ON THE MINIMALITY OF A BOUNDEDLY COMPLETE SUFFICIENT SUB FIELD

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This note, written for its pedagogical interest, attempts at a simplification of a proof due to R. R. Bahadur (1957) of the minimality of a boundedly complete sufficient sub-field.

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SUMMARY

This note, written for its pedagogical interest, attempts at a simplification of a proof due to R. R. Bahadur (1957) of the minimality of a boundedly complete sufficient sub-field[†].

NOTATIONS AND DEFINITIONS

Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be our probability structure. That is, $\mathcal{P}$ is a family $\{P\}$ of probability measures on a σ - field $\mathcal{A}$ of sub-sets of a sample space $\mathcal{X}$.

Definition 1: The set $N \in \mathcal{A}$ is said to be $\mathcal{P}$-null if

$$P(N) = 0 \text{ for all } P \in \mathcal{P} .$$

Definition 2: (a) The two sets A and B belonging to $\mathcal{A}$ are said to be $\mathcal{P}$- equivalent if their symmetric difference $A \Delta B$ is $\mathcal{P}$ -null

(b) The two $\mathcal{A}$ -mble functions f and g are said to be $\mathcal{P}$- equivalent if the set $\{x \mid f(x) \neq g(x)\}$ is $\mathcal{P}$ -null.

Definition 3: An $\mathcal{A}$ -mble function f is said to be $\mathcal{P}$ -integrable if

$$\int |f| \, dP < \infty \text{ for all } P \in \mathcal{P} .$$

[1]This research was supported by the Mathematics Division of the Air Force Office of Scientific Research.

[†]As usual we use the term sub-field to mean a sub-σ-field.

Definition 4: A sub-field $\mathcal{A}_*$ of $\mathcal{A}$ is said to be sufficient if corresponding to each $\mathcal{P}$ - integrable, $\mathcal{A}$ -mble f there exists an $\mathcal{A}_*$-mble f_* such that, for all $B \in \mathcal{A}_*$ and $P \in \mathcal{P}$,

$$\int_B f \, d P = \int_B f_* \, d P \, .$$

The function f_* is then called the conditional expectation of f given $\mathcal{A}_*$ and is determined upto a $\mathcal{P}$ -equivalence.

Definition 5: The sub-field $\mathcal{A}_o$ is said to be boundedly complete if the only bounded $\mathcal{A}_o$-mble functions satisfying the identity

$$\int f \, d P \equiv 0 \quad \text{for all } P \in \mathcal{P}$$

are those that are $\mathcal{P}$ -equivalent to zero.

Definition 6: $\mathcal{A}_o$ is said to be a minimal sufficient sub-field if each member of $\mathcal{A}_o$ is $\mathcal{P}$ -equivalent to some member of every alternative sufficient sub-field $\mathcal{A}_*$.

Now if $\mathcal{A}_*$ be sufficient then for each $\mathcal{A}$ -mble and square $\mathcal{P}$ -integrable f the conditional expection f_* is also square $\mathcal{P}$ -integrable and we have in addition

$$\int f^2 \, d P = \int f_*^2 \, dP + \int (f-f_*)^2 dP \quad \text{for all } P \in \mathcal{P} \, .$$

In other words,

$$\int f^2 dP \geq \int f_*^2 \, dP \qquad \text{for all } P \in \mathcal{P}$$

the sign of equalty holding for all $P \in \mathcal{P}$ if and only if f and f_* are $\mathcal{P}$ - equivalent.

THE THEOREM

<u>Theorem:</u> If $\mathcal{A}_o$ be a boundedly complete sufficient sub-field then $\mathcal{A}_o$ is a minimal sufficient sub-field.

<u>Proof</u>: Let $\mathcal{A}_*$ be any alternative sufficient sub-field and let A be an arbitrary member of $\mathcal{A}_o$. We have to prove the existence of a set $B \in \mathcal{A}_*$ such that A and B are $\mathcal{P}$-equivalent.

Let f be the indicator of A and let f_* be the conditional expectation of f given $\mathcal{A}_*$ and f_{*o} the conditional expectation of f_* given $\mathcal{A}_o$. Since f is bounded we can, without any loss of generality, assume that both f_* and f_{*o} are bounded.

Now, from definition 4 we have, for each $P \in \mathcal{P}$,

$$\int f \, dP = \int f_* \, dP = \int f_{*o} \, dP$$

Thus,

$$\int (f - f_{*o}) \, dP \equiv 0 \text{ for all } P \in \mathcal{P}$$

and $f - f_{*o}$ is a bounded $\mathcal{A}_o$-mble function. From the bounded completeness of $\mathcal{A}_o$ it then follows that f and f_{*o} are $\mathcal{P}$-equivalent and hence

$$\int f^2 \, dP = \int f_{*o}^2 \, dP \quad \text{for all} \quad p \in \mathcal{P}$$

But we know that for all $P \in \mathcal{P}$

$$\int f^2 \, dP \geq \int f_*^2 \, dP \geq \int f_{*o}^2 \, dP$$

Therefore, $\int f^2 dP \equiv \int f_*^2 \, dP$ for all $P \in \mathcal{P}$. and hence f and f_* are $\mathcal{P}$-equivalent.

Thus, the set

$$A = \{x \mid f(x) = 1\}$$

is $\mathcal{P}$-equivalent to the set

$$B = \{x \mid f_*(x) = 1\}$$

and so B is the $\mathcal{A}_*$-mble set we are searching after.

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