac{1}{2}} - (\lambda\,\underset{\sim}{a}'\underset{\sim}{Q}\,\underset{\sim}{a})^{\frac{1}{2}} &\leq [\underset{\sim}{a}'(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})(\underset{\sim}{V}'\,\underset{\sim}{R}\,\underset{\sim}{V})^{-1}(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})'\,\underset{\sim}{a}]^{\frac{1}{2}} \\ &\leq [\underset{\sim}{a}'(\underset{\sim}{C}\underset{\sim}{\eta}\,\underset{\sim}{V})(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})'\ \ \underset{\sim}{a}\ ]^{\frac{1}{2}}\quad ch_1^{\frac{1}{2}}(\underset{\sim}{V}'\,\underset{\sim}{R}\,\underset{\sim}{V})^{-1}\ .\end{aligned}$$

[Note that if we want to have confidence bounds on $\underset{\sim}{a}'(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})[\underset{\sim}{V}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{V}]^{-1}(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})'\underset{\sim}{a}$, we have to replace $ch_j(\underset{\sim}{V}'\underset{\sim}{R}\,\underset{\sim}{V})$ by $1 + ch_j[\{\underset{\sim}{V}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{V}\}^{-1}(\underset{\sim}{Z}'\underset{\sim}{B}^*_1(\underset{\sim}{B}^{*\prime}_1\underset{\sim}{S}\underset{\sim}{B}^*_1)^{-1}\underset{\sim}{B}^{*\prime}_1\underset{\sim}{Z})]$. Hence, we shall not give here the explicit expressions for the confidence bounds on the parametric functions of $(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})[\underset{\sim}{V}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{V}]^{-1}(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})'$.].

Similarly, if we maximize (69) with respect to $\underset{\sim}{a}$, we shall get

$$(70)\qquad [ch_1^{\frac{1}{2}}\underset{\sim}{P}\underset{\sim}{Q}^{-1} - \sqrt{\lambda}\ ](ch_c^{\frac{1}{2}}\,\underset{\sim}{Q})[ch_v^{\frac{1}{2}}(\underset{\sim}{V}'\underset{\sim}{R}\underset{\sim}{V})] \leq\ ch_1^{\frac{1}{2}}\ [(\underset{\sim}{C}\underset{\sim}{\eta}\,\underset{\sim}{V})(\underset{\sim}{C}\underset{\sim}{\eta}\,\underset{\sim}{V})'].$$

Now, for the right hand side of (67), we may keep the same arguments as applied by Roy [11, ch. 14] and finally get the confidence bounds on $ch_1[(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})']$ with ~~greater~~ confidence greater than or equal to $(1-\alpha)$ as

$$(71)\qquad \begin{aligned}(ch_1^{\frac{1}{2}}\,\underset{\sim}{P}\underset{\sim}{Q}^{-1} - \sqrt{\lambda}\ )\ [ch_c^{\frac{1}{2}}\,\underset{\sim}{Q}][ch_v^{\frac{1}{2}}\,(\underset{\sim}{V}'\,\underset{\sim}{R}\,\underset{\sim}{V})] &\leq\ ch_1(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})' \\ &\leq\ ch_1^{\frac{1}{2}}\ [(\underset{\sim}{C}\hat{\underset{\sim}{\xi}}\underset{\sim}{V})(\underset{\sim}{C}\hat{\underset{\sim}{\xi}}\underset{\sim}{V})'] + \{\ \lambda\,(ch_1\,\underset{\sim}{Q})(ch_1\,\underset{\sim}{V}'\,\underset{\sim}{R}\,\underset{\sim}{V})\}^{\frac{1}{2}}\quad .\end{aligned}$$

[Note that right hand side of (71) can be improved, but it is not given here. Also, note that when H_o' is rejected (71) is always positive.].

Now instead of maximizing (69) with respect to $\underset{\sim}{a}$, we minimize the right of (69) with respect to $\underset{\sim}{a}$, and then (69) will imply

$$(72)\qquad [ch_c^{\frac{1}{2}}\ \underset{\sim}{P}\underset{\sim}{Q}^{-1} - \sqrt{\lambda}][ch_c^{\frac{1}{2}}\ \underset{\sim}{Q}][ch_v^{\frac{1}{2}}\ \ \underset{\sim}{V}'\,\underset{\sim}{R}\,\underset{\sim}{V}] \leq\ ch_c^{\frac{1}{2}}\,(\underset{\sim}{C}\underset{\sim}{\eta}\,\underset{\sim}{V})(\underset{\sim}{C}\underset{\sim}{\eta}\underset{\sim}{V})'\ .$$

We may note that if $ch_c\, \underset{\sim}{P}\underset{\sim}{Q}^{-1} > \lambda$, then $ch_c(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})'$ is positive and so $(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})'$ is p.d. We shall get $ch_c\, \underset{\sim}{P}\underset{\sim}{Q}^{-1} > 0$ if $(v \geq c)$. For $v < c$, we first maximize the left of (67) with respect to $\underset{\sim}{a}$, and then minimize with respect to $\underset{\sim}{b}$. By this way, we shall get instead of (72) the following expression when $v < c$:

$$(73) \quad [ch^{\frac{1}{2}}_{c-v}(\underset{\sim}{P}\underset{\sim}{Q}^{-1}) - \sqrt{\lambda}\,]\,(ch^{\frac{1}{2}}_{c}\,\underset{\sim}{Q})(ch^{\frac{1}{2}}_{v}\,\underset{\sim}{V}'\underset{\sim}{R}\,\underset{\sim}{V}) \leq ch^{\frac{1}{2}}_{c-v}(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})' .$$

We give here lower bounds explicitly, for we think that they are important when H_0' is rejected.

It may also be noted that if we maximize (69) [or minimize (69)] with respect to $\underset{\sim}{a}$ subject to some linear constraints, and then minimize [or maximize] with respect to these constraints, we shall get

$$(74) \qquad [ch^{\frac{1}{2}}_{j}\,(\underset{\sim}{P}\underset{\sim}{Q}^{-1}) - \sqrt{\lambda}]\,(ch^{\frac{1}{2}}_{c}\,\underset{\sim}{Q})(ch^{\frac{1}{2}}_{v}\,\underset{\sim}{V}'\underset{\sim}{R}\underset{\sim}{V}) \leq ch^{\frac{1}{2}}_{j}\,(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})' .$$

The similar results for the independence can be given. The result (74) can be utilized for framing the indecision procedures for MANOVA (and independence) on the lines of Roy and Gnanadesikan [13]. This will not be considered here. Including the upper bound for $ch^{\frac{1}{2}}_{j}\,(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})'$, we get with confidence greater than or equal to $(1-\alpha)$,

$$\begin{aligned}(75) \quad [ch^{\frac{1}{2}}_{j}\,(\underset{\sim}{P}\underset{\sim}{Q}^{-1}) -\sqrt{\lambda}\,]\,(ch^{\frac{1}{2}}_{c}\,\underset{\sim}{Q})(ch^{\frac{1}{2}}_{v}\,\underset{\sim}{V}'\underset{\sim}{R}\underset{\sim}{V}) &\leq ch^{\frac{1}{2}}_{j}\,(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})(\underset{\sim}{C}\,\underset{\sim}{\eta}\,\underset{\sim}{V})' \\ &\leq ch^{\frac{1}{2}}_{j}\,[(\underset{\sim}{C}\,\underset{\sim}{\hat{\xi}}\,\underset{\sim}{V})(\underset{\sim}{C}\,\underset{\sim}{\hat{\xi}}\,\underset{\sim}{V})'] + [\lambda(ch_1\,\underset{\sim}{Q})\, ch_1(\underset{\sim}{V}'\underset{\sim}{R}\,\underset{\sim}{V})]^{\frac{1}{2}} .\end{aligned}$$

The similar bounds on the partials can be written down easily.

5. Numerical example:

From the data [8b] of measurements on 11 girls and 16 boys at 4 different ages (8, 10, 12, 14), we get $\underset{\sim}{A}$: 2 x 27 to be a matrix composed of 11 (1, 0) columns followed by 16 (0, 1) columns, and other statistics are given in table 1.

(i) Examine the question that the growth curves due to girls and boys are linear. In this case, $\underset{\sim}{B}_1^{*\prime} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$ and $\underset{\sim}{B}' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & +3 \end{pmatrix}$. For this case, we use the test procedures given by (40). Using the values given in table 1, we find

$$\underset{\sim}{Z}'\underset{\sim}{B}_1^* = \begin{pmatrix} .33 & .05 \\ -1.13 & -.81 \end{pmatrix}, \quad \underset{\sim}{B}_1^{*\prime}\underset{\sim}{S}\underset{\sim}{B}_1^* = \begin{pmatrix} 1263.40 & 13.52 \\ 13.52 & 105.12 \end{pmatrix} \text{ and}$$

$$\underset{\sim}{Z}'\underset{\sim}{B}_1^*(\underset{\sim}{B}_1^{*\prime}\underset{\sim}{S}\underset{\sim}{B}_1^*)^{-1}\underset{\sim}{B}_1^{*\prime}\underset{\sim}{Z} = \begin{pmatrix} .00010677 & -.00064836 \\ -.00064836 & .00707551 \end{pmatrix}.$$

Hence $ch_1(\underset{\sim}{P}_1\underset{\sim}{Q}_1^{-1}) = ch_1[\underset{\sim}{A}\underset{\sim}{A}'\underset{\sim}{Z}'\underset{\sim}{B}_1^*(\underset{\sim}{B}_1^{*\prime}\underset{\sim}{S}\underset{\sim}{B}_1^*)^{-1}\underset{\sim}{B}_1^{*\prime}\underset{\sim}{Z}] = 0.113865$ and $w_1 = 0.10223$ with $s^* = 2$, $m^* = -\frac{1}{2}$ and $n^* = 11$, which is insignificant even at more than 20% level. Hence, in the latter part, we shall assume that the growth curves due to boys and girls are linear.

(ii) Now, we examine the question whether the linear growth rate due to girls and boys are equal or not?

Here $\underset{\sim}{C} = (0 \quad 1)$ and $\underset{\sim}{V}' = (-1 \quad 1)$, and we shall use test procedure (24) or (25) or (26). For this purpose, we get from table 1,

$$(\underset{\sim}{B}'\underset{\sim}{S}^{-1}\underset{\sim}{B}) = \begin{pmatrix} .01130476 & -.00664795 \\ -.00664795 & .36588157 \end{pmatrix}, \quad \underset{\sim}{B}'\underset{\sim}{S}^{-1}\underset{\sim}{Z} = \begin{pmatrix} .2530441 & .2763562 \\ .0229005 & .1378823 \end{pmatrix}.$$

$$\hat{\underset{\sim}{\xi}} = \begin{pmatrix} 22.6628 & 24.9340 \\ .474366 & .829833 \end{pmatrix}, \quad \underset{\sim}{R} = \begin{pmatrix} .09101586 & -.00064836 \\ -.00064836 & .06957555 \end{pmatrix}.$$

Then $ch_1\underset{\sim}{P}\underset{\sim}{Q}^{-1} = 0.28262$ and $F_{1,23} = 23 \times 0.28262 = 6.5$ which is significant at 2% level. Hence, we conclude that the growth rates for girls and boys are different and the 95% confidence interval on $\xi_{22} - \xi_{21}$ = difference between two growth rates is

$$0.067 \leq \xi_{22} - \xi_{21} \leq 0.644 .$$

This shows that the girls' and boys' growth curves are different, but they are linear on account of (i).

(iii) Obtain simultaneous confidence bounds on the girls' and boys' growth cruves.

For this purpose, we take $\lambda = (2/22)F_{0.025}(2,22) = 0.401$. Then using $\underset{\sim}{a}' = (1 \quad t)$ where t = age minus eleven, and $\underset{\sim}{b}' = (1 \quad 0)$ for girls' curve while $\underset{\sim}{b}' = (0 \quad 1)$ for boys' curve in (67), we get the simultaneous confidence bounds with probability greater than or equal to 95% (refer Potthoff and Roy [8b] as follows:

$$(22.663 + 0.47437t) \pm 0.19010\,(89.414 + 3.2492t + 2.7731t^2)^{\frac{1}{2}}$$

for girls' growth curve $\xi_{11} + \xi_{21}t$, while

$$(24.934 + 0.82983t) \pm 0.16703\,(89.414 + 3.2492t + 2.7731t^2)^{\frac{1}{2}}$$

for boys' growth curve $\xi_{12} + \xi_{22}t$.

(iv) The results given in (ii) and (iii) are different ~~than~~ from those given by Potthoff and Roy [8b] by taking a special type of matrix $\underset{\sim}{G}$ using previous data. Here, we give below for comparison the results based on the test procedures given by Potthoff and Roy [8] when $\underset{\sim}{G} = \underset{\sim}{I}$.

(a) The 95% confidence interval on $\xi_{22} - \xi_{21}$ is

$$0.024 \leq \xi_{22} - \xi_{21} \leq 0.588$$

while by using the calculations [8b] with their special $\underset{\sim}{G}$, $\xi_{22} - \xi_{21}$ is insignificant.

(b) The 95% simultaneous interval on the girls' and boys' growth curves respectively are

$$(22.648 + .4795t) \pm 0.1809\,(94.415 + 3.145t + 3.0445t^2)$$

and

$$(24.968 + 0.7855t) \pm 0.1500\,(94.415 + 3.145t + 3.0445t^2) \quad .$$

These results are practically similar to ones we have given in (ii) and (iii).

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TABLE 1

Matrix $\underset{\sim}{S}$ (S. P. matrix)				Matrix $\underset{\sim}{Z}$ (means)		Matrix $\underset{\sim}{B}$ (model matrix)		Matrix $\underset{\sim}{B}_1^*$ ($\underset{\sim}{B}_1^{*\prime}\underset{\sim}{B} = 0$)	
137.64	68.20	97.51	67.01	21.18	22.87	1	-3	-1	-1
68.20	104.91	72.93	82.68	22.23	23.81	1	-1	3	1
97.51	72.93	161.11	102.99	23.09	25.72	1	1	-3	1
67.01	82.68	102.99	124.34	24.09	27.47	1	3	1	-1
				Doo-Little method					
137.64	68.20	97.51	67.01	21.18	22.87	1	-3		
1	0.4954955	.7084423	.4868498	.1538797	.1661581	.00726533	-.02179599		
.	71.1172069	24.6142338	49.4768465	11.7354053	12.4780179	.5045045	.4864865		
.	1	.3432386	.6957085	.1650150	.1754571	.007093986	.00684063		
.	.	83.5812362	38.5349180	4.0571480	5.2349872	.1183923	2.9583460		
.	.	1	.4610475	.0485414	.0626335	.001416494	.03539486		
.	.	.	39.5283046	3.7435621	5.2411042	.1075777	2.7581586		
.	.	.	1	.0945050	.1325912	.002721536	.06977680		

From the above, we get

$$\underset{\sim}{B}'\underset{\sim}{S}^{-1}\underset{\sim}{B} = \begin{bmatrix} 1 & .5045045 & .1183923 & .1075777 \\ -3 & .4864865 & 2.9583460 & 2.7581586 \end{bmatrix} \begin{bmatrix} .00726533 & .007093986 & .001416494 & .002721536 \\ -.02179599 & .00684063 & .03539486 & .06977680 \end{bmatrix}'$$

and

$$\underset{\sim}{B}'\underset{\sim}{S}^{-1}\underset{\sim}{Z} = \begin{bmatrix} 1 & .5045045 & .1183923 & .1075777 \\ -3 & .4864865 & 2.9583460 & 2.7581586 \end{bmatrix} \begin{bmatrix} .1538797 & .1650150 & .0485414 & .0945050 \\ .1661581 & .1754571 & .0626335 & .1325912 \end{bmatrix}'.$$

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