A NOTE ON THE CONFIDENCE BOUNDS FOR THE CHARACTERISTIC ROOTS OF DISPERSION MATRICES OF NORMAL VARIATES

C. G. Khatri

Institute of Statistics
Mimeograph Series No. 406
September 1964

UNIVERSITY OF NORTH CAROLINA

Department of Statistics

Chapel Hill, N.C.

A NOTE ON THE CONFIDENCE BOUNDS FOR THE CHARACTERISTIC ROOTS OF DISPERSION MATRICES OF NORMAL VARIATES

by

C. G. Khatri

September 1964

Contract No. AF-AFOSR-84-63

This research was supported by the Mathematics Division of the Air Force Office of Scientific Research.

Institute of Statistics
Mimeo Series No. 406

A NOTE ON THE CONFIDENCE BOUNDS FOR THE CHARACTERISTIC ROOTS
OF DISPERSION MATRICES OF NORMAL VARIATES

by

C. G. Khatri

University of North Carolina and Gujarat University

---

1. Introduction. Roy and Gnanadesikan [6] have considered certain types of different alternatives in the case of two dispersion matrices, of which first three are strong alternatives, in the sense that all characteristic roots of $\Sigma_1\Sigma_2^{-1}$ (i.e. ch $\Sigma_1\Sigma_2^{-1}$) are less than or equal to 1, or all ch $\Sigma_1\Sigma_2^{-1} \geq 1$, or union of these two alternatives. In this note, we derive the shorter confidence bounds than those given by Roy and Gnanadesikan [6] with probability $\geq (1-\alpha)$, when the alternative is true. Moreover, we show that when the alternative is complement of the null hypothesis, then we come across the same types of confidence bounds as derived by Anderson [1] by a different approach. Keeping in view the certain types of alternatives for more than two population dispersion matrices, we give shorter confidence bounds than those derived by Gnanadesikan [2,3], with probability $\geq (1-\alpha)$. All our confidence bounds are based on either X^2 or F distributions. The confidence bounds that we derive have not the purpose in mind to give confidence bounds on the partials and hence, for the partials we cannot say that our confidence bounds will at all be suitable and this is true for Anderson's results [1] too. Hence, we shall not try to give or compare any types of partials from them.

Without loss of generality, we shall assume that $S_i (i=1,2,\ldots,k)$ are independently distributed as Wishart denoted by $W(S_i, n_i, p, \Sigma_i)$, $(i=1,2,\ldots,k)$ whose density function is given by

$$2^{-\frac{1}{2}pn_i}\,\pi^{-\frac{1}{4}p(p-1)}\left\{\prod_{j=1}^{k}\Gamma\left(\frac{n_i-j+1}{2}\right)\right\}|\Sigma_i|^{-\frac{1}{2}n_i}\,|S_i|^{\frac{1}{2}(n_i-p-1)}\exp[-\tfrac{1}{2}\operatorname{tr}\Sigma_i^{-1}S_i].$$

We shall denote $ch_j \mathbf{A}$ the j-th maximum characteristic (max. ch.) root of $\mathbf{A}$: pxp, i.e. $ch_1 \mathbf{A} \geq ch_2 \mathbf{A} \geq \dots \geq ch_p \mathbf{A}$. When $k = 1$, $\mathbf{S}_1$, n_1 and $\mathbf{\Sigma}_1$ will be denoted by $\mathbf{S}$, n, and $\mathbf{\Sigma}$ respectively.

2. Simultaneous confidence bounds on $ch_p \mathbf{\Sigma}$ or $ch_1 \mathbf{\Sigma}$:

Lemma 1. Let $\mathbf{A}$: pxp be symmetric positive definite and $\mathbf{A} = (a_{ij})$, $\mathbf{A}^{-1} = (a^{ij})$. Then

$$ch_p \mathbf{A} \leq [\ a_{ii}\ ,\ (a^{jj})^{-1}\ ] \leq ch_1 \mathbf{A}$$

Proof:- by definition

$$ch_p \mathbf{A} \leq \boldsymbol{\varepsilon}' \mathbf{A} \boldsymbol{\varepsilon} \mid \boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} \leq ch_1 \mathbf{A}$$

for all non-null vectors $\boldsymbol{\varepsilon}$: px1. Let $\boldsymbol{\varepsilon}$ have zero elements except at the i-th place and so we get

$$ch_p \mathbf{A} \leq a_{ii} \leq ch_1 \mathbf{A}.$$

Similarly $ch_p \mathbf{A}^{-1} \leq a^{jj} \leq ch_1 \mathbf{A}^{-1}$. Since $ch_j \mathbf{A}^{-1} = (ch_{p-j+1} \mathbf{A})^{-1}$ for $\mathbf{A}$ is positive definite, we get the lemma 1.

Lemma 2. If $\mathbf{S}^* = (s^*_{ij})$ be distributed as $W(\mathbf{S}^* ; n, p, \mathbf{D}_\gamma)$ where $\mathbf{D}_\gamma = \text{diag.}(\gamma_1, \dots, \gamma_p)$, then (s^*_{ii}/γ_i) and $(\gamma_j s^{*jj})^{-1}$ are independently distributed as χ^2 with n and $n-p+1$ degrees of freedom respectively when $i \neq j$.

This lemma is a special case of Bartlett's decomposition theorem (e.g. see Kshirsagar [4]), and hence the proof is omitted.

Since $\mathbf{\Sigma}$ is symmetric positive definite, we can find an orthogonal matrix $\mathbf{\Delta}$ such that $\mathbf{\Delta} \mathbf{\Sigma} \mathbf{\Delta}' = \mathbf{D}_\gamma$, a diagonal matrix with $\gamma_1 \geq \dots \geq \gamma_p > 0$. Then the distribution of $\mathbf{S}^* = \mathbf{\Delta} \mathbf{S} \mathbf{\Delta}'$ is $W(\mathbf{S}^* ; n, p, \mathbf{D}_\gamma)$.

(2.1) Let us obtain the confidence bound on γ_1 with probability $\geq (1-\alpha)$ by considering the pair of hypotheses $H_0(\gamma_1=1)$ and $H_1(\gamma_1 \leq 1)$. We note by lemma 2, that s^*_{11}/γ_1 and $(\gamma_1 \; s^{*11})^{-1}$ are distributed as X^2 with n and $n-p+1$ d.f. $H_o(\gamma_1=1)$ against $H_1(\gamma_1 \leq 1)$ is tested by the critical region

$s^*_{11} \leq c$, constant, or $(s^{*11})^{-1} \leq c$, constant, and on account of

$$(s^{*11})^{-1} \leq s^*_{11} ,$$

we shall choose the critical region

(1) $$s^*_{11} \leq c$$

where c is determined from

(2) $$P_{\gamma}(X^2_n \geq c) = 1-\alpha .$$

Since s^*_{11} depends on the nuisance parameters of Σ, we cannot carry out the exact test given by (1), but we shall obtain the confidence bound on γ_1 with probability $\geq (1-\alpha)$. We note from (2) that

(3) $$P_{\gamma}(s^*_{11}/\gamma_1 \geq c \mid H_1) = 1-\alpha ,$$

but by lemma 1, $ch_1 S = ch_1 S^* \geq s^*_{11}$. Hence (3) gives us the following confidence bound on γ_1

(4) $$\gamma_1 \leq c^{-1} \, ch_1 S$$

with probability $\geq (1-\alpha)$, and it will be less than 1 when $ch_1 S \leq c$.

(2.2) Applying arguments similar to (2.1), we obtain the confidence bounds on γ_p with probability $\geq (1-\alpha)$ by considering the pair of hypotheses $H_o'(\gamma_p = 1)$ and $H_2(\gamma_p \geq 1)$ as

(5) $$\gamma_p \geq b^{-1} \, ch_p S$$

where b is determined from

(7) $$P_{\gamma}(X^2_{n-p+1} \leq b) = 1-\alpha .$$

(2.3) Now let us consider the pair of hypotheses $H_0(\gamma_p = 1 = \gamma_1)$ and $H_3 = H_1 \cup H_2$.

By union-intersection principle [7], the critical region is

(8) $$\{ b_1 \le (s^{*pp})^{-1} \} \cup \{ s^{*}_{11} \le c_1 \}$$

where, using lemma 2, b_1 and c_1 are obtained from

(9) $$P_{\mathfrak{r}} (X^2_{n-p+1} \le b_1) \, P_{\mathfrak{r}} (X^2_n \ge c_1) = 1-\alpha \ .$$

The test procedure (8) cannot be carried out in practice. Hence, noting lemma 1, we find the simultaneous confidence bounds on γ_1 or γ_p with probability $\ge (1-\alpha)$ as

(10) $$\gamma_1 \le c_1^{-1} \, ch_1 \, S \quad \text{or} \quad \gamma_p \ge b_1^{-1} \, ch_p \, S,$$

where b_1 and c_1 are given by (9).

We note that the confidence bounds given by (4), (5) and (10) are shorter than those which can be derived by Roy and Gnanadesikan's technique [5].

(2.4) Now, we note that Anderson's confidence bound [1] on all ch Σ can be derived from the following considerations.

$$(1-\alpha) = P_{\mathfrak{r}} \{ X^2_{n-p+1} \le b_1 \} \, P_{\mathfrak{r}} (X^2_n \ge c_1) = P_{\mathfrak{r}} [(\gamma_p \; s^{*pp})^{-1} \le b_1] \, P_{\mathfrak{r}} (s^{*}_{11}/\gamma_1 \ge c_1)$$

$$= P_{\mathfrak{r}} [c_1^{-1} (s^{*pp})^{-1} \le \text{all ch} (\Sigma) \le b_1^{-1} \, s^{*}_{11}] \text{ by lemma 2,}$$

and so using lemma 1, we have

(11) $$1-\alpha \le P_{\mathfrak{r}} [b_1^{-1} \, ch_p S \le \text{all ch}(\Sigma) \le c_1^{-1} \, ch_1 S].$$

3. <u>Simultaneous confidence bounds on</u> $ch_p \, \Sigma, \Sigma_2^{-1}$ <u>or</u> $ch_1 \, \Sigma_1 \Sigma_2^{-1}$.

Lemma 2. Let A and B be two pxp symmetric positive definite matrices, and let δ_1 and δ_p be two unit ch. vectors corresponding to the max. and minimum (min.) ch. roots of A. If $A = (a_{ij})$, $A^{-1} = (a^{ij})$, $t_i = \delta_i' B \delta_i$ and $t^i = \delta_i' B^{-1} \delta_i$

for $i=1,p$, then

$$ch_p \underset{\sim}{A}\underset{\sim}{B}^{-1} \leq [\, a_{ii}t^p \;,\; a_{ii}/t_p \;,\; t^p/a^{jj} \;,\; (t_p a^{jj})^{-1} \,]$$

and

$$ch_1 \underset{\sim}{A}\underset{\sim}{B}^{-1} \geq [\, a_{ii}t^1 \;,\; a_{ii}/t_1 \;,\; t^1/a^{jj} \;,\; (t_1 a^{jj})^{-1} \,] .$$

Proof. Let $\underset{\sim}{\Delta} : p\times p$ be an orthogonal matrix whose first column is $\underset{\sim}{\delta}_1$ and the last column is $\underset{\sim}{\delta}_p$ such that $\underset{\sim}{\Delta}'\underset{\sim}{A}\underset{\sim}{\Delta} = \underset{\sim}{D}_z$ is a diagonal matrix. Let $\underset{\sim}{U} = \underset{\sim}{\Delta}'\underset{\sim}{B}\underset{\sim}{\Delta}$. Then $u_{ii}=t_i$ and $u^{ii}= t^i$ for $i=1, p$. Now, we shall only prove the first part of the lemma 3, for the other part can similarly be proved. We note that

$$ch_p \underset{\sim}{A}\underset{\sim}{B}^{-1} = ch_p(\underset{\sim}{D}_z^{\frac{1}{2}} \; \underset{\sim}{U}^{-1} \underset{\sim}{D}_z^{\frac{1}{2}}) \leq [z_p t^p \;,\; z_p/t_p] \text{ by using lemma 1.}$$

Moreover, by lemma 1,

$$ch_p \underset{\sim}{A} = z_p \leq [a_{ii} \;,\; a^{jj}]$$

and so the first part of the lemma 3 is proved, and this proves the lemma.

Since $\underset{\sim}{\Sigma}_i$ (i=1,2) are symmetric positive definite, there exists a non-singular matrix $\underset{\sim}{C}$ such that

(12) $\quad \underset{\sim}{C}\underset{\sim}{\Sigma}_1\underset{\sim}{C}' = \underset{\sim}{D}_\gamma$,a diagonal matrix, and $\underset{\sim}{C}\underset{\sim}{\Sigma}_2\underset{\sim}{C}' = \underset{\sim}{I}$,

where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_p > 0$, $\gamma_i = ch_i \underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1}$. Let $\underset{\sim}{S}_1^* = \underset{\sim}{C}\underset{\sim}{S}_1\underset{\sim}{C}'$ and $\underset{\sim}{S}_2^* = \underset{\sim}{C}\underset{\sim}{S}_2\underset{\sim}{C}'$. Then $\underset{\sim}{S}_1^*$ and $\underset{\sim}{S}_2^*$ are independently distributed as

(13) $\quad W(\underset{\sim}{S}_1^* \,;\, n_1,p,\underset{\sim}{D}_\gamma)$ and $W(\underset{\sim}{S}_2^* \,;\, n_2,p,\underset{\sim}{I})$.

Lemma 4. Let $\underset{\sim}{\epsilon}_1$ and $\underset{\sim}{\epsilon}_p$ be the unit ch. vectors corresponding to the max. and min. ch. root of $S_1^*=(s^*_{ij})$, and let $\underset{\sim}{S}_1^*$ and $\underset{\sim}{S}_2^*$ be distributed as (13). Then $s^*_{11}(\underset{\sim}{\epsilon}_1'\underset{\sim}{S}_2^{*-1}\underset{\sim}{\epsilon}_1)(n_2-p+1)/n_1\gamma_1$ and $n_2/\{s^{*pp}(\underset{\sim}{\epsilon}_p'\underset{\sim}{S}_2^{*}\underset{\sim}{\epsilon}_p)\,\gamma_p(n_1-p+1)\}$,

where $\underset{\sim}{S}_1^{*-1} = (s^{*ij})$, are indpendently distributed as F with n_1,n_2-p+1 and

n_1-p+1 , n_2 d.f. respectively.

Proof. Let $\underset{\sim}{\epsilon}$ be an orthogonal matrix, whose first and the last columns are $\underset{\sim}{\epsilon}_1$ and $\underset{\sim}{\epsilon}_p$, such that $\underset{\sim}{\epsilon}'\underset{\sim}{S}^*_1\underset{\sim}{\epsilon}$ is a diagonal matrix. Then it is easy to see that $\underset{\sim}{S}^*_1$ and $\underset{\sim}{\epsilon}'\underset{\sim}{S}^*_2\underset{\sim}{\epsilon} = \underset{\sim}{U}$ are independently distributed as

$$W(\underset{\sim}{S}^*_1 ; n_1, p, \underset{\sim}{D}_\gamma) \quad \text{and} \quad W(\underset{\sim}{U}; n_2, p, \underset{\sim}{I}) \quad .$$

We note that if $\underset{\sim}{U}^{-1}=(u^{ij})$ and $\underset{\sim}{U} = (u_{ij})$, $u^{11} = (\underset{\sim}{\epsilon}'_1\underset{\sim}{S}^{*-1}_2\underset{\sim}{\epsilon}_1)$ and $u_{pp} = \underset{\sim}{\epsilon}'_p\underset{\sim}{S}^*_2\underset{\sim}{\epsilon}_p$. Using lemma 2, it can be seen that $(u^{11})^{-1}$, u_{pp}, s^*_{11}/γ_1, $(s^{*pp}\gamma_p)^{-1}$ are independently distributed as X^2 with respective d.f. n_2-p+1, n_2, n_1, n_2-p+1. From this, lemma 4 is obvious.

(3.1) Let us obtain the confidence bound on $\gamma_1 = ch_1\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ by considering the pair of hypotheses $H_o(\gamma_1=1)$ and $H_1(\gamma_1 \leq 1)$. Then, the critical region as in section (2.1) is

$$s^*_{11} \leq \text{ a constant} .$$

Here, we note that s^*_{11} contains unknown parameters depending on $\underset{\sim}{\Sigma}_2$ and so, we require a function depending on $\underset{\sim}{S}^*_2$. On account of $(\underset{\sim}{\epsilon}'_1\underset{\sim}{S}^{*-1}_2\underset{\sim}{\epsilon}_1) \geq (\underset{\sim}{\epsilon}'_1\underset{\sim}{S}^*_2\underset{\sim}{\epsilon}_1)^{-1}$, and lemma 3, we shall choose the critical region

$$(14) \qquad s^*_{11}\,(\underset{\sim}{\epsilon}'_1\underset{\sim}{S}^{*-1}_2\underset{\sim}{\epsilon}_1)\,(n_2-p+1)/n_1 \leq c$$

where c is to be determined from

$$(15) \qquad Pr\,(F_{n_1, n_2-p+1} \geq c.) = 1-\alpha \quad .$$

We note that even though we cannot carry out the test procedure (14), but we can make a confidence statement on γ_1 with probability $\geq (1-\alpha)$. By lemma 4, we have

$$(16) \qquad (1-\alpha) = Pr\,[\,s^*_{11}\,(\underset{\sim}{\epsilon}'_1\underset{\sim}{S}^{*-1}_2\underset{\sim}{\epsilon}_1)(n_2-p+1)/n_1\gamma_1 \geq c\,]$$

$$\leq Pr\,[(ch_1\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1})(n_2-p+1)/n_1c \geq \gamma_1] \text{ , by lemma 3 .}$$

Thus, (16) gives the confidence bound on γ_1 with probability $\geq (1-\alpha)$ considering the pair $H_o(\gamma_1=1)$ and $H_1(\gamma_1 \leq 1)$.

(3.2) Similarly, we obtain the confidence bound on $\gamma_p = ch_p(\Sigma_1 \Sigma_2^{-1})$ with probability $\geq (1-\alpha)$ by considering the pair of hypotheses $H_o'(\gamma_p=1)$ and $H_2(\gamma_p \geq 1)$ as

(17) $$(ch_p S_1 S_2^{-1})\, n_2/b(n_1-p+1) \leq \gamma_p$$

where b is to be determined from

(18) $$P_{\gamma}(F_{n_1-p+1,n_2} \leq b) = 1-\alpha \; .$$

(3.3). Here, we give the confidence bound on γ_1 or γ_p with probability $\geq (1-\alpha)$ by considering the hypotheses $H_o(\gamma_1=\gamma_p=1)$ and $H_3 = H_1 U H_2$ as

(19) $$\gamma_1 \leq c_1^{-1}(ch_1 S_1 S_2^{-1})(n_2-p+1)n_1^{-1} \text{ or } \gamma_p \geq b_1^{-1}(n_1-p+1)^{-1} n_2 (ch_p S_1 S_2^{-1})$$

where c_1 and b_1 are given by

(20) $$P_{\gamma}(F_{n_1-p+1,n_2} \leq b_1)\; P_{\gamma}(F_{n_1,n_2-p+1} \geq c_1) = 1-\alpha$$

The confidence bounds given by (16), (17) and (19) are shorter than those given by Roy and Gnanadesikan [6].

(3.4) Now, we note that Anderson's result [1] on all $(ch\, \Sigma_1 \Sigma_2^{-1})$ can be derived from the following considerations.

$$(1-\alpha) = P_{\gamma}(F_{n_1-p+1,n_2} \leq b_1)\; P_{\gamma}(F_{n_1,n_2-p+1} \geq c_1)$$

$$= P_{\gamma}[n_2/\{(n_1-p+1)\gamma_p s^{*pp}(\epsilon_p' S_2^* \epsilon_p)\} \leq b_1]\; P_{\gamma}[s_{11}^*(\epsilon_1' S_2^{*-1}\epsilon_1)(n_2-p+1)/n_1\gamma_1 \geq c_1]$$

$$= P_{\gamma}[n_2/\{(n_1-p+1)s^{*pp}(\epsilon_p' S_2^* \epsilon_p)b_1\} \leq \text{all } ch(\Sigma_1\Sigma_2^{-1}) \leq s_{11}^*(\epsilon_1' S_2^{*-1}\epsilon_1)(n_2-p+1)/n_1 c_1]$$

Hence, using lemma 3 we have with probability $\geq (1-\alpha)$,

(21) $\quad (ch_p \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1})\, n_2/(n_2-p+1)b_1 \leq \text{all ch } (\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1}) \leq (ch_1 \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1})(n_2-p+1)/n_1 c_1$,

where b_1 and c_1 are given by (20).

4. Simultaneous confidence bounds on $ch_t \underset{\sim}{\Sigma}_i$ $(i=1,2,\ldots,k;\ t=1,p)$.

Since $\underset{\sim}{\Sigma}_i$ is symmetric positive definite, there exists an orthogonal matrix $\underset{\sim}{\Delta}_i$ such that $\underset{\sim}{\Delta}_i \underset{\sim}{\Sigma}_i \underset{\sim}{\Delta}_i' = \underset{\sim}{D}_{i,\beta}$, a diagonal matrix, with diagonal elements $\beta_{i,1} \geq \beta_{i,2} \geq \cdots \geq \beta_{i,p} > 0$ $(i=1,2,\ldots,k)$. Let $\underset{\sim}{S}_i^* = \underset{\sim}{\Delta}_i \underset{\sim}{S}_i \underset{\sim}{\Delta}_i'$. Then $\underset{\sim}{S}_i^*$ are independently distributed as $W(\underset{\sim}{S}_i^* ;\, n_i, p\, \underset{\sim}{D}_{i,\beta})$, $(i=1,2,\ldots,k)$.

(4.1) Let us suppose that the k-th population is standard. We shall try to obtain the confidence bounds on $\beta_{i,1}|\beta_{k,p}$ $(i=1,2,\ldots,k-1)$ with probability $\geq (1-\alpha)$ by considering the hypotheses $H_o(\beta_{1,j} = \beta_{k,p}\ \ j = 1,2,\ldots,k-1)$ and $H_1 = \bigcup_{i=1}^{k-1} H_{1,i}\ (\beta_{i,1} \leq \beta_{k,p})$. We note that for testing $H_{o,i}\,(\beta_{i,1} = \beta_{k,p})$ against $H_{1,i}\ (\beta_{i,1} \leq \beta_{k,p})$, we have the critical region

(22) $$w_i :\ s_{i,11}^{*}\ s_k^{*pp}\ (n_k-p+1)/n_i \leq d_i'$$

where $\underset{\sim}{S}_k^{*-1} = (s_k^{*ij})$ and d'_i is to be determined from

(23) $$P_{\mathfrak{r}}\ (F_{n_i, n_k-p+1} \geq d_i') = 1-\alpha .$$

By the union intersection principle [7], the critical region for testing H_o against H_1 is

(24) $$w = \bigcup_{i=1}^{k} w_i \quad \text{such that} \quad P_{\mathfrak{r}}\,(\underset{\sim}{x} \in w \mid H_o) = \alpha .$$

Note that we cannot carry out the test procedure (24). Hence, we obtain the confidence bounds on $(\beta_{i,1}|\beta_{k,p})$ with probability $\geq (1-\alpha)$ from (24) as

(25) $$\beta_{i,1}|\beta_{k,p} \leq d_i^{-1}\ (ch_1 S_i)(ch_p S_k)^{-1}(n_k-p+1)n_i^{-1}\ ,\ i=1,\ldots,k-1$$

where $d_1, d_2, \ldots, d_{k-1}$ are to be calculated from

(26) $$P_{\not{\kappa}} \left(F_{n_i, n_k-p+1} \geq \alpha_i \ ; \ i=1,2,\ldots, k-1\right) = 1-\alpha$$

(4.2) Similarly, the confidence bounds on $(\beta_{i,p} | \beta_{k,1})$ with probability $\geq (1-\alpha)$ by considering the hypotheses $H_o(\beta_{i,p} = \beta_{k,1}, \ i=1,2,\ldots,k)$ and $H_2 = \bigcup_{i=1}^{k-1} H_{2,i}(\beta_{i,p} \geq \beta_{k,1})$, can be given by

(27) $$\beta_{i,p} | \beta_{k,1} \geq e_i^{-1} (ch_p S_i)(ch_1 S_k)^{-1} n_k (n_i-p+1)^{-1} \ , \ i= 1 \text{ or } 2 \text{ or } \ldots.$$

where $e_1, \ldots, e_{k-1}$ are to be determined from

(28) $$P_{\not{\kappa}} \left(F_{n_i-p+1, n_k} \leq e_i \ ; \ i=1,2,\ldots,k-1\right) = 1-\alpha$$

The values of d_i and e_i, $(i=1,2,\ldots,k-1)$ can be determined from Nair's tables [5,p.164] in some cases. The result similar to (2.3) and (3.3) can be written down for this case too, but we are not giving it, because it is very straightforward.

(4.3) Let us consider

(29) $$1-\alpha = P_{\not{\kappa}} \left(F_{n_i-p+1, n_k} \leq e_i \ ; \ i=1,2,\ldots,k-1\right) P_{\not{\kappa}} \left(F_{n_i, n_k-p+1} \geq d_i ; \ i=1,2,\ldots,k-1\right)$$

$$= P_{\not{\kappa}} \left[n_k | \{ s_i^{*pp} s_{k,11}^{*} (n_i-p+1) e_i \} \leq \beta_{i,p} | \beta_{k,1} \ , i=1,\ldots,k-1 \right]$$

$$P_{\not{\kappa}} \left[\beta_{i,1} | \beta_{k,p} \leq s_{i,11}^{*} s_k^{*pp} (n_p-p+1)(n_i d_i)^{-1}, i=1,2,\ldots,k-1 \right]$$

$$= P_{\not{\kappa}} \left[\frac{n_k}{s_i^{*pp} s_{k,11} (n_i-p+1) e_i} \leq \frac{\beta_{i,p}}{\beta_{k,1}} \ \left(\leq \beta_{i,1} | \beta_{k,p} \right) \leq \frac{s_{i,11}^{*} s_k^{*pp} (n_k-p+1)}{n_i d_i} \ , \ i=1,2 \ldots k-1 \right].$$

Hence using lemma 1, we have with probability $\geq (1-\alpha)$,

(30) $$\frac{n_k \ ch_p S_i}{e_i (n_i-p+1) ch_1 S_k} \leq \frac{\beta_{i,p}}{\beta_{k,1}} \leq \frac{\beta_{i,1}}{\beta_{k,p}} \leq \frac{(n_k-p+1) \ ch_1 S_i}{d_i n_i \ ch_p S_k} \quad \text{for } i=1,2,\ldots,k-1.$$

where e_i and d_i $(i=1, 2, \ldots, k-1)$ are given by (29).

We note that the confidence bounds given by (30) are shorter than those given by Gnanadesikan [2,3]. Our confidence bounds are not meant for deriving the confidence bounds on the partials. The confidence bounds given by (30) are in some sense better than those given by Anderson [1].

REFERENCES

[1] Anderson, T. W. (1963). Some optimum confidence bounds for roots of determinantal equations. Mimeo series, Columbia University, New York.

[2] Gnanadesikan, R. (1959). Equality of more than two variances and more than two dispersion matrices against certain alternatives. Ann. Math. Stat., 30, 177-184.

[3] Gnanadesikan, R. (1960). Correction to and comment on "Equality of more than two variances and of more than two dispersion matrices against certain alternatives", Ann. Math. Statist. 31 227-228.

[4] Kshirsagar, A. M. (1959). Bartlett Decomposition and Wishart distribution. Ann. Math. Statist. 30, 239-241.

[5] Pearson, E.S. and Hartley, H.O. (1958). Biometrika Tables for Statisticians, Vol. I. Second edition, Cambridge University press.

[6] Roy, S.N. and Gnanadesikan, R. (1962). Two sample, comparisons of dispersion matrices for alternatives of intermediate specificity. Ann. Math. Statist. 33, 432-437.

[7] Roy, S.N. (1953). On a heuristic method of test construction and its use in multivariate Analysis. Ann. Math. Statist. 24, 220-228.

NORTH CAROLINA STATE
OF THE UNIVERSITY OF NORTH CAROLINA
AT RALEIGH

INSTITUTE OF STATISTICS

(Mimeo Series available for distribution at cost)

344. Roberts, Charles D. An asymptotically optimal sequential design for comparing several experimental categories with a standard or control. 1963.
345. Novick, M. R. A Bayesian indifference procedure. 1963.
346. Johnson, N. L. Cumulative sum control charts for the folded normal distribution. 1963.
347. Potthoff, Richard F. On testing for independence of unbiased coin tosses lumped in groups too small to use χ^2.
348. Novick, M. R. A Bayesian approach to the analysis of data for clinical trials. 1963.
349. Sethuraman, J. Some limit distributions connected with fixed interval analysis. 1963.
350. Sethuraman, J. Fixed interval analysis and fractile analysis. 1963.
351. Potthoff, Richard F. On the Johnson-Neyman technique and some extensions thereof. 1963.
352. Smith, Walter L. On the elementary renewal theorem for non-identically distributed variables. 1963.
353. Naor, P. and Yadin, M. Queueing systems with a removable service stations. 1963.
354. Page, E. S. On Monte Carlo methods in congestion problems—I. Searching for an optimum in discrete situations. February, 1963.
355. Page, E. S. On Monte Carlo methods in congestion problems—II. Simulation of queueing systems. February, 1963.
356. Page, E. S. Controlling the standard deviation by cusums and warning lines.
357. Page, E. S. A note on assignment problems. March, 1963.
358. Bose, R. C. Strongly regular graphs, partial geometries and partially balanced designs. March, 1963.
359. Bose, R. C. and J. N. Srivastava. On a bound useful in the theory of factorial designs and error correcting codes. 1963.
360. Mudholkar, G. S. Some contributions to the theory of univariate and multivariate statistical analysis. 1963.
361. Johnson, N. L. Quota Fulfilment in finite populations. 1963.
362. O'Fallon, Judith R. Studies in sampling with probabilities depending on size. 1963.
363. Avi-Itzhak, B. and P. Naor. Multi-purpose service stations in queueing problems. 1963.
364. Brosh, I. and P. Naor. On optimal disciplines in priority queueing. 1963.
365. Yniguez, A. D'B., R. J. Monroe, and T. D. Wallace. Some estimation problems of a non-linear supply model implied by a minimum decision rule. 1963.
366. Dall'Aglio, G. Present value of a renewal process. 1963.
367. Kotz, S. and J. W. Adams. On a distribution of sum of identically distributed correlated gamma variables. 1963.
368. Sethuraman, J. On the probability of large deviations of families of sample means. 1963.
369. Ray, S. N. Some sequential Bayes procedures for comparing two binomial parameters when observations are taken in pairs. 1963.
370. Shimi, I. N. Inventory problems concerning compound products. 1963.
371. Heathcote, C. R. A second renewal theorem for nonidentically distributed variables and its application to the theory of queues. 1963.
372. Portman, R. M., H. L. Lucas, R. C. Elston and B. G. Greenberg. Estimation of time, age, and cohort effects. 1963.
373. Bose, R. C. and J. N. Srivastava. Analysis of irregular factorial fractions. 1963.
374. Potthoff, R. F. Some Scheffe-Type tests for some Behrens-Fisher type regression problems. 1963.